

AN ALGORITHM FOR NONDOMINANT SOLUTIONS OF LINEAR SECOND-ORDER INHOMOGENEOUS DIFFERENCE EQUATIONS

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ABSTRACT. An algorithm is given for computing a weighted sum of a nondominant solution of a linear second-order inhomogeneous difference equation to a prescribed accuracy by estimating the truncation error. The present method is an extension of both the stable numerical method due to Olver and Sookne and a summation technique due to Deuflhard for computing minimal solutions of a homogeneous difference equation. The method is illustrated by numerical examples.

1. INTRODUCTION

Let a_n, b_n, c_n, d_n be given sequences of real or complex numbers and let

$$(1.1) \quad a_n y_{n-1} + b_n y_n + c_n y_{n+1} = d_n, \quad n = 1, 2, \dots,$$

be a given linear second-order inhomogeneous difference equation. In this paper, we shall extend the stable numerical method due to Olver and Sookne [20] for computing minimal solutions of the homogeneous equation associated with (1.1) (i.e., $d_n = 0$ for all n):

$$(1.2) \quad a_n y_{n-1} + b_n y_n + c_n y_{n+1} = 0, \quad n = 1, 2, \dots,$$

satisfying a normalizing condition of the type

$$(1.3) \quad \sum_{m=0}^{\infty} \lambda_m y_m = s,$$

where λ_m is a given sequence of real or complex numbers and s is a constant, to the computation of a nondominant solution of the inhomogeneous equation (1.1).

We also extend and incorporate a summation technique due to Deuflhard [5] for a minimal solution of the homogeneous equation (1.2). Indeed, for a positive integer K and given constants $\xi_m, m = 0, 1, \dots, K$, we compute

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the weighted sum

$$(1.4) \quad S_K = \sum_{m=0}^K \xi_m y_m$$

of a nondominant solution of the inhomogeneous difference equation (1.1) to a prescribed accuracy by estimating the truncation error [9]. For applications that need a sum of the type (1.4), such as integration formulas of interpolatory type based on Chebyshev polynomials, we refer to [4, 10, 12]. See also §5.2.

Suppose that the homogeneous equation (1.2), where a_n and c_n do not vanish, has a pair of complementary solutions f_n and g_n such that $\lim_{n \rightarrow \infty} f_n/g_n = 0$. Then f_n and g_n are referred to as minimal and dominant, respectively [6]. Let the inhomogeneous equation (1.1) have a particular solution h_n for which $\lim_{n \rightarrow \infty} h_n/g_n = 0$, so that the general solution of (1.1) may be written as

$$(1.5) \quad y_n = \alpha f_n + \beta g_n + h_n,$$

where α and β are arbitrary constants. Nondominant solutions of (1.1) in the form $\alpha f_n + h_n$ cannot be found simply by forward and backward recurrence of the difference equation, owing to strong instability; see Gautschi [6], Jones and Thron [13, p.163, p.395] and Wimp [23] for surveys of nondominant solutions and the relationship between difference equations and continued fractions. See also Levrie and Bultheel [14].

Olver [18] introduced a stable method having a built-in error estimation technique [17] for the computation of nondominant solutions of (1.1). In Olver's method the original problem is replaced by an equivalent boundary value problem, which is solved by Gaussian elimination without pivoting; see §2.1.

Since then, extensions or reformulations of Olver's method have been made. Indeed, Van der Cruyssen [22] reformulates Olver's method by solving the boundary value problem by LU decomposition instead of Gaussian elimination to reduce both the amount of computational effort and the possible occurrence of overflow. An extension to high-order recurrence relations is made by Cash [2, 3], where the boundary value problem is solved either by using Gaussian elimination [2] or by using LU decomposition [3]. See Lozier [15] for a more detailed discussion. On the other hand, Wimp [23, p.98] formulates the case using a normalizing condition of the type (1.3) by using Gaussian elimination to solve a system of linear equations in which the coefficient matrix is a band matrix except for the first row representing (1.3).

In either method, whether the one based on Gaussian elimination or on LU decomposition, significant loss in accuracy may occasionally occur in the course of computation [22], [23, p.92]. Olver and Sookne [20] showed that in the case of a homogeneous difference equation it is possible to avoid the loss of accuracy in the elimination stage by combining Miller's algorithm and Olver's algorithm and by using a normalizing condition of the type (1.3). In §2.2, we briefly review their method [20].

In §3 we modify both the method due to Olver and Sookne and the summation technique due to Deuffhard to make them applicable in the more general case of inhomogeneous equations. Indeed, we develop a numerically stable al-

gorithm for computing the sum (1.4) of a nondominant solution of (1.1) by avoiding possible loss of accuracy.

Further, our method is an extension of the methods due to Olver [18] and Van der Cruyssen [22] for initial value problems to the case using the normalizing condition (1.3) as well as a generalization of the method given in Wimp [23, p.98]. In fact, our method also uses the LU decomposition scheme, along with a rank-one updating technique [8, p.592], to solve a system of linear equations which is obtained from the original difference equation (1.1) and the normalizing condition (1.3). The coefficient matrix is a band matrix except for having a row representing the normalizing condition (1.3) in an appropriate place in the matrix determined so that the loss of accuracy in the elimination stage may be avoided. The linear system is successively solved with the size of the system being increased until an estimated truncation error of the required solution is within the prescribed accuracy.

In §4, a convergence result for the present algorithm is discussed. Numerical examples are given in §5.

2. OLVER'S METHOD AND ITS MODIFICATION BY OLVER AND SOOKNE

2.1. Olver's algorithm and loss of accuracy. For simplicity, we assume that the initial value of the solution y_0 of (1.1) is given, although in Olver's method no initial values of the desired solution are required and a case of a more general normalizing condition (1.3) is discussed. Beginning with $p_0 = 0$ and $p_1 = 1$, one computes the solution p_n of (1.2) for $n = 2, 3, \dots$ and a sequence $\{q_n\}$ defined by

$$(2.1) \quad q_0 = y_0, \quad q_n = (a_n q_{n-1} - d_n p_n)/c_n, \quad n = 1, 2, \dots$$

The computation of p_n and q_n is terminated automatically at a certain value N of n , which is described below. Let ϵ be a given tolerance in the approximation $y_m^{(N)}$ ($m < N$) to the wanted solution y_m . Then the value N is determined so that the following stopping criterion is satisfied

$$(2.2) \quad \left| y_m^{(N+1)} - y_m^{(N)} \right| = |p_m q_N / (p_N p_{N+1})| \leq \epsilon.$$

The approximations $y_n^{(N)}$ can be generated by computing

$$y_n^{(N)} = (q_n + p_n y_{n+1}^{(N)}) / p_{n+1}, \quad n = N - 1, \dots, 1,$$

with starting value $y_N^{(N)} = 0$.

Significant loss in accuracy may occasionally occur in the course of the computation of p_n , especially when the value of $|f_0|$ is very small [23, p.92]. This can be seen as follows. Since p_n is expressed as

$$(2.3) \quad p_n = (f_0 g_n - g_0 f_n) / (f_0 g_1 - g_0 f_1),$$

it increases ultimately in proportion to the dominant solution g_n . If the value of $|f_0|$ is small compared with $|g_0 f_1 / g_1|$, however, initially p_n may behave more like a multiple of the minimal solution f_n but soon is contaminated with rounding error growing like g_n which causes a steady loss in significant figures.

Olver and Sookne [20, p.943] suggest that although there is a loss of significance if the condition prescribes the value of f_0 , this loss in accuracy often does not occur if a more general condition of the form (1.3) is used. In the next subsection we outline their method.

2.2. Modification by Olver and Sookne. Olver and Sookne show that the above drawback in Olver's method, when applied to the homogeneous equation (1.2), can be overcome by incorporating Miller's algorithm along with the normalizing condition (1.3). Their modification applies when the dominant solution g_n tends to infinity with n in such a way that ultimately $|g_n|$ is monotonic. In this connection, the following lemma due to Olver and Sookne [20] plays an important role.

Lemma 2.1 (Olver and Sookne). *For a positive integer M , let the coefficients a_n , b_n and c_n in (1.2) satisfy*

$$(2.4) \quad |b_n| \geq |a_n| + |c_n| \quad \text{when} \quad n \geq M + 1,$$

and $|b_n| < |a_n| + |c_n|$ otherwise. Let p_n be the solution of (1.2) satisfying $p_M = 0$ and $p_{M+1} = 1$. Then $|p_n| \geq |p_{n-1}|$ for all n greater than M .

Remark 2.1. The computed solution p_n above, which is dominant, is free from the numerical difficulty mentioned in the previous subsection.

By using the solution p_n ($n = M + 1, M + 2, \dots$) obtained in Lemma 2.1, with starting value $q_M = 1$ instead of $q_0 = y_0$ Olver and Sookne compute the sequence $\{q_n\}$ defined by (2.1), where d_n are replaced by zero for all n , until the stopping criterion (2.2) is satisfied. Once the value N has been determined in this way, they apply Miller's method; the backward recursion of (1.2) is made with starting values $y_N = 0$ and $y_{N-1} = q_{N-1}/p_N$ to generate y_n , $n = N, N - 1, \dots, 0$, which are finally renormalized by using the normalizing condition (1.3).

As is mentioned in Olver and Sookne [20], this scheme applies only to homogeneous difference equations; in the case of an inhomogeneous equation, both forward and backward recursion may be unstable. In the next section we consider the case of an inhomogeneous equation, for which we require some assumptions in addition to the condition (2.4).

3. STATEMENT OF THE ALGORITHM

Similarly to Olver and Sookne, here and henceforth we assume that the coefficients in (1.1) satisfy the condition (2.4). Further, we need some definition and assumptions on the relative behaviors of f_n , g_n , and h_n in a finite range of n as well as their asymptotic behaviors mentioned previously. We begin with the definition of dominance provided by Oliver [16] with some modification. Let x_n and z_n be two sequences. Then we say that x_n dominates z_n (or z_n is dominated by x_n) in the range $r \leq n \leq s$ if there exist $r \geq 0$ and $s \geq r$ such that $|x_{n+1}/x_n| > |z_{n+1}/z_n|$ for all n in the range $r \leq n \leq s$. Oliver's original definition is the case where $s \rightarrow \infty$. Now, we assume that in the range $0 \leq n \leq M$ the wanted solution $y_n = \alpha f_n + h_n$ does not dominate the complementary solutions f_n and g_n , whereas y_n dominates f_n and is dominated by g_n in the range $n > M$. Recall that since g_n is a dominant

$[\lambda_0, \dots, \lambda_{M-1}]^T$, where μ_k and λ_k are elements in the $(M + 1)$ th rows of L (3.8) and $A^{(N)}$ (3.2), respectively. Let U_M be the leading principal submatrix of order M of U (3.9)

$$U_M = \begin{bmatrix} a_1 & b_1 & c_1 & \dots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & c_{M-2} \\ \vdots & & & \ddots & b_{M-1} \\ 0 & \dots & & & a_M \end{bmatrix}.$$

Then one can see from (3.2) and (3.7) that $\mu^T U_M = \lambda^T$, i.e., $U_M^T \mu = \lambda$, which is easily solved for μ by forward substitution as follows:

$$(3.12) \quad \mu_i = \frac{\lambda_i - b_i \mu_{i-1} - c_{i-1} \mu_{i-2}}{a_{i+1}}, \quad i = 0, 1, \dots, M - 1,$$

with starting values $\mu_{-1} = \mu_{-2} = 0$. By using the values of $\mu = [\mu_0, \dots, \mu_{M-1}]^T$ obtained above, the elements ζ , l_j ($j = M + 1, \dots, N$) and u_j ($j = M, \dots, N$) in L (3.8) and U (3.9) are computed as follows:

$$(3.13) \quad \zeta = \lambda_{M+1} + c_M \mu_{M-1}, \quad u_M = \lambda_M - b_M \mu_{M-1} - c_{M-1} \mu_{M-2},$$

$$(3.14) \quad l_{M+1} = \frac{a_{M+1}}{u_M}, \quad u_{M+1} = b_{M+1} - \zeta l_{M+1},$$

$$(3.15) \quad l_j = \frac{a_j}{u_{j-1}}, \quad u_j = b_j - c_{j-1} l_j, \quad j = M + 2, \dots, N.$$

Now, we evaluate each component of the vectors \mathbf{z} , \mathbf{h} , \mathbf{v} and \mathbf{w} , for which we set $\mathbf{z} = [z_0, \dots, z_N]^T$, $\mathbf{h} = [h_0, \dots, h_N]^T$, $\mathbf{v} = [v_0, \dots, v_N]^T$ and $\mathbf{w} = [w_0, \dots, w_N]^T$, respectively. From the equations $L\mathbf{z} = \mathbf{d}^{(N)}$ and $L\mathbf{h} = \mathbf{e}$ we have

$$\begin{aligned} z_i &= d_{i+1}, & h_i &= 0, & i &= 0, \dots, M - 1, \\ z_M &= s - \sum_{k=0}^{M-1} \mu_k z_k, & h_M &= 1, \\ z_j &= d_j - l_j z_{j-1}, & h_j &= -l_j h_{j-1}, & j &= M + 1, \dots, N. \end{aligned}$$

Similarly, from $U^T \mathbf{v} = \mathbf{r}$ and $U^T \mathbf{w} = \boldsymbol{\xi}$ we have

$$w_{-1} = w_{-2} = 0, \quad w_i = \frac{\xi_i - b_i w_{i-1} - c_{i-1} w_{i-2}}{a_{i+1}}, \quad i = 0, 1, \dots, M - 1,$$

$$w_M = \frac{\xi_M - b_M w_{M-1} - c_{M-1} w_{M-2}}{u_M}, \quad w_{M+1} = \frac{\xi_{M+1} - \zeta w_M - c_M w_{M-1}}{u_{M+1}},$$

$$v_0 = v_1 = \dots = v_{M+1} = 0,$$

$$w_j = \frac{\xi_j - c_{j-1} w_{j-1}}{u_j}, \quad v_j = \frac{\lambda_j - c_{j-1} v_{j-1}}{u_j}, \quad j = M+2, \dots, N,$$

where we have set $\xi_i \equiv 0$ ($i > K$).

From (3.10) we can compute the values of x and $S_K^{(N)}$ as follows:

$$(3.16) \quad \begin{aligned} x_i &= z_i, \quad i = 0, \dots, M-1, & x_j &= z_j - h_j \phi(N), \quad j = M, \dots, N, \\ S_K^{(N)} &= \sum_{k=0}^N w_k x_k = \sum_{k=0}^N w_k z_k - \phi(N) \sum_{k=M}^N w_k h_k, \end{aligned}$$

where $\phi(N)$ is defined by

$$\phi(N) = \frac{\sum_{k=M+2}^N v_k z_k}{1 + \sum_{k=M+2}^N v_k h_k}.$$

It follows immediately from (3.16) that

$$(3.17) \quad S_K^{(N+1)} - S_K^{(N)} = w_{N+1} z_{N+1} - \phi(N+1) \sum_{k=M}^{N+1} w_k h_k + \phi(N) \sum_{k=M}^N w_k h_k,$$

and the truncation error in $S_K^{(N)}$ can be written in the form

$$(3.18) \quad \begin{aligned} S_K - S_K^{(N)} &= \sum_{n=N}^{\infty} \{S_K^{(n+1)} - S_K^{(n)}\} \\ &= \sum_{k=N+1}^{\infty} w_k z_k - \phi(\infty) \sum_{k=M}^{\infty} w_k h_k + \phi(N) \sum_{k=M}^N w_k h_k. \end{aligned}$$

Generally, the right-hand side of (3.17) converges to zero rapidly as N increases and therefore the truncation error (3.18) is of the same order of magnitude as the first term $S_K^{(N+1)} - S_K^{(N)}$. Here, we consider the following two cases: if we wish to compute $S_K^{(N)}$ to the required absolute accuracy ε_a , then we apply the recurrence relation for values of n past both K and M until

$$(3.19) \quad |S_K^{(n+1)} - S_K^{(n)}| \leq \varepsilon_a,$$

is satisfied, and we then set $N = n$. On the other hand, if we wish to compute $S_K^{(N)}$ to the relative accuracy ε_r , then we examine the condition

$$(3.20) \quad |(S_K^{(n+1)} - S_K^{(n)}) / S_K^{(n)}| \leq \varepsilon_r,$$

instead of (3.19). If the series (3.18) does not converge rapidly, the stopping criteria above could fail. To enhance the reliability of the criteria, it might be advisable to continue the computation until the condition (3.19) (or (3.20)) is satisfied several times, at least twice consecutively. In the numerical examples below we checked the criterion (3.19) twice.

Now we note that the appropriate choice of M in the present algorithm plays an important role in reducing the number of computations required as well as in

reducing the possible occurrence of loss of accuracy mentioned in §2.1. Indeed, the present algorithm requires less computational effort if M can be set as large as possible so as to satisfy the condition (2.4). To see this, we now count the number of computations required in the present algorithm. Factorization into an LU product requires $N + M$ multiplications and N divisions. To solve for \mathbf{v} (and \mathbf{w}), $N - M - 1$ (and $N + M$) multiplications and $N - M - 1$ (and $N + 1$) divisions are required. Similarly, N and $N - M$ multiplications are required in solving for \mathbf{z} and \mathbf{h} , respectively. Further, $4N - 3M + 1$ multiplications and a single division are required to compute $S_K^{(N)}$. To sum up, we need $9N - 3M - 1$ multiplications and $3N - M + 1$ divisions.

On the other hand, for the matrix $A^{(N)}$ (3.2) in the special case where $M = 0$ and $\lambda_1 = 0$, the computation of (3.15) might occasionally suffer from loss of accuracy similar to §2.1. The reason is simply seen as follows. Now, we assume that $\lambda_0 \neq 0$ in addition to $M = 0$ and $\lambda_1 = 0$. Then we see from (3.13) and (3.14) that $\zeta = 0$, $u_0 = \lambda_0$, $l_1 = a_1/\lambda_0$ and $u_1 = b_1$. Further, if we denote $u_n = -c_n \eta_{n+1}/\eta_n$, $n \geq 1$, using a sequence $\{\eta_n\}$, then it is seen from above and (3.15) that η_n , $n \geq 1$, satisfies the recurrence relation (1.2) with starting value $\eta_0 = 0$ and any $\eta_1 \neq 0$. This means that η_n is expressed as a multiple of the right-hand side of (2.3), and therefore the computation in (3.15) becomes very susceptible to loss of significance when f_0 happens to be very small in modulus compared to g_0 ; see Table 1 below.

Finally, the solution $\mathbf{y}^{(N)}$ can also be obtained by solving $U\mathbf{y}^{(N)} = \mathbf{x}$ by backward substitution as follows:

$$(3.21) \quad y_N^{(N)} = \frac{x_N}{u_N}, \quad y_j^{(N)} = \frac{x_j - c_{j+1}y_{j+1}^{(N)}}{u_j}, \quad j = N - 1, \dots, M + 1,$$

$$y_M^{(N)} = \frac{x_M - \zeta y_{M+1}^{(N)}}{u_M}, \quad y_i^{(N)} = \frac{x_i - b_{i+1}y_{i+1}^{(N)} - c_{i+1}y_{i+2}^{(N)}}{a_{i+1}}, \quad i = M - 1, \dots, 0.$$

We conclude this section with some remarks on the relationship between the present algorithm and other existing methods. The algorithm given in this section includes the method of Deuffhard, the method of Van der Cruyssen, and that given in Wimp as particular cases. Indeed, we can see that the Deuffhard scheme [5] coincides with the present one in the special case where $d_n \equiv 0$, $n = 0, \dots, N$ and $s = 0$ in (3.1) and (3.3) and $M = N$ in (3.2), provided the last diagonal elements in L (3.8) and U (3.9) are replaced with μ_N and a_{N+1} , respectively. Here μ_N is determined from the relation (3.12) with i set to N .

Another case of our problem, where $M = 0$ and $\xi_i = 0$ ($i = 0, \dots, K - 1$) $\xi_K = 1$, agrees with that given in Wimp [23, p.98], for which the Gaussian elimination technique is applied instead of LU decomposition with rank-one updating technique. Further, the present scheme reduces to the method due to Van der Cruyssen [22] for the initial value problem, where we set $M = 0$ and $\lambda_i = 0$ ($i = 1, \dots, N$) in (3.2) as well as $\xi_i = 0$ for all $i \neq K$ but $\xi_K = 1$ in (3.4), while Cash's algorithm [3] is an extension of the Van der Cruyssen method to high-order difference equations.

4. CONVERGENCE RESULTS

We discuss convergence as $N \rightarrow \infty$ of the algorithm given in the previous section, in particular, of $y_n^{(N)}$ and therefore of $S_K^{(N)}$ (3.10). We seek nondominant solutions y_n of (1.1) in the form $y_n = \alpha f_n + h_n$ with a constant α , which is determined so as to satisfy the normalizing condition (1.3), i.e.,

$$(4.1) \quad \alpha = \left(s - \sum_{m=0}^{\infty} \lambda_m h_m \right) / \sum_{m=0}^{\infty} \lambda_m f_m.$$

We now examine the convergence of $y_n^{(N)}$ to y_n as $N \rightarrow \infty$.

It is seen from (1.5) that the solution $y_n^{(N)}$ of (3.1) can be written as

$$(4.2) \quad y_n^{(N)} = \zeta_1^{(N)} f_n + \zeta_2^{(N)} g_n + h_n,$$

for some constants $\zeta_1^{(N)}$, $\zeta_2^{(N)}$. On the other hand, the solution $y_n^{(N)}$ must satisfy the conditions

$$y_{N+1}^{(N)} = 0, \quad \sum_{m=0}^N \lambda_m y_m^{(N)} = s.$$

The above relations along with (4.2) yield

$$(4.3) \quad \zeta_1^{(N)} F_N + \zeta_2^{(N)} G_N + H_N = s,$$

$$(4.4) \quad \zeta_1^{(N)} f_{N+1} + \zeta_2^{(N)} g_{N+1} + h_{N+1} = 0,$$

where F_N , G_N and H_N are defined, respectively, by

$$(4.5) \quad F_N = \sum_{m=0}^N \lambda_m f_m, \quad G_N = \sum_{m=0}^N \lambda_m g_m, \quad H_N = \sum_{m=0}^N \lambda_m h_m.$$

Solving (4.3) and (4.4) for $\zeta_1^{(N)}$ and $\zeta_2^{(N)}$ gives

$$(4.6) \quad \zeta_1^{(N)} = -\{h_{N+1} G_N + g_{N+1}(s - H_N)\} / (f_{N+1} G_N - g_{N+1} F_N),$$

$$(4.7) \quad \zeta_2^{(N)} = \{f_{N+1}(s - H_N) + h_{N+1} F_N\} / (f_{N+1} G_N - g_{N+1} F_N).$$

Theorem 4.1. *Let f_n , g_n and h_n be solutions of (1.1) having the properties given in §1, that is, $\lim_{n \rightarrow \infty} f_n/g_n = 0$ and $\lim_{n \rightarrow \infty} h_n/g_n = 0$. Further, let F_N , G_N and H_N be defined by (4.5), respectively, with the properties*

$$\lim_{N \rightarrow \infty} f_{N+1} G_N / g_{N+1} = 0, \quad \lim_{N \rightarrow \infty} h_{N+1} G_N / g_{N+1} = 0,$$

and F_N and H_N do not tend to 0. Then the algorithm given in the previous section converges and

$$(4.8) \quad \lim_{N \rightarrow \infty} y_n^{(N)} = y_n.$$

Proof. To show (4.8), it is sufficient to verify in (4.2) that $\lim_{N \rightarrow \infty} \zeta_1^{(N)} = \alpha$ and $\lim_{N \rightarrow \infty} \zeta_2^{(N)} = 0$, which follow simply by using the assumption of Theorem 4.1 in (4.6) and (4.7), respectively, and by noting (4.1). \square

5. NUMERICAL EXAMPLES

Examples in this section were computed on the SONY NEWS 3860 in double precision; the machine precision is $\epsilon = 2.22 \dots \times 10^{-16}$.

5.1. Example 1. We compute a finite sum $S_K = \sum_{n=0}^K y_n$ of the nondominant solution $y_n = 2^{-n}$ ($n = 0, 1, \dots$) of the following linear second-order difference equation

$$(5.1) \quad y_{n-1} - \frac{2n}{x} y_n + y_{n+1} = \left(2.5 - \frac{2n}{x}\right) 2^{-n}, \quad n \geq 1.$$

Here, the minimal and dominant solutions of (5.1) are Bessel functions of the first and second kind $J_n(x)$ and $Y_n(x)$, respectively, both of which are of oscillatory type [19] for $0 \leq n \leq [x]$. For $n \geq [x]$, on the other hand, $|J_n|$ and $|1/Y_n|$ decrease monotonically to zero as n increases. The condition (2.4) is satisfied if we choose $M = [x]$.

As mentioned in §3, when x in (5.1) is a zero of the minimal solution $J_0(x)$, numerical instability may occur in our algorithm in the special case where $\lambda_1 = 0$ and $M = 0$ in (3.2), although this numerical difficulty could be eliminated by setting M to be any positive integer $\leq [x]$. To see this, we choose $x = 8.653727912911012$, in Table 1 (next page). Namely we choose the third positive zero of $J_0(x)$, and a normalizing condition of the type $y_0 + 2(y_2 + y_3 + \dots) = 2$.

In Table 1 we list the computed solutions $y_n^{(N)}$ and the approximate sum $S_K^{(N)} = \sum_{n=0}^K y_n^{(N)}$ with $K = 14$ satisfying the condition (3.19) for the required tolerance $\epsilon_a = 10^{-10}$. As is shown in the second and fourth columns in Table 1, the estimated N for which $|S_{14}^{(N+1)} - S_{14}^{(N)}| \leq 10^{-10}$ is $N = 40$ when we incorrectly set $M = 0$ in (3.2), but $N = 35$ when $M = [x] = 8$. The computed value of $S_{14}^{(N)}$ in the last row of Table 1 is seen to satisfy the tolerance $\epsilon_a = 10^{-10}$ for $M = 8$, while the value of $S_{14}^{(40)}$ in the column $M = 0$ fails, owing to numerical instability, as expected.

It should be noted that the particular solution y_n ($n \geq 0$) wanted is dominated by J_n only for $0 \leq n \leq [x]$ and dominates J_n otherwise, while Y_n dominates y_n for all $n \geq 0$. This relative dominance was crucial for the present algorithm to work. If this relative dominance is violated, the present method fails. To see this, consider the recurrence relation of the type (5.1) having the right-hand side so that the particular solution $y_n = a^n$ if $0 \leq n \leq [x]$ and $y_n = a^{2[x]-n}$ if $n > [x]$, with some constant $a > 1$. Choose the normalizing condition $y_0 + 2(y_2 + y_3 + \dots) = 1 + 2((a+1)a^{[x]} - a^2)/(a-1)$ and let $x = 20.5$ and $a = 5$. Then the computed value of the weighted sum $S_3 = \sum_{n=0}^3 y_n$ is $S_3^{(N)} = 155.99948883$ for the required accuracy $\epsilon_a = 10^{-10}$ and the estimated N is $N = 47$. The actual error of $S_3^{(N)}$ is 5×10^{-4} , which is much greater than ϵ_a ($= 10^{-10}$).

TABLE 1. Computed values $y_n^{(N)}$ of nondominant solutions $y_n = 2^{-n}$ of the recurrence relation (5.1) and approximate sum $S_K^{(N)} = \sum_{n=0}^K y_n^{(N)}$ with $K = 14$ for the required absolute accuracy $\epsilon_a = 10^{-10}$. *Italic digits are incorrect digits in the computed $y_n^{(N)}$ and $S_{14}^{(N)}$.* Complementary solutions of (5.1) are Bessel functions $J_n(x)$ and $Y_n(x)$. Setting $x = 8.653727912911012$, the third positive zero of $J_0(x)$, causes numerical instability, seen in the second column, when we choose $M = 0$ in (3.2), while with an appropriate choice of $M = [x] = 8$ no numerical difficulty is detected, as shown in the fourth column

n	$M = 0$ ♣		$M = [x] = 8$ ◇	
	$y_n^{(N)} (= y_n^{(40)})$	error	$y_n^{(N)} (= y_n^{(35)})$	error
♣ 0	1.00000000E 00	2E-16	1.00000000000000E 00	0E 00
1	<i>5.40487415</i> E-01	4E-02	5.0000000000982E-01	1E-11
2	<i>2.59357219</i> E-01	9E-03	2.5000000000227E-01	2E-12
3	<i>8.88377584</i> E-02	4E-02	1.2499999999123E-01	9E-12
4	<i>2.80699505</i> E-02	4E-02	6.2499999991649E-02	8E-12
5	<i>3.55831387</i> E-02	4E-03	3.1250000001051E-02	1E-12
6	<i>5.50623004</i> E-02	4E-02	1.5625000009565E-02	1E-11
7	<i>5.81664984</i> E-02	5E-02	7.8125000122127E-03	1E-11
◇ 8	<i>4.59316423</i> E-02	4E-02	3.9062500101927E-03	1E-11
9	<i>2.93004691</i> E-02	3E-02	1.9531250066327E-03	7E-12
10	<i>1.58344232</i> E-02	2E-02	9.7656250360358E-04	4E-12
11	<i>7.47957323</i> E-03	7E-03	4.8828125169565E-04	2E-12
12	<i>3.15994063</i> E-03	3E-03	2.4414062570719E-04	7E-13
13	<i>1.21737417</i> E-03	1E-03	1.2207031276565E-04	3E-13
14	<i>4.36059716</i> E-04	4E-04	6.1035156340957E-05	9E-14
15	<i>1.48643095</i> E-04	1E-04	3.0517578153650E-05	3E-14
16	<i>4.97416161</i> E-05	4E-05	1.5258789070863E-05	8E-15
17	<i>1.70154514</i> E-05	9E-06	7.6293945335265E-06	2E-15
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
32	2.32830597E-10	5E-17	2.3282628627229E-10	4E-15
33	1.16415310E-10	1E-17	1.1638371709852E-10	3E-14
34	5.82076593E-11	2E-18	5.7970976272870E-11	2E-13
35	2.91038293E-11	1E-18	2.7275594464178E-11	2E-12
36	1.45519059E-11	9E-18		
37	7.27588098E-12	8E-17		
38	3.63733286E-12	7E-16		
39	1.81339310E-12	6E-15		
40	8.59698592E-13	5E-14		
$S_{14}^{(N)} =$	2.00000000E 00	6E-05	1.9999389648847E 00	4E-11

5.2. Example 2. The present method has practical applications in constructing interpolatory integration rules for an integral of highly oscillatory functions, which is difficult to evaluate with ordinary quadrature rules. In particular, our scheme could be effectively used to obtain a set of approximations to an indefinite integral, such as $\int_{-1}^x e^{i\omega t} f(t) dt$ with a set of values of x and a set of values of ω , where $f(t)$ is a given smooth function and $|x| \leq 1$.

Suppose that $f(t)$ is expanded in terms of Chebyshev polynomials $T_n(t) = \cos n\theta$ ($t = \cos \theta$) as follows:

$$(5.2) \quad f(t) = \sum_{n=0}' a_n T_n(t), \quad |t| \leq 1,$$

where the prime denotes a summation whose first term is halved. In practice, $f(t)$ can be efficiently approximated by a finite sum of Chebyshev polynomials $p_m(t) = \sum_{n=0}^m a_n^{(m)} T_n(t)$ by using the Fast Fourier Transform (FFT) [1, 7, 11]. If $f(t)$ is a smooth function, the approximation p_m converges to f rapidly as m increases. For the function $f(t)$ given in (5.2) (or the approximation $p_m(t)$) we prove the following lemma in Appendix A.

Lemma 5.1. *For the function $f(t)$ given by (5.2), write the indefinite integral $\int_{-1}^x e^{i\omega t} f(t) dt$ in the form [10, 12]*

$$(5.3) \quad I(x, \omega; f) \equiv \int_{-1}^x e^{i\omega t} f(t) dt = \frac{1}{i\omega} e^{i\omega x} g(x), \quad |x| \leq 1, \quad \omega > 0,$$

and expand $g(x)$ in terms of Chebyshev polynomials

$$(5.4) \quad g(x) = \sum_{n=0}' y_n T_n(x).$$

Then, the coefficients y_n in (5.4) satisfy the inhomogeneous difference equation

$$(5.5) \quad y_{n-1} - i \frac{2n}{\omega} y_n - y_{n+1} = a_{n-1} - a_{n+1}, \quad n \geq 1,$$

and a normalizing condition of the type $\sum_{n=0}' y_n (-1)^n = 0$.

Remark 5.1. For the approximation $p_m(t)$ instead of $f(t)$, it suffices to replace a_{n-1} and a_{n+1} in the right-hand side of (5.5) by $a_{n-1}^{(m)}$ and $a_{n+1}^{(m)}$, respectively, where we assume that $a_n^{(m)} = 0$ for all $n > m$.

The coefficient y_n in (5.4) is a particular solution of (5.5). The complementary solutions of (5.5) are modified Bessel functions $I_n(-i\omega)$ and $(-1)^n K_n(-i\omega)$ ($= K_n(i\omega)$). In the range $n \leq [\omega]$, both of $|I_n|$ and $|K_n|$ are of oscillatory type, whereas for $n > [\omega]$, $|I_n|$ and $|1/K_n|$ decrease rapidly to zero as n increases.

Here we show that with the choice of $M = [\omega]$ in (2.4) the present scheme could effectively evaluate the particular solutions y_n and the weighted sum $g(x)$ (5.4) to a prescribed accuracy ϵ_a . In reality, instead of $g(x)$ given in (5.4) it is sufficient to obtain a truncated Chebyshev series

$$(5.6) \quad g(x) \approx g_K(x) = \sum_{n=0}^K y_n T_n(x),$$

where K is determined so that the truncation error of $g_K(x)$ is at the levels of the rounding error of the computer, which is usually less than the required

accuracy ϵ_a for y_n or the weighted sum $g(x)$ of y_n . In practice, it might be allowed to set K to the smallest n such that $|y_{n+1}| \leq \epsilon$, provided $|y_n| = O(r^n)$ for a positive constant $r < 1$ and for sufficiently large n , because then the error $|\sum_{n=K+1}^{\infty} y_n T_n(x)| \leq \sum_{n=K+1}^{\infty} |y_n| \approx |y_{K+1}|/(1-r)$. If the required accuracy ϵ_a is much larger than ϵ , then we might determine the smallest K so that $|y_{K+1}| \leq c \epsilon_a$ for some small constant $c > 0$, say, $c = 0.1$ at most. In incorporating this scheme into the present algorithm, we approximate $\{y_n\}$ by $\{y_n^{(n)}\} = \{x_n/u_n\}$, the first relation in (3.21).

Now, we compute the weighted sum $S_K \equiv g_K(x = 1) = \sum_{n=0}^K y_n$ (5.6) to the prescribed accuracy $\epsilon_a = 10^{-10}$ for the indefinite integral (5.3) of $f(t)$, where in particular we choose

$$f(t) = \frac{1 - \alpha^2}{1 - 2\alpha t + \alpha^2} = 2 \sum_{n=0}^{\infty} \alpha^n T_n(t), \quad \alpha = 0.9,$$

and $\omega = 150$. In this case the right-hand side of (5.5) is $2(\alpha^{-1} - \alpha)\alpha^n$ for $n \geq 1$. We choose $M = [\omega] = 150$ in (3.2) to satisfy the condition (2.4). The estimated value of the smallest K such that $|y_{K+1}| \leq 0.1 \times \epsilon_a = 0.1 \times 10^{-10}$ is $K = 223$ and the estimated N for which $|S_{223}^{(N+1)} - S_{223}^{(N)}| \leq 10^{-10}$ is $N = 225$. The computed $S_{223}^{(N)} \equiv g_{223}(x = 1) = 10.83928930523 + i 6.550524798738$ and the error is estimated to be $E \equiv g(1) - g_{223}(1) = (0.8 + i 7) \times 10^{-11}$ by using the value of $g(x = 1)$ evaluated as follows. From (5.3) we have $g(x) = i\omega e^{-i\omega x}(\Re I + i\Im I)$, where $\Re I$ and $\Im I$ are $\Re I \equiv \Re I(x, \omega; f) = \int_{-1}^x \cos(\omega t) f(t) dt$ and $\Im I \equiv \Im I(x, \omega; f) = \int_{-1}^x \sin(\omega t) f(t) dt$, respectively. To compute $\Re I(x = 1, \omega = 150; f)$ and $\Im I(x = 1, \omega = 150; f)$ we used the QUADPACK [21] routine DQAWO with the required absolute accuracy = 10^{-14} .

TABLE 2. Weighted sums $g_{223}(x) = y_0 T_0(x)/2 + y_1 T_1(x) + \dots + y_{223} T_{223}(x)$ ($x = 0.1, 0.3, \dots, 0.9$) of the computed nondominant solutions y_n of the recurrence relation (5.5) for the required absolute accuracy $\epsilon_a = 10^{-10}$ and their actual errors. Clenshaw's algorithm is used to evaluate the summation. Italic digits are incorrect digits

x	$g_{223}(x) = \Re g + i\Im g$		Error = $\Re E + i\Im E$	
	$\Re g$	$\Im g$	$\Re E$	$\Im E$
0.1	0.1198685750311	0.05338395512740	-3E-12	-3E-12
0.3	0.09819379394844	0.01279216097213	-2E-12	-3E-12
0.5	0.1895492858980	-0.04626511161379	9E-13	-1E-11
0.7	0.3905994887744	-0.01898488674630	2E-12	-1E-11
0.9	1.025563752769	0.1025908390260	-9E-14	3E-12

Once all the values of $y_n^{(N)}$ ($n = 0, 1, \dots, 223$) are computed from (3.21) after the weighted sum $S_{223}^{(N)} = g_{223}(x = 1)$ of $y_n^{(N)}$ is obtained, any value of $g_{223}(x)$ for many x , where $|x| \leq 1$, can be easily evaluated by using Clenshaw's algorithm. Table 2 lists the computed values of $g_{223}(x_\nu)$, where $x_\nu = 0.1, 0.3, \dots, 0.9$ and their errors, which are estimated by using the

QUADPACK in a similar way mentioned above. The computed $g_{223}(x)$'s listed in the second and third columns are seen to satisfy the required accuracy $\varepsilon_a = 10^{-10}$.

APPENDIX A

Here we prove Lemma 5.1. Differentiating both sides of (5.3) with respect to x and changing the variable x into t yield

$$f(t) = g(t) + g'(t)/(i\omega).$$

Integrating this relation on $[-1, x]$ gives

$$(5.7) \quad \int_{-1}^x f(t) dt = \int_{-1}^x g(t) dt + \{g(x) - g(-1)\}/(i\omega).$$

Substituting the Chebyshev series expansions (5.2) and (5.4) into (5.7) and noting

$$2 \int_{-1}^x T_n(t) dt = \frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1} - (-1)^n \frac{2}{n^2-1}, \quad n \geq 2,$$

we obtain the relation (5.5) by comparing the coefficients of $T_n(x)$. For (5.3) to hold with $x = -1$ requires the condition $g(-1) = 0$, which yields along with (5.4) the normalizing condition $\sum_{n=0}^{\infty} y_n (-1)^n = 0$.

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