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ON YOKOI'S CONJECTURE

MING-YAO ZHANG

ABSTRACT. In this paper we obtain a lower bound for those discriminants of real quadratic fields $\mathbb{Q}(\sqrt{D})$ with $D = m^2 + 4$ and $h(D) = 1$.

In 1986 Yokoi [11] posed the following conjecture.

Conjecture. *Let $D = m^2+4$ be a square-free rational integer and m be a positive integer. Then there exist exactly six real quadratic fields $\mathbb{Q}(\sqrt{D})$ with $h(D) = 1$, i.e., $(D, m) = (5, 1), (13, 3), (29, 5), (53, 7), (173, 13), (293, 17)$.*

In 1987, by using Tatuzawa [10], Huyn Kwang Kim et al. [2] proved that there exists at most one discriminant $D = m^2 + 4 \geq e^{16}$ with $h(D) = 1$. Later, Mollin and Williams [5] generalized this result to arbitrary ERD types, i.e., those radicands of the form $D = l^2 + r$ where $r \mid 4l$. Furthermore, they extended their techniques in [6], where they were able to determine (with the only possible exception ruled out by GRH) all real quadratic fields with class number one and continued fraction period length of the principal class less than 25.

In this paper, by combining the ideas of Stark [9] and Hecke [1], we obtain the following lower bound.

Theorem. *Let $m > 17$ be a positive integer and $D = m^2 + 4$ be a square-free rational integer. If $h(D) = 1$, then we have $D > \exp(3.7 \times 10^8)$.*

For proving the result in the case $293 < D \leq 10^{13}$, we use a computer and the following lemma, which is an immediate consequence of Theorem 2.1 in Mollin and Williams [4].

Lemma 1 (see also [3]). *Let $D = m^2 + 4$ be a square-free integer and m be a positive integer. Then the following four statements are equivalent:*

(i) $h(D) = 1$.
(ii) $f_D(x) = -x^2 + x + (D-1)/4 \not\equiv 0 \pmod{p}$ *for all integers x and primes p such that $0 \leq x < p < \sqrt{D-1}/2$.*
(iii) $f_D(x)$ *is prime for all integers x with $1 < x < \sqrt{D-1}/2$.*
(iv) $(\frac{D}{p}) = -1$ *for all primes $p < \sqrt{D-1}/2$.*

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Corollary 1. *If $D = m^2 + 4$ is square-free and D is composite, then we have $h(D) > 1$.*

Proof. Case I: D has at least three different odd prime divisors. Let p denote the smallest divisor. Then we have $p < D^{\frac{1}{3}} < \sqrt{D-1}/2$. Taking $x = (p+1)/2$, we obtain $f_D(\frac{p+1}{2}) = \frac{D-p^2}{4} \equiv 0 \pmod{p}$, which contradicts (ii) of Lemma 1.

Case II: $D = pq$ ($p < q$, p and q are odd primes). We can easily show that $p < \sqrt{D-1} - 1$. Taking $x = (p+1)/2$ leads to $f_D(\frac{p+1}{2}) = p(q-p)/4$, which cannot be a prime. By (iii) of Lemma 1, we also have $h(D) > 1$. □

Corollary 2. *If $D = m^2 + 4$ is square-free and m is composite, then we have $h(D) > 1$.*

Proof. Taking $x = (m-1)/2$ in (iii) of Lemma 1 leads to the result immediately. □

We now turn to the proof in the case $293 < D \le 10^{13}$.

Lemma 2. *Let $D = m^2 + 4$ be a square-free integer and m be an odd prime such that $293 < D \le 10^{13}$. Then we have $h(D) > 1$.*

Sketch of the Proof. We have $m < 10^{6.5}$ for $D \le 10^{13}$. Let k be a natural number to be chosen later. Let S be the set of the first k primes q_j with $q_1 = 5$. For such $q_j \in S$, tabulate all those integers m_{ij} satisfying

$$0 \le m_{ij} \le q_j - 1, \qquad \left(\frac{m_{ij}^2 + 4}{q_j}\right) = 1,$$

where $(\frac{*}{*})$ denotes the Legendre symbol. This can be easily done by using the tables from p. 437 to p. 444 of Riesel [8].

Let m_0 be an integer such that $17 < m_0 < 10^{6.5}$. If there exist a prime $q_j \in S$ and some m_{ij} such that

$$q_j < \sqrt{m_0^2 + 3}/2, \qquad m_0 \equiv m_{ij} \pmod{q_j},$$

then, by (iv) of Lemma 1, we have $h(D_0) > 1$ for $D_0 = m_0^2 + 4$. Thus, such an m_0 could be eliminated. It can be easily seen that approximately half of the m's are eliminated for each q_j. To insure success, we take $k = 100$ instead of the least $k = 22$ satisfying $2^k > 10^{6.5}$. The corresponding m_{ij} to $q_j > 101$ must be calculated by trial. We use a personal computer to sieve out as many m with $17 < m < 10^{6.5}$ as possible. The computation shows that for $17 < m < 10^{6.5}$ and prime $D = m^2 + 4$, there exist some $q_j < \sqrt{m^2 + 3}/2$ and $q_j \in S$ such that $(\frac{D}{q_j}) = 1$. This completes the proof. □

The next lemma gives the first 20 convergents for the ratio of the imaginary parts of the first two nontrivial zeros in the upper half-plane of $\zeta(s)$.

Lemma 3. *Let $a = 1.487\,262\,003\,298\,890\,048$. Then its expansion in continued fraction is $a = [1; 2, 19, 7, 1, 10, 1, 20, 1, 1, 26, 1, 1, 6, 1, 3, 1, 1, 1, 8, 3, 1, 2, 1, 1, 25, 1, 1, 7, 5, 1, 3, 1, 2, 2]$ and its first 20 convergents are $\alpha_1 = 1$, $\alpha_2 = \frac{3}{2}$, $\alpha_3 = \frac{58}{39}$, $\alpha_4 = \frac{409}{275}$; $\alpha_5 = \frac{467}{314}$, $\alpha_6 = \frac{5079}{3415}$, $\alpha_7 = \frac{5546}{3729}$, $\alpha_8 = \frac{115999}{77995}$, $\alpha_9 = \frac{121545}{81724}$, $\alpha_{10} = \frac{237544}{159719}$, $\alpha_{11} = \frac{6297689}{4234418}$, $\alpha_{12} = \frac{6535233}{4394137}$,*

$\alpha_{13} = \frac{12832922}{8628555}$, $\alpha_{14} = \frac{83532765}{56165467}$, $\alpha_{15} = \frac{96365687}{64794022}$, $\alpha_{16} = \frac{372629826}{250547533}$, $\alpha_{17} = \frac{468995513}{315341555}$, $\alpha_{18} = \frac{841625339}{565889088}$, $\alpha_{19} = \frac{1310620852}{881230643}$, $\alpha_{20} = \frac{11326592155}{7615734232}$.

The following integral is one of our major analytic tools, which was first used in 1917 by Hecke [1] to obtain a Kronecker limit formula for real quadratic fields.

Lemma 4. *Define*

$$c(s) = \int_{-\infty}^{\infty} \frac{dv}{(e^v + e^{-v})^s}, \qquad \mathrm{Re}\, s > 0. \tag{1}$$

Then $c(s)$ can be continued to a meromorphic function over the s-plane with the only singularity $s = 0$ (a pole of order 1). Besides, for any s we have

$$c(s) = \Gamma\left(\frac{s}{2}\right)^2 / (2\Gamma(s)). \tag{2}$$

Proof. For $\mathrm{Re}\, s > 0$ we have

$$c(s) = \int_0^{\frac{1}{2}} t^{\frac{1}{2}s-1}(1-t)^{\frac{1}{2}s-1}\, dt = \frac{1}{2}B\left(\frac{1}{2}s, \frac{1}{2}s\right) = \Gamma\left(\frac{1}{2}s\right)^2 / (2\Gamma(s)),$$

which leads to the desired result. □

The following lemma is similar to the one used in [9]. Here, however, $c(s)$ is used to turn our real quadratic fields into imaginary ones, which can be treated by Stark's method.

Lemma 5. *Let D be a square-free integer with $h(D) = 1$. Define the L-function*

$$L_D(s) = \sum_{n=1}^{\infty} \left(\frac{D}{n}\right) n^{-s} \quad (\mathrm{Re}\, s > 1), \tag{3}$$

where $(\frac{D}{n})$ denotes the Kronecker symbol. Then, for any s, we have

$$\zeta(s)L_D(s)c(s) = \zeta(2s)c(\varepsilon, s) + \sqrt{\pi}D^{\frac{1}{2}-s}\frac{\zeta(2s-1)\Gamma(s-\frac{1}{2})}{\Gamma(s)}c(\varepsilon, 1-s) + R_0(s), \tag{4}$$

where $\varepsilon = (m+\sqrt{D})/2$ is the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{D})$,

$$R_0(s) = \int_{-\ln \varepsilon}^{\ln \varepsilon} R_v(s)\, dv, \tag{5}$$

$$R_v(s) = \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \left(y - [y] - \frac{1}{2}\right) \frac{d}{dy}(Q_v(k, y)^{-s})\, dy, \tag{6}$$

$$Q_v(k, y) = Ak^2 + Bky + Cy^2, \tag{7}$$

$$A = (1+\sqrt{D})^2 e^v/4 + (1-\sqrt{D})^2 e^{-v}/4, \quad B = (1+\sqrt{D})e^v + (1-\sqrt{D})e^{-v}, \qquad C = e^v + e^{-v}, \tag{8}$$

$$B^2 - 4AC = -4D, \tag{9}$$

$$c(\varepsilon, z) = \int_{-\ln \varepsilon}^{\ln \varepsilon} \frac{dv}{(e^v + e^{-v})^z}. \tag{10}$$

Proof. We only need to prove this for $\operatorname{Re} s > 1$. Obviously, $f(x, y) = x^2 + xy - (D-1)y^2/4$ is a quadratic form with discriminant D. Let $\mathfrak{U}$ be the integral ideal corresponding to $f(x, y)$. Then we can take $\mathfrak{U} = [1, \omega]$, where $\omega = (1+\sqrt{D})/2$. Some $h(D) = 1$, we can easily show that

$$\zeta(s)L_D(s) = \frac{1}{2}\sum_{\lambda\in\mathfrak{U}/\varepsilon}{}' 1/|\lambda\lambda'|^s, \tag{11}$$

where $\sum'$ means the summation is taken over all nonzero ideals λ,

$$\left.\begin{aligned} \lambda = k\omega + r, \qquad \lambda' = k\omega' + r, \\ \omega = (1+\sqrt{D})/2, \quad \omega' = (1-\sqrt{D})/2, \qquad k, r \in \mathbb{Z} \end{aligned}\right\} \tag{12}$$

and $\zeta(s)$ denotes the Riemann zeta function.

By the definition of $c(s)$ we obtain that

$$\begin{aligned} \zeta(s)L_D(s)c(s) &= \frac{1}{2}\sum_{\lambda\in\mathfrak{U}/\varepsilon}{}' \int_{-\infty}^{\infty} \frac{dv}{(\lambda^2 e^v + \lambda'^2 e^{-v})^s} \\ &= \frac{1}{2}\sum_{\lambda\in\mathfrak{U}}{}' \int_{-\ln\varepsilon}^{\ln\varepsilon} \frac{dv}{Q_v(k, r)^s} = \int_{-\ln\varepsilon}^{\ln\varepsilon} M_v(k, r)\,dv, \end{aligned} \tag{13}$$

where

$$M_v(k, r) = \frac{1}{2}\sum_{(k, r)\neq(0, 0)} \frac{1}{Q_v(k, r)^s}, \tag{14}$$

$Q_v(k, r)$ is defined in (7), and (9) can be easily verified.

From Euler's summation formula it follows that

$$\begin{aligned} M_v(k, r) &= \frac{\zeta(2s)}{(e^v + e^{-v})^s} + \sum_{k=1}^{\infty}\sum_{r=-\infty}^{\infty} Q_v(k, r)^{-s} \\ &= \frac{\zeta(2s)}{(e^v + e^{-v})^s} + \sum_{k=1}^{\infty}\int_{-\infty}^{\infty} \frac{dy}{Q_v(k, y)^s} \\ &\quad + \sum_{k=1}^{\infty}\int_{-\infty}^{\infty} \left(y - [y] - \frac{1}{2}\right)\frac{d}{dy}(Q_v(k, y)^{-s})\,dy. \end{aligned} \tag{15}$$

A direct calculation gives

$$\sum_{k=1}^{\infty}\int_{-\infty}^{\infty} \frac{dy}{Q_v(k, y)^s} = \frac{\sqrt{\pi}D^{\frac{1}{2}-s}}{(e^v + e^{-v})^{1-s}} \frac{\zeta(2s-1)\Gamma(s-\frac{1}{2})}{\Gamma(s)}. \tag{16}$$

This completes the proof. □

Combining Lemma 5 with our reiteration method leads to the following result, which secures the theorem.

Lemma 6. *Let* $D = m^2 + 4 > 10^{13}$ *be a square-free integer with* $h(D) = 1$. *Then we have*

$$D > \exp(371815978). \tag{17}$$

Proof. Let $B_j(u)$ denote the jth Bernoulli polynomial. We have

$$B_1(u) = u - \frac{1}{2}, \qquad B_2(u) = u^2 - u + \frac{1}{6}. \tag{18}$$

By (2.6) of Rademacher [7] and the substitution $t = (2Cy + Bk)/(2\sqrt{D}k)$ we obtain

$$\int_{-\infty}^{\infty} B_1(y - [y]) \frac{d}{dy}(Q_v(k, y)^{-s})\, dy = \frac{C^{s+1}}{D^{s+\frac{1}{2}} k^{2s+1}} (J_1(s) + J_2(s)), \tag{19}$$

where

$$J_1(s) = \frac{1}{48} \int_{-\infty}^{\infty} \frac{d^2}{dt^2}((t^2+1)^{-s})\, dt, \tag{20}$$

$$J_2(s) = \int_{-\infty}^{\infty} \frac{B_2(y - [y]) - \frac{1}{24}}{2} \frac{d^2}{dt^2}((t^2+1)^{-s})\, dt. \tag{21}$$

We can easily obtain

$$J_1(s) = \frac{\sqrt{\pi}}{24} \frac{\Gamma(s + \frac{1}{2})}{\Gamma(s)} \tag{22}$$

and

$$-\frac{1}{8} \le B_2(y - [y]) - \frac{1}{24} < \frac{1}{8}. \tag{23}$$

Let ν_j be the ordinate of the jth nontrivial zero in the upper half-plane of the Riemann zeta function. In [6] it was shown that

$$\nu_1 = 14.134\,725\,141\,734\,693\,790\,457 + 10^{-21}\theta, \tag{24}$$
$$\nu_2 = 21.022\,039\,638\,771\,554\,992\,628 + 10^{-21}\theta, \qquad (|\theta| \le 1). \tag{25}$$

Thus we have

$$\begin{aligned} \left|J_2\left(\frac{1}{2} + i\nu_j\right)\right| &\le \frac{1}{8} \int_0^{\infty} \left|\frac{d}{dt^2}((t^2+1)^{-s})\right| dt \\ &\le \frac{1}{6} \left|\frac{1}{2} + i\nu_j\right| \left|\frac{3}{2} + i\nu_j\right| < \begin{cases} 33.5065, & j = 1, \\ 73.8644, & j = 2. \end{cases} \end{aligned} \tag{26}$$

From (5), (19), (22) and (26) it follows that

$$\begin{aligned} R_0\left(\frac{1}{2} + i\nu_j\right) &= \frac{\sqrt{\pi}\Gamma(1 + i\nu_j)}{24 D^{1+i\nu_j} \Gamma\left(\frac{1}{2} + i\nu_j\right)} \zeta(2 + 2i\nu_j) c\left(\varepsilon, -\frac{3}{2} - i\nu_j\right) \\ &\quad + \frac{\alpha_j \zeta(2) c\left(\varepsilon, -\frac{\varepsilon}{2}\right)\theta}{D} \quad (|\theta| \le 1), \end{aligned} \tag{27}$$

where

$$\alpha_j = \begin{cases} 33.5065, & j = 1, \\ 73.8644, & j = 2. \end{cases} \tag{28}$$

It can be readily shown that

$$\begin{aligned} c\left(\varepsilon, \frac{1}{2} + i\nu_j\right) &= c\left(\frac{1}{2} + i\nu_j\right) - 2\int_{\ln\varepsilon}^{\infty} \frac{dv}{(e^v + e^{-v})^{\frac{1}{2}+i\nu_j}} \\ &= \frac{\Gamma\left(\frac{1}{4} + \frac{1}{2}i\nu_j\right)^2}{2\Gamma\left(\frac{1}{2} + i\nu_j\right)} + \frac{4\theta}{\sqrt{\varepsilon}} \quad (|\theta| \le 1). \end{aligned} \tag{29}$$

From (5) it follows that

$$R_0\left(\frac{1}{2}+i\nu_j\right) = -\zeta(1+2i\nu_j)c\left(\varepsilon\,,\,\frac{1}{2}+i\nu_j\right) - \sqrt{\pi}D^{-i\nu_j}\frac{\zeta(2i\nu_j)\Gamma(i\nu_j)}{\Gamma\left(\frac{1}{2}+i\nu_j\right)}c\left(\varepsilon\,,\,\frac{1}{2}-i\nu_j\right). \tag{30}$$

By well-known formulae for the Riemann zeta function and gamma function we have

$$\zeta(2i\nu_j)\Gamma(i\nu_j) = \pi^{-\frac{1}{2}+2i\nu_j}\zeta(1-2i\nu_j)\Gamma\left(\frac{1}{2}-i\nu_j\right). \tag{31}$$

From (27), (30), (29) and (31) it follows that

$$\left(\frac{D}{\pi^2}\right)^{i\nu_j} = -\frac{\Gamma\left(\frac{1}{4}-\frac{1}{2}i\nu_j\right)^2\zeta(1-2i\nu_j)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}i\nu_j\right)^2\zeta(1+2i\nu_j)} + \sum_{k=1}^{3}J_k^*\left(\frac{1}{2}+i\nu_j\right), \qquad j=1\,,\,2\,, \tag{32}$$

where

$$J_1^*\left(\frac{1}{2}+i\nu_j\right) = -\frac{16}{\sqrt{\varepsilon}}\left(\frac{D}{\pi^2}\right)^{i\nu_j}\frac{\Gamma\left(\frac{1}{2}+i\nu_j\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}i\nu_j\right)^2}\theta, \qquad j=1\,,\,2\ (|\theta|\le 1), \tag{33}$$

$$J_2^*\left(\frac{1}{2}+i\nu_j\right) = -\frac{\pi^{\frac{1}{2}-2i\nu_j}\Gamma(1+i\nu_j)\zeta(2+2i\nu_j)c(\varepsilon\,,\,-\frac{3}{2}-i\nu_j)}{12D\Gamma\left(\frac{1}{4}+\frac{1}{2}i\nu_j\right)^2\zeta(1+2i\nu_j)}, \qquad j=1\,,\,2\,, \tag{34}$$

$$J_3^*\left(\frac{1}{2}+i\nu_j\right) = -\frac{1}{3}\left(\frac{\pi^2}{D}\right)^{1-i\nu_j}\alpha_j\frac{\Gamma\left(\frac{1}{2}+i\nu_j\right)c\left(\varepsilon\,,\,-\frac{3}{2}\right)\theta}{\Gamma\left(\frac{1}{4}+\frac{1}{2}i\nu_j\right)^2\zeta(1+2i\nu_j)}, \qquad j=1\,,\,2\ (|\theta|\le 1). \tag{35}$$

A direct calculation gives

$$\left|\Gamma\left(\frac{1}{2}+i\nu_j\right)\right| > \begin{cases} 5.708835\times 10^{-10}, & j=1\,,\\ 1.413149\times 10^{-14}, & j=2\,,\end{cases} \tag{36}$$

$$\left|\Gamma\left(\frac{3}{4}+\frac{1}{2}i\nu_j\right)\right| < \begin{cases} \exp(-9.6937176), & j=1\,,\\ \exp(-15.0036975), & j=2\,,\end{cases} \tag{37}$$

$$|\Gamma(1+i\nu_j)| < \begin{cases} 2.158099\times 10^{-9}, & j=1\,,\\ 5.261146\times 10^{-4}, & j=2\,,\end{cases} \tag{38}$$

$$|\zeta(2+2i\nu_j)| < \begin{cases} 1.4229, & j=1\,,\\ 0.9162, & j=2.\end{cases} \tag{39}$$

By (36), (37) and the multiplication formula for the gamma function we have

$$\left|\Gamma\left(\frac{1}{4}+\frac{1}{2}i\nu_j\right)\right|^2 > \begin{cases} 5.3843214\times 10^{-10}, & j=1\,,\\ 8.8395796\times 10^{-15}, & j=2.\end{cases} \tag{40}$$

By [9] we have

$$|\zeta(1+2i\nu_j)| > \begin{cases} 1.948757, & j=1\,,\\ 0.830962, & j=2.\end{cases} \tag{41}$$

For $D > 10^{13}$ we easily have

(42)
$$c\left(\varepsilon, -\frac{3}{2}\right) < \frac{4}{3}\left\{\left(1+\frac{4}{\varepsilon^2}\right)^{3/2} + \left(\left(1+\frac{1}{\varepsilon}\right)^{3/2} - 1\right)2^{-3/2} + (2^{3/2}-1)\varepsilon^{-3/4}\right\}D^{3/4}$$
$$< 1.33336607D^{3/4}.$$

Therefore, for $D > 10^{13}$ we have

(43) $$\left|J_1^*\left(\frac{1}{2}+i\nu_j\right)\right| < \begin{cases} 16.9643217D^{-1/4}, & j=1,\\ 20.6914635D^{-1/4}, & j=2,\end{cases}$$

(44) $$\left|J_2^*\left(\frac{1}{2}+i\nu_j\right)\right| < \begin{cases} 0.57636882D^{-1/4}, & j=1,\\ 1.2924129D^{-1/4}, & j=2,\end{cases}$$

(45) $$\left|J_3^*\left(\frac{1}{2}+i\nu_j\right)\right| < \begin{cases} 79.967973D^{-1/4}, & j=1,\\ 504.258735D^{-1/4}, & j=2.\end{cases}$$

From (32), (43), (44) and (45) it follows that for $D > 10^{13}$ we have

(46)
(47)
$$\left(\frac{D}{\pi^2}\right)^{i\nu_j} = -\frac{\Gamma\left(\frac{1}{4}-\frac{1}{2}i\nu_j\right)^2\zeta(1-2i\nu_j)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}i\nu_j\right)^2\zeta(1+2i\nu_j)} + \begin{cases} 97.50866360\theta D^{-\frac{1}{4}}, & j=1,\\ 526.2426120\theta D^{-\frac{1}{4}}, & j=2,\end{cases} \quad (|\theta| \le 1).$$

Taking the arguments on both sides of (46) and (47), and noticing that $D > 10^{13}$, we obtain

(48)
(49)
$$\nu_j \ln(D/\pi^2) = a_j + 2\pi x_j + \begin{cases} 0.0548331520\theta, & j=1,\\ 0.2959279680\theta, & j=2,\end{cases}$$

where x_j are nonnegative integers and

(50)
$$a_j \equiv \pi - 4\arg\Gamma\left(\frac{1}{4}+\frac{1}{2}i\nu_j\right) - 2\arg\zeta(1+2i\nu_j) \pmod{2\pi},$$
$$0 \le a_j < 2\pi, \ j = 1, 2.$$

From (48) and (49) it follows that

(51) $$x_2 = \frac{\nu_2}{\nu_1}x_1 + a + R^*,$$

where

(52) $$a = \frac{1}{2\pi}\left(\frac{\nu_2}{\nu_1}a_1 - a_2\right)$$

and

(53) $$|R^*| \le \frac{1}{2\pi}\left(\frac{\nu_2}{\nu_1}(0.054833152) + 0.295927968\right) < 0.0600776857.$$

In [9] it is shown that

(54) $$\frac{\nu_2}{\nu_1} = 1.487\,262\,003\,892\,890\,048 + 10^{-18}\theta,$$

$$\begin{matrix}(55)\\(56)\end{matrix}\quad \frac{1}{\pi}\arg\Gamma\left(\frac{1}{2}+i\nu_j\right)=\begin{cases} 7.418\,512\,651\,985\,173+2\times 10^{-13}\theta, & j=1,\\ 13.688\,619\,111\,000\,235\ +2\times 10^{-13}\theta, & j=2,\end{cases}$$

$$\begin{matrix}(57)\\(58)\end{matrix}\quad \frac{1}{\pi}\arg\zeta(1+2i\nu_j)=\begin{cases} -0.108\,452\,737\,083\,095+10^{-10}\theta\ (\mathrm{mod}\ 2), & j=1,\\ 0.067\,103\,865\,503\,910+10^{-10}\theta\ (\mathrm{mod}\ 2), & j=2.\end{cases}$$

It can be easily seen that

$$\frac{\Gamma\left(\frac{1}{4}+\frac{1}{2}i\nu_j\right)}{\Gamma\left(\frac{1}{4}-\frac{1}{2}i\nu_j\right)}=2^{\frac{1}{2}-i\nu_j}\Gamma\left(\frac{1}{2}+i\nu_j\right)\sin\left(\left(\frac{1}{4}-\frac{1}{2}i\nu_j\right)\pi\right)/\sqrt{\pi}, \tag{59}$$

from which it follows that

(60)

$$\arg\Gamma\left(\frac{1}{4}+\frac{1}{2}i\nu_j\right)=\frac{1}{2}\left(\arg\Gamma\left(\frac{1}{2}+i\nu_j\right)-\nu_j\ln 2+\arg\left\{\sin\left(\left(\frac{1}{4}-\frac{1}{2}i\nu_j\right)\pi\right)\right\}\right).$$

By (60), (55) and (56) we have

$$\begin{aligned}&\frac{2}{\pi}\arg\Gamma\left(\frac{1}{4}+\frac{1}{2}i\nu_j\right)\\ &\quad=\begin{cases}4.049\,889\,087\,345\,757\,85+2.1\times 10^{-13}\theta, & j=1.\\ 8.800\,408\,778\,867\,03\qquad +2.1\times 10^{-13}\theta, & j=2.\end{cases}\end{aligned}\qquad \begin{matrix}(61)\\(62)\end{matrix}$$

From (50), (57), (58), (61) and (62) we obtain

$$\begin{matrix}(63)\\(64)\end{matrix}\quad \frac{a_j}{2\pi}=\begin{cases}0.558\,563\,649\,737\,337\,15+1.003\times 10^{10}\theta, & j=1.\\ 0.632\,487\,355\,629\,06\qquad +1.003\times 10^{-10}\theta, & j=2,\end{cases}$$

$$a=0.198\,243\,137\,455\,44+1.76\times 10^{-10}\theta. \tag{65}$$

Our next device is the following reiteration. By (48) and $D>10^{13}$ we first have $x_1>61$. Then, by using this reiteration, we push x_1 to a much greater value which corresponds to a much greater lower bound for D.

Let us now prove that

$$x_1\geq 84. \tag{66}$$

Taking $x_1=7$ in (51) gives

$$10.410\,834\,027\,250\,230\,336+7\times 10^{-18}\theta=7\frac{\nu_2}{\nu_1}+a+R^*+1.77\times 10^{-10}\theta. \tag{67}$$

By (51) and (67) we have

$$x_2-10=\frac{\nu_2}{\nu_1}(x_1-7)-b_1+R_1, \tag{68}$$

where

$$b_1=0.4108,\qquad |R_1|<0.1201893989. \tag{69}$$

Take

$$p_1=58,\qquad q_1=39, \tag{70}$$

for which we have

$$\left|\frac{\nu_2}{\nu_1}-\frac{p_1}{q_1}\right|<8.2517\times 10^{-5}. \tag{71}$$

Assume that Q and R $(0 \le R < q_1)$ are two integers such that

$$Q + \frac{R}{q_1} = \frac{p_1}{q_1}(x_1 - 7). \tag{72}$$

By (68) and (72) we have

$$x_2 - Q - 10 = \left(\frac{\nu_2}{\nu_1} - \frac{p_1}{q_1}\right)(x_1 - 7) + \left(\frac{R}{q_1} - b_1\right) + 0.12018939 89\theta. \tag{73}$$

If $x_1 \le 83$, by (71) and (73) we have

$$|x_2 - \theta - 10| \le 76 \times 8.2517 \times 10^{-5} + (1 - b_1) + 0.121 < 1. \tag{74}$$

On the other hand, we have

$$b_1 q_1 = (0.4108)(39) = 16.0212. \tag{75}$$

Thus, for $R \notin [12, 20]$ we have

$$|x_2 - Q - 10| > 0.12766 - 76 \times 8.2517 \times 10^{-5} - 0.121 > 0. \tag{76}$$

which contradicts (74).

Furthermore, we easily have

$$3q_1 - 2p_1 = 1, \tag{77}$$

from which and (72) it follows that

$$x_1 \equiv 7 + 37R \pmod{q_1}. \tag{78}$$

For $61 < x_1 \le 83$ and $R \in [12, 20]$ a direct calculation shows that there is no such pair of x_1 and R satisfying (78). This proves (66).

Now we prove that

$$x_1 \ge 92. \tag{79}$$

By (66) and (48) we have

$$D > 2.07447887 \times 10^{17}. \tag{80}$$

By (80) and (46), (47) we easily reduce (48) and (49) to

$$\nu_j \ln(D/\pi^2) = a_j + 2\pi x_j + \begin{cases} 0.00456894 53\theta, & j = 1, \quad (81) \\ 0.02465805 17\theta, & j = 2, \quad (82) \end{cases}$$

and (73) because

$$x_2 - Q - 10 = \left(\frac{\nu_2}{\nu_1} - \frac{p_2}{q_2}\right)(x_1 - 7) + \left(\frac{R}{q_2} - b_1\right) + 0.01004591 45\theta, \tag{83}$$

where we choose

$$p_2 = 409, \qquad q_2 = 275. \tag{84}$$

We easily have

$$\left|\frac{\nu_2}{\nu_1} - \frac{p_2}{q_2}\right| < 1.0724 \times 10^{-5} \tag{85}$$

and

$$39p_2 - 58q_2 = 1, \qquad b_1 q_2 = 112.97. \tag{86}$$

If (79) is false, we should have

(87) $$84 \le x_1 \le 91.$$

By (83), (85) and (87) we have

(88) $$|x_2 - Q - 10| \le (84)(1.0724 \times 10^{-5}) + (1 - b_1) + 0.01005 < 1.$$

For $R \notin [110, 115]$ we have

(89) $$\left|\frac{R}{q_2} - b_1\right| \ge \frac{3.03}{q_2} > 0.01101818.$$

Hence, for $R \notin [110, 115]$ we have

$$|x_2 - Q - 10| \ge 0.01101818 - (84)(1.0724 \times 10^{-5}) - 0.010046 > 0,$$

which contradicts (88). Here we have assumed that Q and R $(0 \le R < q_2)$ are two integers defined by

(90) $$Q + \frac{R}{q_2} = \frac{p_2}{q_2}(x_1 - 7).$$

By (86) and (90) we have

(91) $$x_1 \equiv 7 + 39R \pmod{q_2}.$$

A direct calculation shows that there is no such pair of x_1 and R, $84 \le x_1 \le 91$, $R \in [110, 115]$. This proves (79).

By (79) we have

(92)
(93) $$\nu_j \ln(D/\pi^2) = a_j + 2\pi x_j + \begin{cases} 0.00187639230\theta, & j = 1, \\ 0.01012666470\theta, & j = 2. \end{cases}$$

By taking

(94) $$p_3 = 467, \qquad q_3 = 314$$

and defining Q and R $(0 \le R < q_3)$ by

(95) $$Q + \frac{R}{q_3} = \frac{p_3}{q_3}(x_1 - 7),$$

we have

(96) $$x_2 - Q - 10 - \left(\frac{\nu_2}{\nu_1} - \frac{p_3}{q_3}\right)(x_1 - 7) + \left(\frac{R}{q_3} - b_1\right) + 0.00414574820\theta.$$

This easily leads to

(97) $$x_1 \ge 289.$$

By (97) and taking

(98) $$p_4 = 468995513, \qquad q_4 = 315341555,$$

we are led to

(99) $$x_1 \ge 315341562.$$

By (99) and taking

(100) $$p_5 = 1310620852, \qquad q_5 = 881230643,$$

we are led to

$$x_1 \geq 836441460. \tag{101}$$

By (101) we easily have

$$D > \exp(371815978),$$

which is the desired conclusion. □

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DEPARTMENT OF MATHEMATICS OF USTC, HEFEI 230026, ANHUI PROVINCE, P. R. CHINA

Current address: Department of Mathematics and Physics, College of Science & Technology of Hainan University, Haikou 570228, Hainan Province, P. R. China