MATHEMATICS OF COMPUTATION
VOLUME 64, NUMBER 212
OCTOBER 1995, PAGES 1421-1431

A LOCAL PROJECTION OPERATOR FOR QUADRILATERAL FINITE ELEMENTS

V. GIRAULT

ABSTRACT. This note studies the approximation error of a local projection operator on polynomials of total degree k defined on quadrilaterals. Among other applications, this projection operator permits to derive easily error estimates for quadrature formulas.

1. INTRODUCTION

Let T denote a convex and nondegenerate quadrilateral (i.e., not reduced to a triangle) and let k be a nonnegative integer. We denote by $\mathbb{P}_k$ the set of polynomials in two variables of total degree k, i.e., spanned by all products of the form $x_1^{i_1}x_2^{i_2}$ with $0 \le i_1 \le k$, $0 \le i_2 \le k$ and $i_1 + i_2 \le k$. Then, for any function u in $L^1(T)$, we define its local projection $\widetilde{I}_T^k(u) \in \mathbb{P}_k$ by

$$\forall r \in \mathbb{P}_k, \quad \int_T \widetilde{I}_T^k(u) r \, d\mathbf{x} = \int_T u r \, d\mathbf{x}. \tag{1}$$

Obviously, (1) defines uniquely $\widetilde{I}_T^k(u)$, but deriving error estimates for this operator is not altogether straightforward, because the polynomial space $\mathbb{P}_k$ is well adapted to triangular finite elements but not to quadrilateral finite elements. Indeed, in the case of quadrilateral finite elements, the polynomials are first defined on the reference square $\widehat{T} = [0, 1]^2$ in the reference $(\hat{x}_1, \hat{x}_2)$-space and they belong to the space $\widehat{Q}_k$ of polynomials of degree k in *each* variable, i.e., spanned by all products of the form $\hat{x}_1^{i_1}\hat{x}_2^{i_2}$ with $0 \le i_1 \le k$ and $0 \le i_2 \le k$. Then they are transformed into functions (generally, not polynomials) defined on T by a transformation that maps T onto $\widehat{T}$. More precisely, as T is convex and nondegenerate, there exists an invertible bilinear mapping F_T that maps $\widehat{T}$ onto T (cf. Ciarlet [2]); then we define the function space $\mathcal{Q}_k(T)$ by

$$\mathcal{Q}_k(T) = \{q = \hat{q} \circ F_T^{-1} ; \forall \hat{q} \in \widehat{Q}_k\}.$$

It turns out that this is the "good" space for interpolating functions on quadrilaterals because, unlike the space $\mathbb{P}_k$, it yields optimal interpolation error estimates. We refer to Ciarlet [2] for a study of these quadrilateral finite element spaces and isoparametric finite element spaces in general.

Received by the editor March 22, 1994 and, in revised form, November 23, 1994.
1991 *Mathematics Subject Classification.* Primary 65D05, 65N30; Secondary 65D32.
Key words and phrases. Isoparametric quadrilateral finite elements, local projections, quadrature formulas.

However, because $\widetilde{I}_T^k$ is a projection operator, and thus minimizes the L^2 norm, it does satisfy optimal error estimates. To my knowledge, the first L^2 estimate for this operator was established in Girault and Raviart [5], following an original idea of C. Bernardi, where it was applied to analyze the $\mathscr{Q}_k - \mathbb{P}_{k-1}$ element for the Stokes problem. The purpose of the present paper is to extend the above result first to L^p, and next to $W^{1,p}$ estimates. As an application, we shall use this projection operator to show in particular that, when $k = 1$, the four-point quadrature rule is of order one, a result very difficult to establish otherwise on an arbitrary quadrilateral (cf. Ciarlet and Raviart [3] and Raviart [8]).

We conclude this introduction by recalling some notations and properties of Sobolev spaces that we shall use further on; they can be found in Adams [1] or Nečas [7]. Let Ω denote a bounded and connected open subset of $\mathbb{R}^2$ with a Lipschitz continuous boundary. For any nonnegative integer k and number p with $1 \le p \le \infty$, recall the standard Sobolev space

$$W^{k,p}(\Omega) = \{v \in L^p(\Omega)\,;\, \partial^\alpha v \in L^p(\Omega) \text{ for } 0 \le |\alpha| \le k\},$$

where α denotes any pair of nonnegative integers (α_1, α_2), $\partial^\alpha v = \partial^{|\alpha|} v / \partial x_1^{\alpha_1} \partial x_2^{\alpha_2}$ and $|\alpha| = \alpha_1 + \alpha_2$. It is a Banach space for the norm

$$\|v\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|=0}^{k} \sum_{\alpha} \|\partial^\alpha v\|_{L^p(\Omega)}^p \right)^{1/p},$$

with the usual modification when $p = \infty$. When $p = 2$, this space is denoted simply by $H^k(\Omega)$. We also define the seminorm

$$|v|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|=k} \|\partial^\alpha v\|_{L^p(\Omega)}^p \right)^{1/p}.$$

By interpolating between two consecutive values of k, we can extend the definition of Sobolev spaces to noninteger values of k (cf. Lions and Magenes [6]).

Finally, we recall a fundamental result on polynomial interpolation. On any bounded domain K, for any nonnegative integers k and l, the polynomial space $\mathbb{P}_k$ is contained in $W^{l,p}(K)$ and we can define the quotient space $W^{l,p}(K)/\mathbb{P}_k$, which is also a Banach space equipped with the quotient norm

$$\forall \dot{v} \in W^{l,p}(K)/\mathbb{P}_k, \quad \|\dot{v}\|_{W^{l,p}(K)/\mathbb{P}_k} = \inf_{r \in \mathbb{P}_k} \|v + r\|_{W^{l,p}(K)}.$$

This quotient space has the following property proved by Deny and Lions [4] (cf. also Nečas [7]).

Theorem 1. *Assume that K is a bounded and connected open set of $\mathbb{R}^2$ with a Lipschitz continuous boundary. For each integer $k \ge 0$ and number p with $1 \le p \le \infty$, there exists a constant C such that*

$$\forall \dot{v} \in W^{k+1,p}(K)/\mathbb{P}_k, \quad \|\dot{v}\|_{W^{k+1,p}(K)/\mathbb{P}_k} \le C |v|_{W^{k+1,p}(K)}.$$

2. An L^p-estimate

To simplify the discussion and avoid the technical difficulties related to curved boundaries, we assume from now on that Ω is a polygonal domain.

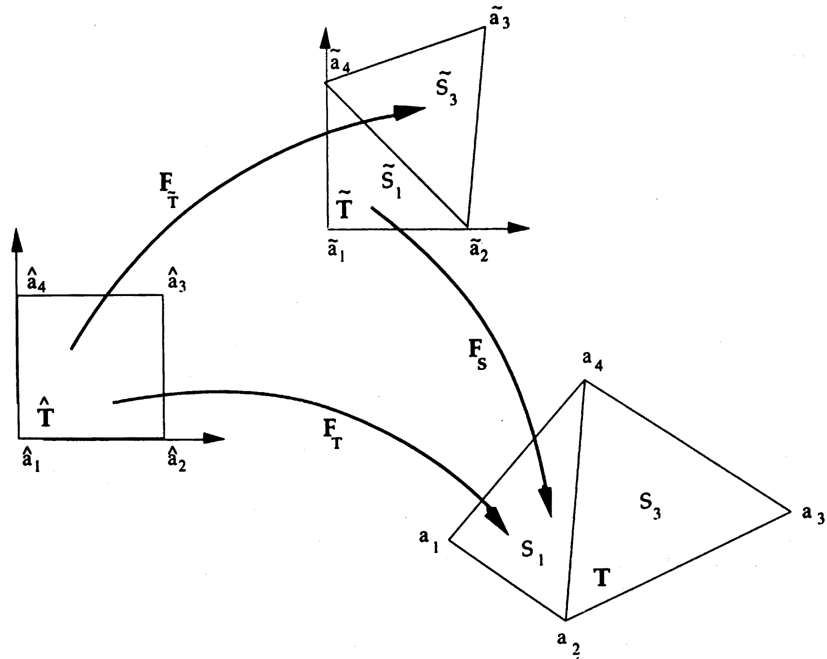

FIGURE 1

Let h be a parameter that will tend to zero, and for each value of h, let $\mathcal{T}_h$ be a quadrangulation of $\overline{\Omega}$ made of convex and nondegenerate quadrilaterals, with diameter bounded by h. Let T be one of these quadrilaterals, let $\mathbf{a}_i$ be its vertices, for $1 \leq i \leq 4$, and let S_i denote its subtriangle with vertices $\mathbf{a}_{i-1}$, $\mathbf{a}_i$, $\mathbf{a}_{i+1}$, the indices being numbered modulo four, as in Figure 1. Let h_i be the diameter of S_i and ρ_i the diameter of its inscribed circle. We set

$$h_T = \sup_{1\leq i\leq 4} h_i, \quad \rho_T = 2 \inf_{1\leq i\leq 4} \rho_i \quad \text{and} \quad \sigma_T = \frac{h_T}{\rho_T}.$$

Clearly, h_T is the diameter of T and σ_T is a measure of the nondegeneracy of T.

In order to study $\widetilde{I}_T^k$, it will be useful to introduce the reference unit triangle $\widetilde{S}_1$ with vertices $\tilde{\mathbf{a}}_1 = \hat{\mathbf{a}}_1 = (0, 0)$, $\tilde{\mathbf{a}}_2 = \hat{\mathbf{a}}_2 = (1, 0)$, $\tilde{\mathbf{a}}_4 = \hat{\mathbf{a}}_4 = (0, 1)$, and the *affine* invertible mapping

$$F_S(\tilde{\mathbf{x}}) = B_S\tilde{\mathbf{x}} + \mathbf{b},$$

defined by $F_S(\tilde{\mathbf{a}}_i) = \mathbf{a}_i$ for $i = 1, 2, 4$. (For strictly consistent notations, we should denote this mapping by $F_{S_1}(\tilde{\mathbf{x}})$, but for the sake of simplicity, we agree to drop the index on S.) As T is nondegenerate, this mapping is unique and it maps $\widetilde{S}_1$ onto S_1. Furthermore, the matrix B_S satisfies the following bounds:

$$\|B_S\| \leq \frac{h_1}{2-\sqrt{2}} < 2h_1 \leq 2h_T, \qquad \|B_S^{-1}\| \leq \frac{\sqrt{2}}{\rho_1} \leq \frac{2\sqrt{2}}{\rho_T},$$

$$|\det(B_S)| = 2\,\mathrm{meas}(S_1) \quad \text{and} \quad \frac{\pi}{8}\rho_T^2 < |\det(B_S)| \leq \frac{\sqrt{3}}{2}h_T^2.$$

Then we set $\widetilde{S}_3 = F_S^{-1}(S_3)$ and we associate with T an auxiliary reference quadrilateral, $\widetilde{T}$, which generally does not coincide with the reference square $\widehat{T}$, but is convex because T is convex:

$$\widetilde{T} = \widetilde{S}_1 \cup \widetilde{S}_3 = F_S^{-1}(T). \tag{2}$$

In particular, we set $\tilde{\mathbf{a}}_3 = F_S^{-1}(\mathbf{a}_3)$, and the diameter $h_{\widetilde{T}}$ of $\widetilde{T}$ satisfies

$$h_{\widetilde{T}} \le \|B_S^{-1}\| h_T \le 2\sqrt{2}\sigma_T.$$

As $\widetilde{T}$ is a convex and nondegenerate quadrilateral, there exists a (unique) invertible bilinear mapping $F_{\widetilde{T}}$ that maps the unit square $\widehat{T}$ onto $\widetilde{T}$ and such that for $1 \le i \le 4$,

$$F_{\widetilde{T}}(\hat{\mathbf{a}}_i) = \tilde{\mathbf{a}}_i,$$

where $\hat{\mathbf{a}}_3 = (1, 1)$. It is interesting to observe that

$$F_S \circ F_{\widetilde{T}} = F_T,$$

where F_T is the bilinear mapping (mentioned in the introduction) satisfying $\mathbf{a}_i = F_T(\hat{\mathbf{a}}_i)$. Let J_T and $J_{\widetilde{T}}$ denote respectively the Jacobians of F_T and $F_{\widetilde{T}}$; we have

$$\|J_T\|_{L^\infty(\widehat{T})} = 2 \sup_{1\le i\le 4} \operatorname{meas}(S_i) \le \frac{\sqrt{3}}{2} h_T^2,$$

$$\|J_T^{-1}\|_{L^\infty(T)} = \frac{1}{2\inf_{1\le i\le 4}\operatorname{meas}(S_i)} < \frac{8}{\pi\rho_T^2},$$

$$\|J_{\widetilde{T}}\|_{L^\infty(\widehat{T})} = \sup_{1\le i\le 4} \frac{\operatorname{meas}(S_i)}{\operatorname{meas}(S_1)} < \frac{4\sqrt{3}}{\pi}\sigma_T^2,$$

$$\|J_{\widetilde{T}}^{-1}\|_{L^\infty(\widetilde{T})} = \frac{\operatorname{meas}(S_1)}{\inf_{1\le i\le 4}\operatorname{meas}(S_i)} < \frac{4\sqrt{3}}{\pi}\sigma_T^2,$$

$$\|DF_T\|_{L^\infty(\widehat{T})} \le C_1 h_T, \quad \|DF_{\widetilde{T}}\|_{L^\infty(\widehat{T})} \le C_2 h_{\widetilde{T}} \le C_3\sigma_T,$$

$$\|DF_T^{-1}\|_{L^\infty(T)} \le C_4\frac{\sigma_T}{\rho_T}.$$

Our first lemma shows that the operator $\widetilde{I}_T^k$ is stable in $L^p(T)$.

Lemma 2. *For any integer $k \ge 0$ and any number p with $1 \le p \le \infty$, there exists a constant C, independent of the geometry of T, such that*

$$\forall u \in L^p(T), \quad \|\widetilde{I}_T^k(u)\|_{L^p(T)} \le C\sigma_T^4 \|u\|_{L^p(T)}.$$

Proof. Make the change of variable $\mathbf{x} = F_S(\tilde{\mathbf{x}})$ in definition (1). As the mapping F_S is affine, we have

$$\forall r \in \mathbb{P}_k, \quad |\det(B_S)| \int_{\widetilde{T}} (\widetilde{I}_T^k(u) - u)\circ F_S(\tilde{\mathbf{x}}) r\circ F_S(\tilde{\mathbf{x}})\, d\tilde{\mathbf{x}} = 0.$$

But both the space $\mathbb{P}_k$ and the operator $\widetilde{I}_T^k$ are invariant by affine transformation, i.e., setting $\tilde{u} = u\circ F_S$, we have

$$\widetilde{I}_T^k(u)\circ F_S = \widetilde{I}_{\widetilde{T}}^k(\tilde{u}).$$

Hence, we can write

$$\forall r \in \mathbb{P}_k, \quad \int_{\widetilde{T}} (\widetilde{I}^k_{\widetilde{T}}(\tilde{u}) - \tilde{u}) r \, d\tilde{\mathbf{x}} = 0.$$

Therefore, choosing $r = \widetilde{I}^k_{\widetilde{T}}(\tilde{u})$ and letting q denote the conjugate exponent of p, $1/p + 1/q = 1$, we obtain

$$\|\widetilde{I}^k_{\widetilde{T}}(\tilde{u})\|^2_{L^2(\widetilde{T})} \leq \|\tilde{u}\|_{L^p(\widetilde{T})} \|\widetilde{I}^k_{\widetilde{T}}(\tilde{u})\|_{L^q(\widetilde{T})}. \tag{3}$$

But

$$\|\widetilde{I}^k_{\widetilde{T}}(\tilde{u})\|_{L^q(\widetilde{T})} \leq \|J_{\widetilde{T}}\|^{1/q}_{L^\infty(\widehat{T})} \|\widetilde{I}^k_{\widetilde{T}}(\tilde{u}) \circ F_{\widetilde{T}}\|_{L^q(\widehat{T})},$$

and $\widetilde{I}^k_{\widetilde{T}}(\tilde{u}) \circ F_{\widetilde{T}}$ belongs to $\widehat{Q}_k$, a finite-dimensional space on which all norms are equivalent. Hence, there exists a constant C_1 which depends only on the geometry of $\widehat{T}$ and on the degree k, such that

$$\begin{aligned}\|\widetilde{I}^k_{\widetilde{T}}(\tilde{u})\|_{L^q(\widetilde{T})} &\leq C_1 \|J_{\widetilde{T}}\|^{1/q}_{L^\infty(\widehat{T})} \|\widetilde{I}^k_{\widetilde{T}}(\tilde{u}) \circ F_{\widetilde{T}}\|_{L^2(\widehat{T})} \\ &\leq C_1 \|J_{\widetilde{T}}\|^{1/q}_{L^\infty(\widehat{T})} \|J^{-1}_{\widetilde{T}}\|^{1/2}_{L^\infty(\widetilde{T})} \|\widetilde{I}^k_{\widetilde{T}}(\tilde{u})\|_{L^2(\widetilde{T})}.\end{aligned}$$

Therefore, for any real number q, we have

$$\|\widetilde{I}^k_{\widetilde{T}}(\tilde{u})\|_{L^q(\widetilde{T})} \leq C_2 \sigma_T^{2(1/q+1/2)} \|\widetilde{I}^k_{\widetilde{T}}(\tilde{u})\|_{L^2(\widetilde{T})}. \tag{4}$$

Substituting (4) into (3), first with q and next with p, we obtain

$$\|\widetilde{I}^k_{\widetilde{T}}(\tilde{u})\|_{L^p(\widetilde{T})} \leq C_3 \sigma_T^4 \|\tilde{u}\|_{L^p(\widetilde{T})}. \tag{5}$$

Hence,

$$\begin{aligned}\|\widetilde{I}^k_T(u)\|_{L^p(T)} &= |\det(B_S)|^{1/p} \|\widetilde{I}^k_{\widetilde{T}}(\tilde{u})\|_{L^p(\widetilde{T})} \leq C_3 \sigma_T^4 |\det(B_S)|^{1/p} \|\tilde{u}\|_{L^p(\widetilde{T})} \\ &\leq C_3 \sigma_T^4 |\det(B_S)|^{1/p} |\det(B_S)|^{-1/p} \|u\|_{L^p(T)}. \quad \square\end{aligned}$$

Theorem 3. *For any integers $k \geq 0$ and $l \geq 0$ with $l \leq k+1$, and for all numbers p with $1 \leq p \leq \infty$, there exists a constant C, independent of the geometry of T, such that*

$$\forall u \in W^{l,p}(T), \quad \|u - \widetilde{I}^k_T(u)\|_{L^p(T)} \leq C \sigma_T^{l(4/p+1)+4} h_T^l |u|_{W^{l,p}(T)}. \tag{6}$$

Proof. By virtue of Lemma 2, it suffices to consider the case where $l \geq 1$. Here again, we can write

$$\|u - \widetilde{I}^k_T(u)\|_{L^p(T)} = |\det(B_S)|^{1/p} \|\tilde{u} - \widetilde{I}^k_{\widetilde{T}}(\tilde{u})\|_{L^p(\widetilde{T})}.$$

But since the operator $\widetilde{I}^k_{\widetilde{T}}$ preserves all polynomials in $\mathbb{P}_k$, we have for any r in $\mathbb{P}_k$,

$$\begin{aligned}\|\tilde{u} - \widetilde{I}^k_{\widetilde{T}}(\tilde{u})\|_{L^p(\widetilde{T})} &= \|\tilde{u} - r - \widetilde{I}^k_{\widetilde{T}}(\tilde{u} - r)\|_{L^p(\widetilde{T})} \leq \|\tilde{u} - r\|_{L^p(\widetilde{T})} + \|\widetilde{I}^k_{\widetilde{T}}(\tilde{u} - r)\|_{L^p(\widetilde{T})} \\ &\leq (1 + C_1 \sigma_T^4) \|\tilde{u} - r\|_{L^p(\widetilde{T})},\end{aligned}$$

where C_1 denotes the constant of (5). Therefore,

$$\|u - \widetilde{I}^k_T(u)\|_{L^p(T)} \leq |\det(B_S)|^{1/p} (1 + C_1 \sigma_T^4) \|\tilde{u}\|_{L^p(\widetilde{T})/\mathbb{P}_k}. \tag{7}$$

It would be tempting to apply Theorem 1 to the right-hand side of (7), but we cannot do it directly here, because $\widetilde{T}$ is a variable quadrilateral and the constant of Theorem 1 depends upon the geometry of the set. Therefore, we must switch to the reference element $\widehat{T}$. Consider first the case where $k = 0$ and which is the only case where $\widehat{Q}_k$ and $\mathbb{P}_k$ coincide on $\widehat{T}$, since they both consist of constants. Thus, applying Theorem 1 in $\widehat{T}$, we obtain

$$\inf_{r\in\mathbb{R}} \|\tilde{u} + r\|_{L^p(\widetilde{T})} \le \|J_{\widetilde{T}}\|^{1/p}_{L^\infty(\widehat{T})} \inf_{r\in\mathbb{R}} \|\tilde{u}\circ F_{\widetilde{T}} + r\|_{L^p(\widehat{T})} \le C_2 \|J_{\widetilde{T}}\|^{1/p}_{L^\infty(\widehat{T})} |\tilde{u}\circ F_{\widetilde{T}}|_{W^{1,p}(\widehat{T})}$$
$$\le C_2 \|J_{\widetilde{T}}\|^{1/p}_{L^\infty(\widehat{T})} \|J^{-1}_{\widetilde{T}}\|^{1/p}_{L^\infty(\widetilde{T})} \|DF_{\widetilde{T}}\|_{L^\infty(\widehat{T})} |\tilde{u}|_{W^{1,p}(\widetilde{T})}.$$

Hence,

$$\|\tilde{u}\|_{L^p(\widetilde{T})/\mathbb{P}_0} \le C_3 \sigma_T^{4/p+1} |\tilde{u}|_{W^{1,p}(\widetilde{T})}. \tag{8}$$

We shall extend (8) to any integer k by induction. Assume that for any integer $j \le l-2$, we have

$$\|\tilde{u}\|_{L^p(\widetilde{T})/\mathbb{P}_j} \le C \sigma_T^{(4/p+1)(j+1)} |\tilde{u}|_{W^{j+1,p}(\widetilde{T})}, \tag{9}$$

and let us prove that (9) is valid for $j+1$. We use the decomposition

$$\mathbb{P}_{j+1} = \mathbb{P}_j \oplus \mathbb{P}^*_{j+1},$$

where $\mathbb{P}^*_{j+1}$ denotes the polynomial space spanned by the $j+2$ terms $x_1^i x_2^{j+1-i}$ for $0 \le i \le j+1$. Therefore,

$$\inf_{r\in\mathbb{P}_{j+1}} \|\tilde{u} + r\|_{L^p(\widetilde{T})} = \inf_{r\in\mathbb{P}_j,\, r^*\in\mathbb{P}^*_{j+1}} \|\tilde{u} + r + r^*\|_{L^p(\widetilde{T})}$$
$$= \inf_{r^*\in\mathbb{P}^*_{j+1}} \inf_{r\in\mathbb{P}_j} \|(\tilde{u} + r^*) + r\|_{L^p(\widetilde{T})}$$
$$\le C_4 \sigma_T^{(4/p+1)(j+1)} \inf_{r^*\in\mathbb{P}^*_{j+1}} |\tilde{u} + r^*|_{W^{j+1,p}(\widetilde{T})},$$

owing to the induction hypothesis. But

$$|\tilde{u} + r^*|_{W^{j+1,p}(\widetilde{T})} = \left(\sum_{|\alpha|=j+1} \|\partial^\alpha \tilde{u} + c_\alpha\|^p_{L^p(\widetilde{T})} \right)^{1/p},$$

for real constants c_α. Hence,

$$\inf_{r^*\in\mathbb{P}^*_{j+1}} |\tilde{u} + r^*|_{W^{j+1,p}(\widetilde{T})} = \left(\sum_{|\alpha|=j+1} \inf_{c\in\mathbb{R}} \|\partial^\alpha \tilde{u} + c\|^p_{L^p(\widetilde{T})} \right)^{1/p}.$$

Now, (8) yields

$$\inf_{c\in\mathbb{R}} \|\partial^\alpha \tilde{u} + c\|_{L^p(\widetilde{T})} \le C_3 \sigma_T^{4/p+1} |\partial^\alpha \tilde{u}|_{W^{1,p}(\widetilde{T})} \le C_3 \sigma_T^{4/p+1} |\tilde{u}|_{W^{j+2,p}(\widetilde{T})}.$$

Thus,

$$\|\tilde{u}\|_{L^p(\widetilde{T})/\mathbb{P}_{j+1}} \le C_5 \sigma_T^{(4/p+1)(j+2)} |\tilde{u}|_{W^{j+2,p}(\widetilde{T})},$$

and (9) is proved by induction. It remains to substitute (9) into (7) with $k = j+1 = l$ and switch back to T:

$$\|u - \widetilde{I}^k_T(u)\|_{L^p(T)} \le (1 + C_1 \sigma_T^4) C \sigma_T^{l(4/p+1)} \|B_S\|^l |u|_{W^{l,p}(T)},$$

thereby proving (6). □

To derive a global estimate from (6), we assume that the family of quadrangulations $\mathcal{T}_h$ is regular as h tends to zero: there exists a constant σ, independent of h, such that

$$\forall T \in \mathcal{T}_h, \quad \sigma_T \leq \sigma .$$

Then we immediately derive the following corollary.

Corollary 4. *Assume that $\mathcal{T}_h$ is regular. For any integers $k \geq 0$ and $l \geq 0$ with $l \leq k+1$, and for all numbers p with $1 \leq p \leq \infty$, there exists a constant C, independent of h, such that*

$$\forall u \in W^{l,p}(\Omega), \quad \|u - \widetilde{I}_T^k(u)\|_{L^p(\Omega)} \leq Ch^l |u|_{W^{l,p}(\Omega)} .$$

3. A $W^{1,p}$-ESTIMATE

The next lemma shows that $\widetilde{I}_T^k$ is stable in $W^{1,p}(T)$.

Lemma 5. *For any integer $k \geq 0$ and any number p with $1 \leq p \leq \infty$, there exists a constant C, independent of the geometry of T, such that*

$$\forall u \in W^{1,p}(T), \quad |\widetilde{I}_T^k(u)|_{W^{1,p}(T)} \leq C\sigma_T^{6/p+7} |u|_{W^{1,p}(T)} . \tag{10}$$

Proof. The result is trivial for $k = 0$, since $\widetilde{I}_T^0(u)$ is a constant. Therefore, we can assume that $k \geq 1$ and we write

$$|\widetilde{I}_T^k(u)|_{W^{1,p}(T)} \leq |\widetilde{I}_T^k(u) - \widetilde{I}_T^0(u)|_{W^{1,p}(T)} + |\widetilde{I}_T^0(u)|_{W^{1,p}(T)} = |\widetilde{I}_T^k(u) - \widetilde{I}_T^0(u)|_{W^{1,p}(T)} ,$$

as $\widetilde{I}_T^0(u)$ is a constant. Now switch to the reference element $\widehat{T}$:

$$|\widetilde{I}_T^k(u) - \widetilde{I}_T^0(u)|_{W^{1,p}(T)} \leq \|J_T\|_{L^\infty(\widehat{T})}^{1/p} \|DF_T^{-1}\|_{L^\infty(T)} |(\widetilde{I}_T^k(u) - \widetilde{I}_T^0(u)) \circ F_T|_{W^{1,p}(\widehat{T})} .$$

Since $(\widetilde{I}_T^k(u) - \widetilde{I}_T^0(u)) \circ F_T$ belongs to the finite-dimensional space $\widehat{Q}_k$ on $\widehat{T}$, the equivalence of norms yields

$$\begin{aligned} |(\widetilde{I}_T^k(u) - \widetilde{I}_T^0(u)) \circ F_T|_{W^{1,p}(\widehat{T})} &\leq C_1 \|(\widetilde{I}_T^k(u) - \widetilde{I}_T^0(u)) \circ F_T\|_{L^p(\widehat{T})} \\ &\leq C_1 \|J_T^{-1}\|_{L^\infty(T)}^{1/p} \|\widetilde{I}_T^k(u) - \widetilde{I}_T^0(u)\|_{L^p(T)} . \end{aligned}$$

Thus,

$$\begin{aligned} &|\widetilde{I}_T^k(u) - \widetilde{I}_T^0(u)|_{W^{1,p}(T)} \\ &\quad \leq C_1 \|J_T\|_{L^\infty(\widehat{T})}^{1/p} \|J_T^{-1}\|_{L^\infty(T)}^{1/p} \|DF_T^{-1}\|_{L^\infty(T)} \|\widetilde{I}_T^k(u) - \widetilde{I}_T^0(u)\|_{L^p(T)} \\ &\quad \leq C_2 \sigma_T^{2/p+1} \frac{1}{\rho_T} \|\widetilde{I}_T^k(u) - \widetilde{I}_T^0(u)\|_{L^p(T)} . \end{aligned}$$

Hence,

$$\begin{aligned} |\widetilde{I}_T^k(u)|_{W^{1,p}(T)} &\leq C_2 \sigma_T^{2/p+1} \frac{1}{\rho_T} (\|\widetilde{I}_T^k(u) - u\|_{L^p(T)} + \|\widetilde{I}_T^0(u) - u\|_{L^p(T)}) \\ &\leq C_3 \sigma_T^{6/p+7} |u|_{W^{1,p}(T)} , \end{aligned}$$

by applying (6) with $l = 1$. □

Theorem 6. *For any integers $k \geq 0$ and $1 \leq l \leq k+1$, and for all numbers p with $1 \leq p \leq \infty$, there exists a constant C which depends on σ_T but is otherwise independent of the geometry of T, such that*

$$\forall u \in W^{l,p}(T), \quad |u - \widetilde{I}_T^k(u)|_{W^{1,p}(T)} \leq C h_T^{l-1} |u|_{W^{l,p}(T)}. \tag{11}$$

Proof. Since $\widetilde{I}_T^k$ preserves all polynomials in $\mathbb{P}_k$, we can write

$$\begin{aligned}\forall r \in \mathbb{P}_k, \quad |u - \widetilde{I}_T^k(u)|_{W^{1,p}(T)} &= |u - r - \widetilde{I}_T^k(u-r)|_{W^{1,p}(T)} \\ &\leq (1 + C_1 \sigma_T^{6/p+7})|u - r|_{W^{1,p}(T)},\end{aligned}$$

where C_1 is the constant of (10). Now the proof follows the lines of Theorem 3:

$$\inf_{r \in \mathbb{P}_k} |u - r|_{W^{1,p}(T)} \leq |\det(B_S)|^{1/p} \|B_S^{-1}\| \inf_{r \in \mathbb{P}_k} |\tilde{u} - r|_{W^{1,p}(\widetilde{T})}.$$

Obviously,

$$|\tilde{u}|_{W^{1,p}(\widetilde{T})/\mathbb{P}_0} = |\tilde{u}|_{W^{1,p}(\widetilde{T})},$$

and we shall prove an upper bound for $\mathbb{P}_k$ by induction. To this end, assume that for any integer $j \leq l-2$, we have

$$|\tilde{u}|_{W^{1,p}(\widetilde{T})/\mathbb{P}_j} \leq C \sigma_T^{(4/p+1)j} |\tilde{u}|_{W^{j+1,p}(\widetilde{T})}. \tag{12}$$

Then

$$\begin{aligned}\inf_{r \in \mathbb{P}_{j+1}} |\tilde{u} + r|_{W^{1,p}(\widetilde{T})} &= \inf_{r^* \in \mathbb{P}_{j+1}^*} \inf_{r \in \mathbb{P}_j} |(\tilde{u} + r^*) + r|_{W^{1,p}(\widetilde{T})} \\ &\leq C_2 \sigma_T^{(4/p+1)j} \inf_{r^* \in \mathbb{P}_{j+1}^*} |\tilde{u} + r^*|_{W^{j+1,p}(\widetilde{T})},\end{aligned}$$

by the induction hypothesis. But we have shown in the proof of Theorem 3 that

$$\inf_{r^* \in \mathbb{P}_{j+1}^*} |\tilde{u} + r^*|_{W^{j+1,p}(\widetilde{T})} \leq C_3 \sigma_T^{4/p+1} |\tilde{u}|_{W^{j+2,p}(\widetilde{T})}.$$

Therefore,

$$|\tilde{u}|_{W^{1,p}(\widetilde{T})/\mathbb{P}_{j+1}} \leq C_4 \sigma_T^{(4/p+1)(j+1)} |\tilde{u}|_{W^{j+2,p}(\widetilde{T})},$$

and (12) is proved by induction. Hence,

$$\begin{aligned}|u - \widetilde{I}_T^k(u)|_{W^{1,p}(T)} &\leq C_5 \sigma_T^{6/p+7} \sigma_T^{(l-1)(4/p+1)} \frac{1}{\rho_T} h_T^l |u|_{W^{l,p}(T)} \\ &\leq C_5 \sigma_T^{l(4/p+1)+2/p+7} h_T^{l-1} |u|_{W^{l,p}(T)}. \quad \square\end{aligned}$$

As an immediate application, we have the global estimate:

Corollary 7. *Assume that $\mathcal{T}_h$ is regular. For any integers $k \geq 0$ and $1 \leq l \leq k+1$, and for all numbers p with $1 \leq p \leq \infty$, there exists a constant C, independent of h, such that*

$$\forall u \in W^{l,p}(\Omega), \quad \left(\sum_{T \in \mathcal{T}_h} |u - \widetilde{I}_T^k(u)|_{W^{1,p}(T)}^p \right)^{1/p} \leq C h^{l-1} |u|_{W^{l,p}(\Omega)}.$$

Finally, by interpolating the estimates of Corollaries 4 and 7 between two consecutive values of k, we can extend their results to noninteger values of k:

Corollary 8. *Assume that $\mathcal{T}_h$ is regular. For any integer $k \geq 0$ and real number s with $0 \leq s \leq k+1$, and for all numbers p with $1 \leq p \leq \infty$, there exists a constant C, independent of h, such that*

$$\forall u \in W^{s,p}(\Omega), \quad \|u - \widetilde{I}_T^k(u)\|_{L^p(\Omega)} \leq Ch^s \|u\|_{W^{s,p}(\Omega)}.$$

Corollary 9. *Assume that $\mathcal{T}_h$ is regular. For any integer $k \geq 0$ and real number s with $1 \leq s \leq k+1$, and for all numbers p with $1 \leq p \leq \infty$, there exists a constant C, independent of h, such that*

$$\forall u \in W^{s,p}(\Omega), \quad \left(\sum_{T \in \mathcal{T}_h} |u - \widetilde{I}_T^k(u)|^p_{W^{1,p}(T)} \right)^{1/p} \leq Ch^{s-1} \|u\|_{W^{s,p}(\Omega)}.$$

4. Application to quadrature formulas

Let $k = 1$ and consider the standard finite element space

$$\Theta_h = \{\theta_h \in \mathscr{C}^0(\overline{\Omega}) ; \forall T \in \mathcal{T}_h , \theta_h|_T \in \mathscr{Q}_1(T)\}.$$

In solving elliptic boundary value problems, one often has to calculate terms of the form $\int_T \nabla u_h \cdot \nabla v_h \, d\mathbf{x}$ with u_h and v_h in Θ_h. The exact computation of this integral is difficult because the integrand in each T involves fractions in two variables. But exact computation is not necessary and we can approximate the integral by an appropriate quadrature formula. The most commonly used quadrature formula in this case is the two-dimensional extension of the "trapezoidal rule", called the "four-point" rule. We propose to show in this section that the error arising from the use of this quadrature formula is comparable to the interpolation error of the space Θ_h, namely $O(h)$.

For any function $\hat{f}$ defined and continuous on $\widehat{T}$, we define the four-point quadrature rule by

$$\widehat{\mathcal{S}_4}(\hat{f}) = \frac{1}{4}(\hat{f}(0,0) + \hat{f}(1,0) + \hat{f}(1,1) + \hat{f}(0,1)). \tag{13}$$

Then, observing that

$$\int_T f \, d\mathbf{x} = \int_{\widehat{T}} J_T f \circ F_T \, d\hat{\mathbf{x}},$$

we define for any continuous function f on T, the quadrature formula

$$\mathcal{S}_{4,T}(f) = \widehat{\mathcal{S}_4}(J_T f \circ F_T). \tag{14}$$

Now, assume that the solution u of the problem we want to solve belongs to $H^2(\Omega)$. Then it is continuous in Ω and we can define first $I_T(u)$ in each T, which is the classical interpolation operator in $\mathscr{Q}_1(T)$ defined by (cf. Ciarlet [2])

$$I_T(u)(\mathbf{a}_i) = u(\mathbf{a}_i) \quad \text{for } 1 \leq i \leq 4.$$

After this, we define $I_h(u) \in \Theta_h$ by

$$\forall T \in \mathcal{T}_h, \quad I_h(u)|_T = I_T(u|_T).$$

The error arising from the integration formula (13), (14) involves in particular the difference

$$\int_T \nabla(I_T(u)) \cdot \nabla v_h \, d\mathbf{x} - \mathcal{S}_{4,T}(\nabla(I_T(u)) \cdot \nabla v_h),$$

and this difference must be bounded in terms of $\|v_h\|_{H^1(T)}$. The next theorem establishes a slightly more general result.

Theorem 10. *Assume that the quadrangulation $\mathcal{T}_h$ is regular. For any number p with $1 < p \le \infty$, there exists a constant C such that for all u in $W^{2,p}(\Omega)$ the following bound holds:*

$$\forall v_h \in \Theta_h,$$

$$\left|\int_\Omega \nabla(I_h(u))\cdot\nabla v_h\,d\mathbf{x} - \sum_{T\in\mathcal{T}_h} \mathcal{S}_{4,T}(\nabla(I_T(u))\cdot\nabla v_h)\right| \le Ch|u|_{W^{2,p}(\Omega)}|v_h|_{W^{1,q}(\Omega)},$$

where $1/p + 1/q = 1$.

Proof. First note that $I_h(u)$ is well defined because $W^{2,p}(\Omega) \subset \mathcal{C}^0(\overline{\Omega})$ for $1 < p \le \infty$. For any T in $\mathcal{T}_h$, we can write

$$\int_T \nabla(I_T(u))\cdot\nabla v_h\,d\mathbf{x} - \mathcal{S}_{4,T}(\nabla(I_T(u))\cdot\nabla v_h)$$
$$= \int_T \nabla(I_T(u) - \widetilde{I}_T^1(u))\cdot\nabla v_h\,d\mathbf{x} - \mathcal{S}_{4,T}(\nabla(I_T(u) - \widetilde{I}_T^1(u))\cdot\nabla v_h),$$

because $\nabla(\widetilde{I}_T^1(u))$ is a constant vector and each component of the vector function ∇v_h is integrated exactly by the quadrature formula:

$$\forall v_h \in \Theta_h, \quad \int_T \nabla v_h\,d\mathbf{x} = \mathcal{S}_{4,T}(\nabla v_h).$$

Now, it can be easily checked that

$$|\mathcal{S}_{4,T}(\nabla(I_T(u) - \widetilde{I}_T^1(u))\cdot\nabla v_h)| \le C_1|I_T(u) - \widetilde{I}_T^1(u)|_{W^{1,p}(T)}|v_h|_{W^{1,q}(T)},$$

with a constant C_1 independent of the geometry of T. Therefore,

$$\left|\int_T \nabla(I_T(u))\cdot\nabla v_h \quad d\mathbf{x} - \mathcal{S}_{4,T}(\nabla(I_T(u))\cdot\nabla v_h)\right|$$
$$\le (1 + C_1)|I_T(u) - \widetilde{I}_T^1(u)|_{W^{1,p}(T)}|v_h|_{W^{1,q}(T)}.$$

On one hand, as $\mathcal{T}_h$ is regular, Theorem 6 applied with $l = 2$ implies that

$$|\widetilde{I}_T^1(u) - u|_{W^{1,p}(T)} \le C_2 h_T|u|_{W^{2,p}(T)},$$

with a constant C_2 independent of the geometry of T. On the other hand, a standard result of finite element interpolation yields (cf. Ciarlet [2])

$$|I_T(u) - u|_{W^{1,p}(T)} \le C_3 h_T|u|_{W^{2,p}(T)},$$

whence the desired result. □

The operator $\widetilde{I}_T^1$ plays a crucial part in this proof. If $\nabla(\widetilde{I}_T^1(u))$ were not constant, the same estimate for the quadrature error would require that the derivative of the Jacobian J_F be small with respect to h^2. This holds if T is nearly a parallelogram but not if T is an arbitrary quadrilateral.

This proof has been written in the particular case where $k = 1$, because finite elements of degree one are most commonly used in practice, but clearly, the above result extends readily to finite elements of degree k and the same type of integration formulas of order k.

ACKNOWLEDGMENT

This work was done while the author was visiting the Indian Institute of Science, Bangalore, India, and was supported by the CEFIPRA contract, number IFC/501-1/93/1553.

BIBLIOGRAPHY

1. R. Adams, *Sobolev spaces*, Academic Press, New York, 1975.
2. P. G. Ciarlet, *The finite element method for elliptic problems*, North-Holland, Amsterdam, 1977.
3. P. G. Ciarlet and P. A. Raviart, *The combined effect of curved boundaries and numerical integration in isoparametric finite element methods*, The Mathematical Foundations of the Finite Element Method with Application to Partial Differential Equations (A. K. Aziz, ed.), Academic Press, New York, 1972, pp. 409–474.
4. J. Deny and J. L. Lions, *Les espaces du type de Beppo Levi*, Ann. Inst. Fourier (Grenoble) **5** (1953–54), 305–370.
5. V. Girault and P. A. Raviart, *Finite element methods for Navier-Stokes equations, Theory and algorithms*, Springer Series in Comput. Math., vol. 5, Springer-Verlag, Berlin, 1986.
6. J. L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*, Dunod, Paris, 1970.
7. J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Masson, Paris, 1967.
8. P. A. Raviart, *Méthode des éléments finis*, Cours de DEA, Lab. d'Analyse Numérique, Université Pierre et Marie Curie, Paris, 1972.

LABORATOIRE D'ANALYSE NUMÉRIQUE - TOUR 55-65, 5ÈME ÉTAGE, UNIVERSITÉ PIERRE ET MARIE CURIE, 4, PLACE JUSSIEU, 75252 PARIS CEDEX 05, FRANCE

E-mail address: `girault@ann.jussieu.fr`