MATHEMATICS OF COMPUTATION
VOLUME 64, NUMBER 210
APRIL 1995, PAGES S19–S22

Supplement to

A NUMERICAL LIAPUNOV-SCHMIDT METHOD WITH APPLICATIONS TO HOPF BIFURCATION ON A SQUARE

PETER ASHWIN, KLAUS BÖHMER, AND MEI ZHEN

4. Computation of the Liapunov-Schmidt reduction

We provide here the details of the computations performed to provide the reduced bifurcation equations for Brusselator Hopf bifurcation. We use splittings and null space projections as defined in § 3. For λ, τ in (37)-(39) and for any

$$\eta := (z, \lambda, \tau) = (\sum_{i=1}^{4} \alpha_i \psi_i, \lambda, \tau) \in \mathrm{Ker}(\partial_u \Phi_0) \times \mathbb{R}^2 = \mathrm{Ker}(\Phi_0'), \tag{53}$$

we find that the truncated bifurcation equations to second order are given by

$$\begin{aligned} B_2(z,\lambda,\tau) &= (I-\hat{Q}) j_2 R(\sum_{i=1}^{4} \alpha_i \psi_i, \lambda, \tau) \\ &= (I-\hat{Q})\Big\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \lambda \sum_{i=1}^{4} \alpha_i \psi_i + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Big[\frac{B_0}{A_0} (\sum_{i=1}^{4} \alpha_i \psi_i^1)^2 \\ &\quad + 2A_0 (\sum_{i=1}^{4} \alpha_i \psi_i^1)(\sum_{i=1}^{4} \alpha_i \psi_i^2) \Big] - \tau \partial_t \sum_{i=1}^{4} \alpha_i \psi_i \Big\}, \end{aligned} \tag{54}$$

where we have used the notation $\psi_i = (\psi_i^1, \psi_i^2)^T$ and j_2 truncates to the polynomial of degree 2 in the Taylor expansion of Φ at $(0,0,0)$ with respect to α_i, λ and τ. In view of the orthogonality properties of $\{\phi_{kl}\}$, we can write

$$\begin{aligned} B_2(z,\lambda,\tau) &= (I-\hat{Q}) \Big[\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \lambda \sum_{i=1}^{4} \alpha_i \psi_i - \tau \partial_t \sum_{i=1}^{4} \alpha_i \psi_i \Big] \\ &= \sum_{j=1}^{4} \Big\langle \psi_j^*, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \lambda \sum_{i=1}^{4} \alpha_i \psi_i^1 - \tau \partial_t (\sum_{i=1}^{4} \alpha_i \psi_i) \Big\rangle \psi_j \\ &= \frac{1}{2} \{ [(\alpha_1 - \alpha_2(P_{11} - P_{12}))\lambda - 2\tau\alpha_2]\psi_1 + [((P_{11} - P_{12})\alpha_1 + \alpha_2)\lambda + 2\tau\alpha_1]\psi_2 \\ &\quad + [(\alpha_3 - \alpha_4(P_{11} - P_{12}))\lambda - 2\tau\alpha_4]\psi_3 + [((P_{11} - P_{12})\alpha_3 + \alpha_4)\lambda + 2\tau\alpha_3]\psi_4 \} \\ &= 0, \end{aligned}$$

which gives

$$(55)\qquad \frac{1}{2}\begin{pmatrix} \lambda & -(P_{11}-P_{12})\lambda - 2\tau \\ (P_{11}-P_{12})\lambda + 2\tau & \lambda \end{pmatrix}\begin{pmatrix} \alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4 \end{pmatrix} = 0.$$

Evidently, $B_2(z,\lambda,\tau)$ is not determined as it has vertically branching bifurcating solutions; we have a linear equation for $\{\alpha_i\}$ that has only zero as a solution unless λ, τ are both zero. To investigate the structure of solution branches of (37) at $(0,0,0)$, the next step of the truncated Liapunov-Schmidt method is needed.

4.1. Calculating w_2. To find the third-order truncation, we first need to solve w_2 from the equation

$$\begin{aligned} \partial_u\Phi(0,0,0)w_2 &= -\hat{Q}j_2R\Big(\sum_{i=1}^{4}\alpha_i\psi_i,\lambda,\tau\Big) \\ &= -\hat{Q}\Big\{\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}\lambda\sum_{i=1}^{4}\alpha_i\psi_i + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\Big[\frac{B_0}{A_0}\Big(\sum_{i=1}^{4}\alpha_i\psi_i^1\Big)^2 \\ &\qquad +2A_0\Big(\sum_{i=1}^{4}\alpha_i\psi_i^1\Big)\Big(\sum_{i=1}^{4}\alpha_i\psi_i^2\Big)\Big] - \tau\partial_t\sum_{i=1}^{4}\alpha_i\psi_i\Big\} \\ &=: f(\alpha,\lambda,\tau). \end{aligned}$$

Note that the right-hand side is a periodic function in t. More precisely, we have

$$(56)\qquad f(\alpha,\lambda,\tau) = f_0 + f_1\cos t + f_2\sin t + f_3\cos 2t + f_4\sin 2t,$$

where $f_i, i = 0,1,\dots,4$, are functions of α,λ,τ and x, y only. Decomposing the expected solution in the form

$$(57)\qquad w_2 = w_2^0 + w_2^1\cos t + w_2^2\sin t + w_2^3\cos 2t + w_2^4\sin 2t,$$

we solve the coefficients from the equations

$$(58a)\qquad L(B_0)w_2^0 = f_0(\alpha,\lambda,\tau),$$

$$(58b)\qquad \begin{pmatrix} L(B_0) & -\omega_0 I \\ \omega_0 I & L(B_0) \end{pmatrix}\begin{pmatrix} w_2^1 \\ w_2^2 \end{pmatrix} = \begin{pmatrix} f_1(\alpha,\lambda,\tau) \\ f_2(\alpha,\lambda,\tau) \end{pmatrix},$$

$$(58c)\qquad \begin{pmatrix} L(B_0) & -2\omega_0 I \\ 2\omega_0 I & L(B_0) \end{pmatrix}\begin{pmatrix} w_2^3 \\ w_2^4 \end{pmatrix} = \begin{pmatrix} f_3(\alpha,\lambda,\tau) \\ f_4(\alpha,\lambda,\tau) \end{pmatrix}.$$

Determining the expressions $f_0,\dots f_4$ in terms of the parameters of the problem and the null space coordinates, and then selecting the coefficients of the various monomials in the α_i gives us a set of linear problems on the square. To obtain these, we study the right-hand sides of the equations (58):

$$(59)\quad f_0(\alpha,\lambda,\tau) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}\sum_{i,j=1}^{4}\alpha_i\alpha_j\Big[\frac{B_0}{2A_0}(\mu_i\mu_j + \xi_i\xi_j) + A_0(\mu_i\nu_j + \xi_i\eta_j)\Big],$$

$$(60)\quad f_1(\alpha,\lambda,\tau) = -\hat{Q}\sum_{i=1}^{4}\alpha_i\Big[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\mu_i\lambda - \tau\begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}\Big],$$

$$(61)\quad f_2(\alpha,\lambda,\tau) = -\hat{Q}\sum_{i=1}^{4}\alpha_i\Big[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\xi_i\lambda + \tau\begin{pmatrix} \mu_i \\ \nu_i \end{pmatrix}\Big],$$

$$(62)\quad f_3(\alpha,\lambda,\tau) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}\sum_{i,j=1}^{4}\alpha_i\alpha_j\Big[-\frac{B_0}{2A_0}(\mu_i\mu_j - \xi_i\xi_j) + A_0(\mu_i\nu_j - \xi_i\eta_j)\Big],$$

$$(63)\quad f_4(\alpha,\lambda,\tau) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}\sum_{i,j=1}^{4}\alpha_i\alpha_j\Big[\frac{B_0}{A_0}\mu_i\xi_j + A_0(\mu_i\eta_j + \xi_i\nu_j)\Big],$$

where we have used the notation

$$\psi_i = \begin{pmatrix} \mu_i \\ \nu_i \end{pmatrix}\cos t + \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}\sin t,\ i = 1,\dots,4.$$

For the given basis of $\mathrm{Ker}(\partial_u\Phi_0)$ in (43), elementary calculations yield

(64)

$$f_0(\alpha,\lambda,\tau) = -\begin{pmatrix} 1 \\ -1 \end{pmatrix}\Big[\frac{B_0}{2A_0}(1+P_{11}^2) + A_0P_{11}P_{21}\Big](\alpha_1^2\phi_{kl}^2 + 2\alpha_1\alpha_3\phi_{kl}\phi_{lk}) + \cdots,$$

while the coefficients of the terms $\alpha_1\alpha_2$, $\alpha_1\alpha_4$ are equal to zero. Similarly,

(65)

$$\begin{pmatrix} f_1(\alpha,\lambda,\tau) \\ f_2(\alpha,\lambda,\tau) \end{pmatrix} = -\hat{Q}\begin{pmatrix} \lambda - \tau P_{11} & -\lambda P_{11} - \tau \\ -\lambda - \tau P_{11} & \lambda P_{11} \\ \lambda P_{11} + \tau & \lambda - \tau P_{11} \\ -\lambda P_{11} & -\lambda - \tau P_{21} \end{pmatrix}\begin{pmatrix} \alpha_1\phi_{kl} + \alpha_3\phi_{lk} \\ \alpha_2\phi_{kl} + \alpha_4\phi_{lk} \end{pmatrix}$$

and

(66)

$$\begin{aligned} f_3(\alpha,\lambda,\tau) = &-\begin{pmatrix} 1 \\ -1 \end{pmatrix}\Big\{\Big[\tfrac{B_0}{2A_0}(1-P_{11}^2) - A_0P_{11}P_{21}\Big](\alpha_1^2\phi_{kl}^2 + 2\alpha_1\alpha_3\phi_{kl}\phi_{lk}) \\ &-2\big(\tfrac{B_0}{A_0}P_{11} + A_0P_{21}\big)(\alpha_1\alpha_2\phi_{kl}^2 + \alpha_1\alpha_4\phi_{kl}\phi_{lk})\Big\} + \cdots, \\ f_4(\alpha,\lambda,\tau) = &-\begin{pmatrix} 1 \\ -1 \end{pmatrix}\Big[\big(\tfrac{B_0}{A_0}P_{11} + A_0P_{21}\big)(\alpha_1^2\phi_{kl}^2 + \alpha_1\alpha_3\phi_{kl}\phi_{lk}) \\ &+\big(\tfrac{B_0}{A_0}(1-P_{11}^2) - 2A_0P_{21}P_{11}\big)(\alpha_1\alpha_2\phi_{kl}^2 + \alpha_1\alpha_4\phi_{kl}\phi_{lk})\Big] + \cdots. \end{aligned}$$

Since f_i are polynomials of $\alpha \in \mathbb{R}^4$ and $\lambda,\tau \in \mathbb{R}$, in particular, f_0, f_3, f_4 are independent of λ,τ, the functions $w_2^i, i = 0,\dots,4$, are also polynomials of α,λ,τ

of the same form as f_i. Their coefficients can be solved from the corresponding equations. For example, the equation for the coefficient independent of t will be

(67) $$L(B_0)w_2^{0ij} = -\begin{pmatrix} 1 \\ -1 \end{pmatrix} \left[\frac{B_0}{2A_0}(\mu_i\mu_j + \xi_i\eta_j) + A_0(\mu_i\nu_j + \xi_i\eta_j)\right]$$

for $i, j = 1, \ldots, 4$, and one gets the solution w_2^0 of (58) as

$$w_2^0 = \sum_{i,j=1}^{4} \alpha_i\alpha_j w_2^{0ij}.$$

In fact, (58b) can be solved analytically in the following way. The restriction of the operator

$$\begin{pmatrix} L(B_0) & -\omega_0 I \\ \omega_0 I & L(B_0) \end{pmatrix}$$

to the subspace span$[\binom{\phi_{kl}}{0}, \binom{0}{\phi_{kl}}, \binom{\phi_{lk}}{0}, \binom{0}{\phi_{kl}}]$ reduces to the matrix

$$\omega_0 \begin{pmatrix} P & -I \\ I & P \end{pmatrix}.$$

Its inverse can be given explicitly (cf. [1]):

$$\omega_0 \hat{Q} \begin{pmatrix} P & -I \\ I & P \end{pmatrix}^{-1} \hat{Q} = -\frac{1}{4\omega_0} \begin{pmatrix} P & -I \\ I & P \end{pmatrix}.$$

Hence, the solution of (58b) is given by

(68)
$$\begin{pmatrix} w_2^1 \\ w_2^2 \end{pmatrix} = \frac{1}{4\omega_0} \begin{pmatrix} P & -I \\ I & P \end{pmatrix} \begin{pmatrix} \lambda - \tau P_{11} & -\lambda P_{11} - \tau \\ -\lambda - \tau P_{21} & \lambda P_{11} \\ \lambda P_{11} + \tau & \lambda - \tau P_{11} \\ -\lambda P_{11} & -\lambda - \tau P_{21} \end{pmatrix} \begin{pmatrix} \alpha_1\phi_{kl} + \alpha_3\phi_{lk} \\ \alpha_2\phi_{kl} + \alpha_4\phi_{lk} \end{pmatrix}.$$

Similarly, owing to the special form of f_0 in (64), the solution of (58a) can be generated by the solutions of the following two equations:

(69a) $$L(B_0)u = -\begin{pmatrix} 1 \\ -1 \end{pmatrix} \left[\frac{B_0}{2A_0}(1 + P_{11}^2) + A_0P_{11}P_{21}\right] \phi_{kl}^2,$$

(69b) $$L(B_0)u = -\begin{pmatrix} 1 \\ -1 \end{pmatrix} \left[\frac{B_0}{2A_0}(1 + P_{11}^2) + A_0P_{11}P_{21}\right] \phi_{kl}\phi_{lk}.$$

Concerning the equation (58c), together with (66), one sees that its solution can be generated from the solutions of the following four equations:

(70) $$\mathcal{L}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \binom{-1}{1}[\frac{B_0}{2A_0}(1 - P_{11}^2) - A_0P_{11}P_{21}]\phi_{kl}^2 \\ \binom{-1}{1}(\frac{B_0}{A_0}P_{11} + A_0P_{21})\phi_{kl}^2 \end{pmatrix},$$

(71) $$\mathcal{L}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -2\binom{-1}{1}(\frac{B_0}{A_0}P_{11} + A_0P_{21})\phi_{kl}^2 \\ \binom{-1}{1}[\frac{B_0}{A_0}(1 - P_{11}^2) - 2A_0P_{21}P_{11}]\phi_{kl}^2 \end{pmatrix},$$

(72) $$\mathcal{L}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2\binom{-1}{1}[\frac{B_0}{2A_0}(1 - P_{11}^2) - A_0P_{11}P_{21}]\phi_{kl}\phi_{lk} \\ 2\binom{-1}{1}(\frac{B_0}{A_0}P_{11} + A_0P_{21})\phi_{kl}\phi_{lk} \end{pmatrix},$$

(73) $$\mathcal{L}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -2\binom{-1}{1}(\frac{B_0}{A_0}P_{11} + A_0P_{21})\phi_{kl}\phi_{lk} \\ \binom{-1}{1}[\frac{B_0}{A_0}(1 - P_{11}^2) - 2A_0P_{21}P_{11}]\phi_{kl}\phi_{lk} \end{pmatrix},$$

where

$$\mathcal{L} := \begin{pmatrix} L(B_0) & -2\omega_0 \\ 2\omega_0 & L(B_0) \end{pmatrix}.$$

As an example, we consider solution of (70). Observing its right-hand side, one sees

$$\begin{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} [\frac{B_0}{2A_0}(1 - P_{11}^2) - A_0P_{11}P_{21}]\phi_{kl}^2 \\ \begin{pmatrix} -1 \\ 1 \end{pmatrix} (\frac{B_0}{A_0}P_{11} + A_0P_{21})\phi_{kl}^2 \end{pmatrix}$$
$$= \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} [\frac{B_0}{2A_0}(1 - P_{11}^2) - A_0P_{11}P_{21}]\phi_{kl}^2 + \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} (\frac{B_0}{A_0}P_{11} + A_0P_{21})\phi_{kl}^2.$$

Thus, solving the systems

(74) $$\begin{pmatrix} L(B_0) & -2\omega_0 \\ 2\omega_0 & L(B_0) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \phi_{kl}^2,$$

(75) $$\begin{pmatrix} L(B_0) & -2\omega_0 \\ 2\omega_0 & L(B_0) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \phi_{kl}^2,$$

we obtain the solution $(u_1, u_2)^T$ of (70) as follows:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = [\frac{B_0}{2A_0}(1 - P_{11}^2) - A_0P_{11}P_{21}] \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + (\frac{B_0}{A_0}P_{11} + A_0P_{21}) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Moreover, we have

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}.$$

Consequently, we need only solve (74) in order to get the solutions of (70) and (71), respectively. Correspondingly, replacing the factors ϕ_{kl}^2 in (74) by $\phi_{kl}\phi_{lk}$, and solving it, one gets solutions of (72) and (73) similarly as above. Thus, we can calculate all the coefficients w_2^{kij} for $k = 0, \dots, 4$ and $i, j = 1, \dots, 4$ from the solutions of the problems (69), (69b) and (74).

These problems are discretized using finite differences and then solved directly using the bandedness of the matrices in the discrete problems to give the coefficients for the various quadratic terms. Thus we can find the solutions w_2^k of (58a), (58c) in the form

$$w_2^k = \sum_{i,j=1}^{4} \alpha_i \alpha_j w_2^{kij}, \ k = 0, 3, 4. \tag{76}$$

The equivariance of the equations in (58) leads to the relations

$$\gamma w_2^i(\alpha) = w_2^i(\gamma\alpha) \ \forall\ \gamma \in \mathbf{D}_4 \times \mathbf{S}^1, \ i = 0, 3, 4.$$

Thus, once the coefficients $w_2^{i11}, w_2^{i12}, w_2^{i13}, w_2^{i14}$ of the terms $\alpha_1^2, \alpha_1\alpha_2, \alpha_1\alpha_3, \alpha_1\alpha_4$ in the expansion of $w_2^i, i = 0, 3, 4$, are known, all other terms can be derived via group actions. Hence to solve the equations (58), we solve four equations corresponding to $(i, j) = (1,1), (1,2), (1,3), (1,4)$ in the expansion of the right-hand side of (58) with respect to $\alpha_1, \dots, \alpha_4$. Even in the computation of these coefficients, the special structure of the systems allows further decomposition of the solutions and reduction of computational effort. In fact, the right-hand sides of (58) consist of products of various eigenfunctions of the Laplacian on the square and possess rich nodal line properties; in view of the underlying periodic boundary conditions of these linear systems, we can often restrict the problem to smaller fundamental domains; see, e.g., Mei [2].

4.2. Calculating B_3. Once again, we can use the symmetries of the problem to considerably reduce the number of calculations required for the bifurcation equations truncated to third order.

Note that we know all the second-order terms from the calculation of B_2, and so we only need to consider the third-order terms from the right-hand side of the equivalent to equation (11). This was performed by a Maple code, which generated the pointwise values of the function (in terms of the w_{2i}) that, when integrated, gave the coefficients of B_3. The Maple expression was then transcribed into (10k byte of) optimized Fortran formula. For simplicity, we briefly give the main idea used to calculate B_3. For $u = (u_1, u_2)^T$, $v = (v_1, v_2)^T \in C_{2\pi}^1$, we define

$$R_1(u, v) := \begin{pmatrix} 1 \\ -1 \end{pmatrix} \left[\frac{B_0}{A_0} u_1 v_1 + 2A_0 \frac{u_1 v_2 + u_2 v_1}{2} \right], \ R_2(u) := \begin{pmatrix} 1 \\ -1 \end{pmatrix} u_1,$$

$$R_3(u, u, u) := \begin{pmatrix} 1 \\ -1 \end{pmatrix} u_1^2 u_2, \ R_4(u, v) := \begin{pmatrix} 1 \\ -1 \end{pmatrix} u_1 v_1 / A_0.$$

Hence, for

$$\eta := (z, \lambda, \tau) = (\sum_{i=1}^{4} \alpha_i \psi_i, \lambda, \tau) \in \mathrm{Ker}(\partial_u \Phi_0) \times \mathbb{R}^2,$$

we have

$$\begin{aligned} B_3(z, \lambda, \tau) &= (I - \hat{Q}) j_3 R(\sum_{i=1}^{4} \alpha_i \psi_i + w_2, \lambda, \tau) \\ &= B_2(z, \lambda, \tau) + (I - \hat{Q})(2R_1(z, w_2(\eta)) + \lambda R_2(w_2(\eta)) \\ &\quad + R_3(z, z, z) + \lambda R_4(z, z) - \tau \partial_t w_2(\eta)). \end{aligned} \tag{77}$$

Again, B_3 is $\mathbf{D}_4 \times \mathbf{S}^1$ equivariant. Once $w_2(\eta)$ is known, $B_3(\eta)$ follows from simple calculations consisting mainly of L^2-products in $\hat{C}_{2\pi}$.

References

1. P. Ashwin and Z. Mei: *Liapunov-Schmidt reduction at Hopf bifurcation of the Brusselator equations on a square.* Preprint, University of Warwick, 1992.
2. Z. Mei: Path following around Corank-2 bifurcation points of a semi-linear elliptic problem with symmetry, *Computing* 47:69-85, 1991.

P. Ashwin, Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK

K. Böhmer, FB Math., Universität Marburg, Lahnberge, D-35032 Marburg, FRG

Z. Mei, FB Math., Universität Marburg, Lahnberge, D-35032 Marburg, FRG and Dept. Math., Xian Jiaotong University, Xi'an 710049, PRC