EFFICIENCY OF A POSTERIORI BEM–ERROR ESTIMATES FOR FIRST-KIND INTEGRAL EQUATIONS ON QUASI–UNIFORM MESHES

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ABSTRACT. In the numerical treatment of integral equations of the first kind using boundary element methods (BEM), the author and E. P. Stephan have derived a posteriori error estimates as tools for both reliable computation and self-adaptive mesh refinement. So far, efficiency of those a posteriori error estimates has been indicated by numerical examples in model situations only. This work affirms efficiency by proving the reverse inequality. Based on best approximation, on inverse inequalities and on stability of the discretization, and complementary to our previous work, an abstract approach yields a converse estimate. This estimate proves efficiency of an a posteriori error estimate in the BEM on quasi–uniform meshes for Symm's integral equation, for a hypersingular equation, and for a transmission problem.

1. INTRODUCTION

In recent decades adaptive mesh refining proved to be a tool of high practical importance in numerical analysis of partial differential equations. The questions of how and where to perform the refinement and whether this is "efficient" (a concept to be defined) is subject of many papers, and we refer, e.g., to [1, 14, 15, 16, 17, 18, 23, 24, 28, 29, 30] and the references quoted therein. The framework of adaptive methods, introduced by Eriksson and Johnson [14, 15] for finite elements, is studied in [3, 4, 5, 6, 7, 8] for boundary element methods (BEM) and covers weakly singular and hypersingular integral equations, integral equations for transmission problems, and the coupling of finite elements and boundary elements. However, the questions of efficiency of the adaptive algorithms and the sharpness of the a posteriori error estimates have been studied by numerical experiments only.

In this paper we focus our attention mainly on quasi–uniform meshes (for reasons which become clear later and are discussed in §8.4) and then prove that one of the a posteriori estimates is sharp, i.e., a complementary inequality holds. We first describe a typical example and state the results obtained; the subsequent sections contain proofs, more abstract results and further applications.

Consider Symm's integral equation: *Given f find ϕ with*

$$(1.1) \qquad V\phi(x) = f(x) \quad (x \in \Gamma).$$

Here, $\Gamma = \partial\Omega$ is a polygon, the boundary of a bounded two-dimensional Lipschitz domain Ω, and

$$(1.2) \qquad V\phi(x) := -\frac{1}{\pi} \int_\Gamma \phi(y) \log|x-y| ds_y$$

is the weakly singular single-layer potential operator.

Equation (1.1) is equivalently related to the Dirichlet problem for the Laplacian

$$(1.3) \qquad \Delta u = 0 \quad \text{in } \Omega \quad , \qquad u = u_0 \quad \text{on } \Gamma,$$

with given Dirichlet data u_0. Indeed, using the right-hand side

$$f(x) = u_0(x) - \frac{1}{\pi} \int_\Gamma u_0(y) \frac{\partial}{\partial n_y} \log|x-y| ds_y \quad (x \in \Gamma)$$

in (1.1), one obtains $\phi = \frac{\partial u}{\partial n}$, i.e., the unknown ϕ in (1.1) is the normal derivative of the solution u of the Dirichlet problem. Moreover, once $u|_\Gamma$ and ϕ are known, u is given via a representation formula (see, e.g., [10]).

Under some assumptions on Ω (cf. §5), the single-layer potential $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is linear, bounded, symmetric and positive definite. ($H^{1/2}(\Gamma)$ is the trace space of $H^1(\Omega)$ and $H^{-1/2}(\Gamma) = H^{1/2}(\Gamma)^*$, see §5 for details.) For $f \in H^1(\Gamma)$, Lax–Milgram's lemma guarantees that (1.1) has a unique solution $\phi \in H^{-1/2}(\Gamma)$, which then belongs to $L^2(\Gamma)$ [9]. Moreover, by Cea's lemma, Galerkin methods like (Galerkin) BEM are quasi–optimally convergent. The simplest of such discretization schemes is described by a partition (a so–called mesh) $\pi = \{\Gamma_1, \ldots, \Gamma_n\}$ of the polygon Γ in intervals (so–called elements) $\Gamma_1, \ldots, \Gamma_n$. Then

$$(1.4) \qquad S_\pi^0(\Gamma) := \{v_h \in L^\infty(\Gamma) : v_h|_{\Gamma_j} \in \mathbb{R} \quad \text{for all } j = 1, \ldots, N\}$$

denotes the linear space of piecewise constant functions and $h(\pi) \in S_\pi^0(\Gamma)$ is defined as the local mesh size, i.e., $h(\pi)|_{\Gamma_j} := |\Gamma_j| :=$ length of Γ_j.

Lax–Milgram's lemma guarantees a unique solution $\phi_h \in S_\pi^0(\Gamma)$ of the Galerkin equations, which are equivalent to

$$R_h := f - V\phi_h \perp S_\pi^0(\Gamma).$$

Here, $\perp$ means orthogonality in $L^2(\Gamma)$, i.e., $\int_\Gamma R_h \psi_h ds = 0$ for all $\psi_h \in S_\pi^0(\Gamma)$. Note that $R_h \in H^1(\Gamma)$, so that the derivative $\frac{\partial}{\partial s} R_h = R_h'$ of R_h along Γ with respect to the arclength exists and belongs to $L^2(\Gamma)$.

Theorem 1 ([4])**.** *There exists a constant $c > 0$ (depending only on Γ) such that for any $s \in [0,1]$ there holds*

$$(1.5) \qquad \|\phi - \phi_h\|_{H^{-s}(\Gamma)} \le c \cdot \|R_h'\|_{L^2(\Gamma)}^{1-s} \|h(\pi) \cdot R_h'\|_{L^2(\Gamma)}^s.$$

An immediate consequence of Theorem 1 is

$$(1.6) \qquad \|\phi - \phi_h\|_{H^{-s}(\Gamma)} \le c \cdot \max h(\pi)^s \cdot \|R_h'\|_{L^2(\Gamma)}$$

with $\max h(\pi) := \|h(\pi)\|_{L^\infty(\Gamma)}$. Compared with (1.5), the estimate (1.6) is reasonable if $\max h(\pi)/\min h(\pi)$ is not too large, $\min h(\pi) := \min\{h(\pi)(x) : x \in \Gamma\}$. That means that there is a global constant c_u such that for all meshes π under consideration,

$$(1.7) \qquad \max h(\pi)/\min h(\pi) = \max_{j \ne k} |\Gamma_j|/|\Gamma_k| \le c_u.$$

We prove in this paper the reverse inequality of (1.6) and hence affirm the sharpness of the estimate in Theorem 1 for quasi-uniform meshes.

Theorem 2. *If f is continuous and smooth on each side of Γ, there exist constants $c_0, h_0 > 0$ (depending only on Γ, f and c_u) such that for all partitions π of Γ with $\max h(\pi) < h_0$ and (1.7), and for $s \in [0,1]$, one has*

$$(1.8) \qquad \max h(\pi)^s \cdot \| R'_h \|_{L^2(\Gamma)} \le c_0 \cdot \| \phi - \phi_h \|_{H^{-s}(\Gamma)}.$$

The rest of this paper is organized as follows: The general framework of a posteriori estimates presented in [4] is recalled and enlarged to cover an abstract complementary estimate in §2. The three ingredients of this estimate are (i) approximation properties (upper and lower bounds for the the best approximation), (ii) inverse assumptions and (iii) stability of the discretization of the trial functions in a certain sense. A more transparent implication is derived in §3 while the proof is given in §4. The abstract results are studied for Symm's integral equation in §5 and Theorem 2 is proved there. Two other applications concerning the hypersingular integral equation and an integral equation for some transmission problem are described in §§6 and 7. The paper concludes with miscellaneous comments in §8, including a comparison of the estimates in [4] and [5].

2. ABSTRACT SETTING

We recall the general approach of [4] for a posteriori error estimates, enlarge it and add a complementary abstract estimate.

Let $X_1 \subset X_0$ and $Y_1 \subset Y_0$ be real Banach spaces, and let $X_\theta := [X_0, X_1]_\theta$ and $Y_\theta := [Y_0, Y_1]_\theta$ be defined by interpolation of these spaces, $0 \le \theta \le 1$, cf. [2]. We briefly list a few properties inherited from interpolation, which are assumed in the sequel: There are positive constants $c_{\theta,X}$ and $c_{\theta,Y}$ such that for all $x \in X_1$ and $y \in Y_1$

$$(2.1)$$
$$\| x \|_{X_\theta} \le c_{\theta,X} \cdot \| x \|_{X_0}^{1-\theta} \cdot \| x \|_{X_1}^{\theta} \quad \text{and} \quad \| y \|_{Y_\theta} \le c_{\theta,Y} \cdot \| y \|_{Y_0}^{1-\theta} \cdot \| y \|_{Y_1}^{\theta}.$$

Let $L(X;Y)$ denote the Banach space of linear bounded mappings between the Banach spaces X and Y, and let $\| \cdot \|_{L(X;Y)}$ be the corresponding operator norm. Then, for each $A_j \in L(X_j;Y_j)$, $j = 0,1$, with $A_0|_{X_1} = A_1$, the restriction $A_\theta = A_0|_{X_\theta}$ belongs to $L(X_\theta;Y_\theta)$ and

$$(2.2) \qquad \| A_\theta \|_{L(X_\theta;Y_\theta)} \le \| A_\theta \|_{L(X_0;Y_0)}^{1-\theta} \cdot \| A_\theta \|_{L(X_1;Y_1)}^{\theta}.$$

Let $A = A_\theta$ be such a mapping and assume, in addition, that $A_\theta : X_\theta \to Y_\theta$ is surjective and injective. Then, fix a right-hand side $f \in Y_1$ and the solution $u \in X_\theta$ of

$$(2.3) \qquad Au = f.$$

We apply the Galerkin method to approximate u. Let $S_h \subset X_1$ and $T_h \subset Y_0^*$ be finite-dimensional subspaces such that there exists some $u_h \in S_h$ satisfying

$$(2.4) \qquad t_h(Au_h) = t_h(f) \quad \text{for all } t_h \in T_h.$$

The residual $R_h := f - Au_h$ and the error $e_h := u - u_h$ are related by the following estimate.

Theorem 3 ([4]). *There exists $\rho \in Y_0^*$ satisfying*

$$(2.5) \qquad \|\rho\|_{Y_0^*}^2 = \| R_h \|_{Y_0}^2 = \rho(R_h),$$

and there holds

$$(2.6) \qquad \|e_h\|_{X_\theta} \leq c_{\theta,Y} \cdot \|A^{-1}\|_{L(Y_\theta; X_\theta)} \cdot \|R_h\|_{Y_1}^\theta \cdot \inf_{t_h \in T_h} \|\rho - t_h\|_{Y_0^*}^{1-\theta}.$$

Proof. To make the presentation self-contained, we give a brief outline of the proof. The existence of the (weighted) peak functional $\rho \in Y_0^*$ satisfying (2.5) follows from a well–known corollary of Hahn–Banach's theorem. Since $A = A_\theta : X_\theta \to Y_\theta$ is bijective, we have $e_h = A^{-1} R_h$ and

$$\|e_h\|_{X_\theta} \leq \|A^{-1}\|_{L(Y_\theta; X_\theta)} \cdot \|R_h\|_{Y_\theta}.$$

By interpolation, $\|R_h\|_{Y_\theta} \leq c_{\theta,Y} \cdot \|R_h\|_{Y_0}^{1-\theta} \cdot \|R_h\|_{Y_1}^\theta$. The proof is concluded by standard duality arguments, using (2.4) and (2.5),

$$\|R_h\|_{Y_0}^2 = \rho(R_h) = (\rho - t_h)(R_h) \leq \|R_h\|_{Y_0} \cdot \|\rho - t_h\|_{Y_0^*} \quad (t_h \in T_h). \quad \square$$

This abstract estimate can be made precise, e.g., Theorem 3 implies Theorem 1 for Symm's integral equation (cf. §5). Since other examples are also covered by the given framework, we state the efficiency result in a related abstract form.

The following concepts (i)–(iii) are essential where we consider a family of Galerkin methods described by a family of discrete subspaces $(S_h : h \in I)$ and $(T_h : h \in I)$ of X_1 and Y_0^*, respectively.

(i) *Approximation property.* Assume that the solution $u \in X_\theta$ of (2.3) also belongs to X_1. Then, for each $h \in I$, let

$$E(u, S_h) := \inf\{\|u - v_h\|_{X_1} : v_h \in S_h\} = \|u - \Pi_h u\|_{X_1}$$

be the best approximation error in the norm of X_1 and let $\Pi_h : X_1 \to S_h$ denote a projection such that $\Pi_h u$ is the best approximate in S_h. Assume $E(u, S_h) > 0$ (provided the Galerkin method converges quasi–optimally, otherwise we obtain $u_h = u$ and this is the exceptional case we are not interested in) and define

$$F(u, S_h) := \frac{\|u - \Pi_h u\|_{X_0}}{\|u - \Pi_h u\|_{X_1}}.$$

(ii) *Inverse assumption.* For each $h \in I$ let

$$G(S_h) := \sup\left\{\frac{\|v_h\|_{X_1}}{\|v_h\|_{X_0}} : v_h \in S_h \setminus \{0\}\right\}.$$

(iii) *Stability.* For each $h \in I$ let

$$P_h : X_0 \to S_h \quad \text{be a projection}$$

such that $P_h \in L(X_0, X_0)$ and $P_h|_{X_1} \in L(X_1, X_1)$ with norms

$$\|P_h\|_j := \sup\left\{\frac{\|P_h v\|_{X_j}}{\|v\|_{X_j}} : v \in X_j \setminus \{0\}\right\}, \qquad j = 0, 1.$$

Then, for $h, H \in I$, define

$$\delta(u, S_h, S_H) := \frac{E(u, S_H)}{E(u, S_h)}$$

$$(2.7) \qquad \qquad \cdot \left(1 + c_{\theta,X} \cdot \|P_H\|_1^\theta \cdot [F(u, S_H) G(S_H) \|P_H\|_0]^{1-\theta}\right).$$

Remark 1. For Symm's equation we have $X_0 = H^{-1}(\Gamma)$, $X_1 = L^2(\Gamma) = Y_0$, $Y_1 = H^1(\Gamma)$, and, as we shall see below, by standard arguments, $E(u, S_h) \simeq F(u, S_h) \simeq h$, $G(S_h) \simeq h^{-1}$. The projection P_h will satisfy $\| P_h \|_0, \| P_h \|_1 \leq C$. In this case, if $H = \epsilon h$,

$$\delta(u, S_h, S_H) \simeq C\epsilon,$$

which is smaller than 1 for sufficiently small ϵ.

Theorem 4. *Let $A \in L(X_1, Y_1)$, $h, H \in I$ with $S_h \subseteq S_H$; consider $u, u_h \in X_1$, $e_h := u - u_h$, $R_h := Ae_h \in Y_1$, and assume* (i)—(iii). *If $\delta(u, S_h, S_H) < 1$, then*

(2.8)

$$\| R_h \|_{Y_1} \leq \frac{\| P_H \|_0^{1-\theta} \cdot \| P_H \|_1^{\theta}}{1 - \delta(u, S_h, S_H)} \cdot \| A \|_{L(X_1; Y_1)} \cdot G(S_H)^{1-\theta} \cdot \| e_h \|_{X_\theta}.$$

The proof of Theorem 4 is given in §4.

Remark 2. The complementary character of the two complementary inequalities of Theorems 3 and 4 should be emphasized: No assumption on the test functions or the residual (neither (2.4) nor bijectivity of $A : X_\theta \to Y_\theta$) is required in Theorem 4, which are essential in Theorem 3. Conversely, the approximation, inverse, and stability properties of the trial spaces, the main ingredients in Theorem 4, play no role in Theorem 3.

3. SPECIALIZED SETTING

Since $E(u, S_h)$, $F(u, S_h)$ and $G(S_h)$ can be bounded if S_h are spline function spaces, we state in this section a more transparent form of the abstract estimate of Theorem 4. Let $(S_h : h \in I)$ be a family of subspaces of X, where the index h is a positive parameter, say, $I \subset (0, 1)$. Suppose $S_h \subset S_H$ for all $h, H \in I$ with $H < h$ and that $\bigcup_{h \in I} S_h$ is dense in X_1. Suppose that there exist positive constants c_α, c_β, c_p and real constants α, β such that for all $h \in I$

(3.1) $$F(u, S_h) \leq c_\alpha \cdot h^\alpha,$$

(3.2) $$G(S_h) \leq c_\beta \cdot h^\beta,$$

(3.3) $$\| P_h \|_0^{1-\theta} \cdot \| P_h \|_1^{\theta} \leq c_p.$$

Corollary 1. *Assume* (3.1)–(3.3) *and $\alpha + \beta \geq 0$. Define*

$$c_1 := 2 + 2c_{\theta,X} \cdot c_p \cdot c_\alpha^{1-\theta} \cdot c_\beta^{1-\theta},$$

$$c_2 := 2c_p \cdot \| A \|_{L(X_1; Y_1)} \cdot c_\beta^{1-\theta}.$$

Then, for each $h \in I$, we can find $H \in I$ with

(3.4) $$E(u, S_H) \leq \frac{1}{c_1} E(u, S_h) \quad and \quad H < h,$$

and we have $\| R_h \|_{Y_1} \leq c_2 \cdot H^{\beta(1-\theta)} \cdot \| e_h \|_{X_\theta}$.

Proof. Use the above notation in Theorem 2 to see that (3.4) implies $\delta(u, S_h, S_H) \leq 1/2$. Then, (2.8) proves the corollary. □

As chosen in Corollary 1, H depends highly on h and we need more information on $E(u, S_h)$ to control this in (3.4).

Corollary 2. *In addition to the assumptions of Corollary 1 let there exist constants η, q, h_0, $0 < \eta, q < 1$, such that for all $h \in I$ with $h < h_0$ we have*

$$(3.5) \qquad E(u, S_{\eta \cdot h}) \leq q \cdot E(u, S_h) \quad and \quad \eta \cdot h \in I.$$

Then, there exists $c_0 > 0$ such that for all $h \in I$ with $h < h_0$

$$(3.6) \qquad \| R_h \|_{Y_1} \leq c_0 \cdot h^{\beta(1-\theta)} \cdot \| e_h \|_{X_\theta}.$$

Proof. Use (3.5) and $H = \eta^k \cdot h$ in Corollary 1 with a natural number $k \geq -\log(c_1)/\log(q)$. $\qquad\qquad\square$

Remark 3. The assumptions on $F(u, S_h)$, $G(S_h)$ as well as an upper bound of $E(u, S_h)$, like $E(u, S_h) \leq c \cdot h^\gamma$, are standard tools in finite element and boundary element analysis (see, e.g., [19]). The assumption on the stability (concerning c_p) is sometimes used implicitly. The only additional assumption is some kind of lower bound of the best approximation error $E(u, S_h)$, e.g., $c' \cdot h^\gamma \leq E(u, S_h)$, such that (3.5) holds: $E(u, S_H) \leq c \cdot H^\gamma \leq c \cdot \eta^\gamma h^\gamma \leq c \cdot \eta^\gamma/c' E(u, S_h)$.

4. PROOF OF THEOREM 4

We need two lemmas to convey the assumptions to the interpolation spaces.

Lemma 1. *For any $v_h \in S_h$, $h \in I$, we have*

$$\| v_h \|_{X_1} \leq \| P_h \|_0^{1-\theta} \| P_h \|_1^\theta \cdot G(S_h)^{1-\theta} \cdot \| v_h \|_{X_\theta}.$$

Proof. By interpolation of the projection P_h as a mapping in $L(X_j, X_1)$ for $j = 0, 1$ we get as in (2.2) (where the image X_1 is fixed) that $P_h \in L(X_\theta, X_1)$ with

$$\| P_h \|_{L(X_\theta; X_1)} \leq \| P_h \|_{L(X_0; X_1)}^{1-\theta} \cdot \| P_h \|_1^\theta.$$

By the inverse assumption (ii), $\| P_h \|_{L(X_0; X_1)} \leq \| P_h \|_0 \cdot G(S_h)$. An application of the bound of $\| P_h \|_{L(X_\theta; X_1)}$ (to $v_h = P_h v_h$) proves the lemma. $\qquad\square$

Lemma 2. *For any $h \in I$ we have*

$$\| u - \Pi_h u \|_{X_\theta} \leq c_{\theta, X} \cdot F(u, S_h)^{1-\theta} \cdot \| u - \Pi_h u \|_{X_1}.$$

Proof. Combine the approximation property with (2.1). $\qquad\qquad\square$

Proof of Theorem 4. Let $h, H \in I$ with $S_h \subset S_H$ and define $w_H := \Pi_H u$. By continuity of $A = A_1 : X_1 \to Y_1$ we obtain

$$(4.1) \qquad \| R_h \|_{Y_1} \leq \| A \|_{L(X_1; Y_1)} \cdot \| e_h \|_{X_1}$$

and then focus on the error $e_h = u - u_h$. By the triangle inequality and Lemma 1 (with $v_H = w_H - u_h \in S_H$) we obtain

$$(4.2)$$
$$\| e_h \|_{X_1} \leq E(u, S_H) + \| P_H \|_0^{1-\theta} \| P_H \|_1^\theta \cdot G(S_H)^{1-\theta} \cdot \| w_H - u_h \|_{X_\theta}.$$

By the triangle inequality and Lemma 2,

$$(4.3) \qquad \| w_H - u_h \|_{X_\theta} \leq c_{\theta, X} \cdot F(u, S_H)^{1-\theta} \cdot E(u, S_H) + \| e_h \|_{X_\theta}.$$

Combining (4.2), (4.3) and

$$E(u, S_H) \leq \frac{E(u, S_H)}{E(u, S_h)} \cdot \| e_h \|_{X_1},$$

one infers

(4.4)
$$\| e_h \|_{X_1} \le \delta(u, S_h, S_H) \cdot \| e_h \|_{X_1}$$
$$+ \ \| P_H \|_0^{1-\theta} \cdot \| P_H \|_1^{\theta} \cdot G(S_H)^{1-\theta} \cdot \| e_h \|_{X_\theta}.$$

Now, (2.8) follows from (4.1) and (4.4). □

5. SYMM'S INTEGRAL EQUATION

We use the notation from §1 and recall that Γ is a polygon in $\mathbb{R}^2$. The Sobolev space $H^s(\mathbb{R}^2)$ is defined as usual [21] and

$$
\begin{aligned}
H^s(\Gamma) &:= \{u|_\Gamma \in L^2(\Gamma) : u \in H^{1/2+s}(\mathbb{R}^2)\}, \qquad s > 0, \\
H^0(\Gamma) &:= L^2(\Gamma), \\
H^s(\Gamma) &:= H^{-s}(\Gamma)^*, \qquad s < 0, \\
H^s_0(\Gamma) &:= \{u \in H^s(\Gamma) : \int_\Gamma u \, ds = 0\}, \\
L^2_0(\Gamma) &:= H^0_0(\Gamma),
\end{aligned}
$$

where $H^s(\Gamma)^*$ is the dual of $H^{-s}(\Gamma)$ with respect to the extended inner product in $L^2(\Gamma)$. Further, $Y_\theta = H^\theta(\Gamma)$ is the interpolation space of $Y_0 = L^2(\Gamma)$ and $Y_1 = H^1(\Gamma)$ and $X_\theta = H^{\theta-1}(\Gamma)$ is the interpolation space of $X_0 = H^{-1}(\Gamma)$ and $X_1 = L^2(\Gamma)$, $0 \le \theta \le 1$, $c_{\theta,X} = c_{\theta,Y} = 1$.

It is also known that $A := V$ is a pseudodifferential operator and is a linear and bounded mapping between $H^{s-1}(\Gamma)$ and $H^s(\Gamma)$ for any $s \in [0,1]$ [9]. Throughout this paper, we assume $\mathrm{cap}(\Gamma) < 1$, so that V defines a positive definite bilinear form on $H^{-1/2}(\Gamma)^2$ [25, 26, 27]. Here, $\mathrm{cap}(\Gamma)$ is the capacity (or conformal radius or transfinite diameter) of Γ and $\mathrm{cap}(\Gamma) < 1$ is satisfied if, for example, Ω is included in a disc with radius < 1, which can be arranged by scaling. Moreover, by the regularity results in [9], $V : L^2(\Gamma) \to H^1(\Gamma)$ is bijective. Hence, V satisfies the assumptions on A in §2, and we obtain in particular Theorem 1.

Proof of Theorem 1. The assumptions of Theorem 3 are satisfied in the present case and $R_h = \rho$ is easily verified. Hence, with Theorem 3 and some standard estimation of the approximation error $R_h - \Pi_h R_h$, one proves (1.5) (cf. [4]). □

In the next steps we prove the estimates described in §3, where it is sufficient that f in (1.1) is continuous and $f|_\omega$ belongs to $H^3(\omega)$ for each side ω of the polygon Γ. Then, $\phi = V^{-1}(f) \in L^2(\Gamma)$ and we are interested in its $L^2(\Gamma)$–best approximation error

(5.1)
$$E(\phi, S_h) = \min\{\| \phi - \psi_h \|_{L^2(\Gamma)} : \psi_h \in S^0_\pi(\Gamma)\} = \| \phi - \Pi_h \phi \|_{L^2(\Gamma)};$$

here, Π_h is the orthogonal projection onto $S_h = S^0_\pi(\Gamma)$ in $L^2(\Gamma)$. If ϕ is constant, then $E(\phi, S_h) = 0$ and $R_h = 0$ indicates $\phi = \phi_h$. Thus, we may, and will, assume ϕ to be nonconstant in the following.

Proposition 1. *Provided ϕ is not constant, there exist positive constants γ, h_0, c_γ, and c'_γ (depending only on Γ, f and c_u in (1.7)) such that $0 < \gamma \le 1$ and either*

(5.2)
$$c'_\gamma \le \max h(\pi)^{-\gamma} \cdot E(\phi, S_h) \le c_\gamma,$$

or $\gamma = 1/2$ and

(5.3) $c'_\gamma \leq -\max h(\pi)^{-1/2} \cdot \log^{-1/2}(\max h(\pi)) \cdot E(\phi, S_h) \leq c_\gamma,$

holds for all meshes π with $\max h(\pi) < h_0$ and satisfying (1.7).

Proof. As it is proved in [10], the solution ϕ of (1.1) has the form

(5.4) $$\phi(x) = \phi_0(x) + \sum_{j=1}^{m} c_j \cdot \phi_j(x), \qquad x \in \Gamma,$$

where $\phi_0 \in H^2(\Gamma)$ and the real constants c_j depend on f, but the singular functions ϕ_j depend on the corners of the polygon only. Here, ϕ_j is of the form

$$\phi_j(x) = r^{\beta_j} \cdot \chi_j(x) \quad \text{or} \quad \phi_j(x) = r^{\beta_j} \cdot \log(r) \cdot \chi_j(x),$$

where r is the Euclidean distance between x and the nearest corner of Γ. Furthermore, χ_j is a smooth cutoff function with a small compact support which is one on a neighborhood of some corner of Γ. The exponent β_j is determined by the interior angle ω_j at the related corner: $\beta_j + 1 = k_j \pi / \omega_j$ for some nonnegative integer k_j. Since $0 < \omega_j < 2\pi$, $\omega_j \neq \pi$, we conclude $\beta_j > -1/2$. Moreover, the second case $\phi_j(x) = r^{\beta_j} \cdot \log r$ appears if $\beta_j + 1 = k_j \pi / \omega_j$ is integer, whence only if $\beta_j \geq 1$.

We split the proof into several steps considering $\phi(x) = r^\beta$, $\phi(x) = r^\beta \log r$, $\phi(x) \in H^2(\Gamma)$ first before we come back to the general case. We analyze the approximation properties locally, so ϕ is approximated on the real unit interval by $S_h^0(0,1)$ on a quasi–uniform mesh on $(0,1)$ described by a partition $0 = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = 1$. Define $m_j = h_j^{-1} \cdot \int_{x_j}^{x_{j+1}} \phi(x)\, dx = \Pi_h \phi|_{(x_j, x_{j+1})}$ and $h_j := x_{j+1} - x_j$ for $j = 0, \ldots, n$.

Case 1. Let $\phi(x) = x^\beta$ for $x \in (0,1)$, and $-1/2 < \beta < 1/2$, $\beta \neq 0$. Let $0 \leq a < a + h \leq 1$, Summary: " and consider the error $\| \phi - m \|_{L^2(a, a+h)}$, where ϕ is approximated by the constant $m = h^{-1} \cdot \int_a^{a+h} \phi(x)\, dx$. Some calculations show

(5.5) $\| \phi - m \|^2_{L^2(a, a+h)} = a^{2\beta+1} \cdot \eta(h/a) \qquad (a > 0),$

(5.6) $\| \phi - m \|^2_{L^2(0, h)} = \dfrac{\beta^2}{(1+\beta)^2(2\beta+1)} \cdot h^{2\beta+1} \qquad (a = 0),$

where

$$\eta(\delta) := \frac{(1+\delta)^{2\beta+1} - 1}{2\beta+1} - \frac{[(1+\delta)^\beta - 1]^2}{\delta(1+\beta)^2} \qquad (\delta > 0).$$

A power series expansion of η in δ shows

(5.7) $\eta(\delta) = c_1 \cdot \delta^3 + \text{ higher-order terms in } \delta$

with a positive constant c_1 (depending only on $\beta > -1/2$). Note that, the first exponent in (5.7) is expected to be three (because of linear convergence when a constant approximates a smooth function). Moreover, one can conclude from (5.7) and (5.5) that

(5.8) $c_2 \leq \eta(\delta) \cdot \delta^{-3} \leq c_3 \qquad \text{for all } \delta \in (0, c_u]$

with constants c_j depending on β and c_u.

According to (5.5) and (5.6),

$$(5.9) \quad \| \phi - \Pi_h \phi \|_{L^2(\Gamma)}^2 = h_0^{2\beta+1} \cdot \Big(\frac{\beta^2}{(1+\beta)^2(2\beta+1)}$$
$$+ \sum_{j=1}^{n} (x_j/h_0)^{2\beta+1} \cdot \eta(h_j/x_j) \Big).$$

Since $2 - 2\beta > 1$, we have with Riemann's Zeta function ζ

$$\sum_{j=1}^{n} j^{2\beta-2} \leq \zeta(2 - 2\beta) = \sum_{j=1}^{\infty} j^{2\beta-2}.$$

Using this, (5.8), (5.9), and (1.7), one finally obtains (5.2) for $\phi(x) = x^\beta$ and $S_h = S_h^0(0,1)$, $\gamma = \beta + 1/2$.

Case 2. Let $\phi(x) = x^\beta$ for $x \in (0,1)$ as before but now $\beta = 1/2$. The above calculations remain true for $\beta = 1/2$, in particular (5.8) and (5.9) hold, but the zeta function has a pole at 1. Using the well–known formula

$$\sum_{j=1}^{n} j^{-1} = \log n + C + O(1/n) \qquad (n \to \infty)$$

involving Euler's constant $C = 0.577...$ instead, the above arguments prove (5.3) for $\phi(x) = x^{1/2}$ and $S_h = S_h^0(0,1)$.

Case 3. Let $\phi \in H^1(0,1)$, e.g., $\phi(x) = x^\beta \log x$ for $\beta \geq 1$. Then, the estimate $\max h(\pi)^{-1} \cdot \| \phi - \Pi_h \phi \|_{L^2(\Gamma)} \leq c_4$ is well known. Conversely, let $\phi \in H^2(\omega)$ for at least a compact subinterval ω of one side of Γ where ϕ is not constant. Then we have a compact nonempty subinterval $[a,b]$ where $\phi' \in C([a,b])$ attains values between c_5 and c_6 with $c_5 \cdot c_6 > 0$. As it is easily seen,

$$(5.10) \qquad c_7 \cdot \min h(\pi) \leq \| \phi - \Pi_h \phi \|_{L^2([a,b])} \leq \| \phi - \Pi_h \phi \|_{L^2(\Gamma)}$$

holds in this case for some positive constant c_7. This proves (5.2) for $\phi = \phi_0$, for $\phi(x) = x \cdot \log x$, and also for $\phi(x) = x^\beta$ with $\beta > 1/2$, $S_h = S_h^0(0,1)$.

So far we proved for each summand in (5.4) an inequality

$$(5.11)$$
$$c_j' \leq -\max h(\pi)^{-\beta_j} \cdot \log(\max h(\pi))^{-\beta_j'} \cdot \| \phi_j - \Pi_h \phi_j \|_{L^2(\Gamma)} \leq c_j$$

with positive $c_j, c_j', \beta_j, \beta_j'$ being independent of h.

The triangle inequality and (5.11) prove (5.2) or (5.3) for $\beta := \min\{\beta_0, \ldots, \beta_m\} > 0$. The proof of the upper bound is immediate while the proof of the lower bound uses that the terms with exponent β are dominant and that $\beta = \beta_j$ is possible at most once at each corner. We omit the details.   □

Remark 4. Upper bounds for the best approximation error $E(\phi, S_h)$ are well known for spline functions. Indeed, regularity of ϕ can be measured in the convergence rate of $E(\phi, S_h)$ (see, e.g., [13, p. 358]):

$$E(\phi, S_h) = O(\max h(\pi)) \quad \text{if and only if} \quad \phi \in H^1(\Gamma)$$

provided we have a uniform mesh (i.e., $c_u = 1$). In particular, this explains that (5.2) is false for $\phi(x) = x^{1/2}$.

Remark 5. A model example is studied in [18] where, in particular, $E(\phi, S_h)$ is computed with $\phi(x) = x^\beta$ and S_h consists of (possibly discontinues) piecewise polynomials of degree p at most on a uniform mesh of $(0,1)$ with meshsize h. In particular it is proved (cf. arguments in [18, p. 603] and [18, Part 2, Corollary 2.1]) that the limits of

$$h^{-p} \cdot E(\phi, S_h), \quad -h^{-p} \cdot \sqrt{\log(h)} \cdot E(\phi, S_h) \quad \text{and} \quad h^{-\beta-1/2} \cdot E(\phi, S_h)$$

exist as positive real numbers as $h \to 0^+$ for $p < \beta - 1/2$, $p = \beta - 1/2$ and $p > \beta - 1/2$, respectively. This confirms Cases 1 and 2 in the proof of Proposition 1 for $p = 0$ for a uniform mesh ($c_u = 1$) and extends the arguments to splines with polynomials of arbitrary degree.

Proposition 2. *There holds* $F(\phi, S_h) \le \max h(\pi)$.

Proof. Note that Π_h is the L^2–projection onto $S_\pi^0(\Gamma)$. Let $\sup_\eta$ denote the supremum among all nonzero η in $H^1(\Gamma)$. Then we have

$$\| \phi - \Pi_h \phi \|_{H^{-1}(\Gamma)} = \sup_\eta \frac{\int_\Gamma (\phi - \Pi_h \phi)(\eta - \pi_h \eta)\, ds}{\| \eta \|_{H^1(\Gamma)}}$$
$$\le \max h(\pi) \cdot \| \phi - \Pi_h \phi \|_{L^2(\Gamma)},$$

using $\| \eta - \Pi_h \eta \|_{L^2(\Gamma)} \le \max h(\pi) \cdot \| \eta' \|_{L^2(\Gamma)}$ for all $\eta \in H^1(\Gamma)$. $\qquad\square$

Let prime or $\frac{\partial}{\partial s}$ denote the derivative along the straight sides of the polygon Γ with respect to the arclength. Conversely, given $f \in L_0^2(\Gamma)$, let $I(f)$ be defined by integrating f along Γ with respect to the arclength. This defines $I(f)$ up to an additive constant on each side of Γ, which is chosen such that $I(f)$ is continuous at each corner. So far, $I(f) \in H^1(\Gamma)$ is defined up to an additive constant, which is fixed by $\int_\Gamma I(f)\, ds = 0$. This defines $I : L_0^2(\Gamma) \to H_0^1(\Gamma)$.

Lemma 3 ([4, Lemma 3]). *For all* $s \in [0,1]$, *I is an isomorphism between* $H_0^{s-1}(\Gamma)$ *and* $H_0^s(\Gamma)$ *with* $\frac{\partial}{\partial s} I(f) = f$ *for all* $f \in H_0^{s-1}(\Gamma)$.

Besides $S_\pi^0(\Gamma)$, define

$$(5.12) \qquad\qquad S_\pi^1(\Gamma) := \{w_h \in H^1(\Gamma) : w_h' \in S_\pi^0(\Gamma)\},$$

the linear space of continuous and piecewise linear functions with respect to a mesh π.

The following result shows $G(S_h) \le C/h$.

Proposition 3. *There exists a constant* $c > 0$ *such that for all meshes* π *with* $\max h(\pi) < 1$

$$(5.13) \quad \| w_h \|_{H^1(\Gamma)} \le c \cdot \min h(\pi)^{-1} \cdot \| w_h \|_{L^2(\Gamma)} \quad \text{for all} \quad w_h \in S_\pi^1(\Gamma),$$

$$(5.14) \quad \| \psi_h \|_{L^2(\Gamma)} \le c \cdot \min h(\pi)^{-1} \cdot \| \psi_h \|_{H^{-1}(\Gamma)} \quad \text{for all} \quad \psi_h \in S_\pi^0(\Gamma).$$

Proof. The inverse inequality (5.13) is well known and easily proved by direct calculations on each element. According to Lemma 3, $I : S_\pi^0(\Gamma) \cap L_0^2(\Gamma) \to S_\pi^1(\Gamma) \cap L_0^2(\Gamma)$ is an isomorphism, and (5.14) follows essentially from (5.13). $\qquad\square$

For each $\psi \in H^{-1}(\Gamma)$ define $P_h \psi \in S_\pi^0(\Gamma)$ by

$$P_h \psi := \psi^0 + \frac{\partial}{\partial s} \Pi_h^1 I(\psi - \psi^0) \quad \text{for} \quad \psi^0 := |\Gamma|^{-1} \int_\Gamma \psi\, ds$$

where Π_h^1 is the L^2–projection onto $S_\pi^1(\Gamma)$ (see [6] for a similar construction).

Proposition 4. *The operator P_h is a projection onto $S_\pi^0(\Gamma)$ which is bounded as a mapping between $H^{-1}(\Gamma)$ and $H^{-1}(\Gamma)$ or between $L^2(\Gamma)$ and $L^2(\Gamma)$.*

Proof. It is known that Π_h^1 is continuous as a mapping between $L^2(\Gamma)$ and $L^2(\Gamma)$ and between $H^1(\Gamma)$ and $H^1(\Gamma)$ [12]. According to this and Lemma 3, P_h is linear and bounded as a mapping between $H^{-1}(\Gamma)$ and $H^{-1}(\Gamma)$ and between $L^2(\Gamma)$ and $L^2(\Gamma)$. Finally, direct calculations show $P_h \psi_h = \psi_h$ for each $\psi_h \in S_\pi^0$, i.e., P_h is a projection onto $S_\pi^0(\Gamma)$. □

Proof of Theorem 2. Using the above notation and that of §3 the Propositions 2–4 show (3.1)–(3.3) where, as usual, the index parameter h is identified with max $h(\pi)$ for a mesh π satisfying (1.7).

Given a mesh π, let $\eta = 1/k$ for an integer k and define a new mesh by dividing each element Γ_j of π in k pieces of length $|\Gamma_j|/k$. Then, the new mesh satisfies (1.7) as π does. Moreover, according to Proposition 1, we obtain (3.5) with some q which depends on k, c_γ, c_γ', and γ. A simple calculation, indicated in Remark 3, shows that we obtain $0 < q < 1$ by choosing k large enough. Note that η depends only on c_γ, c_γ' and γ. Therefore, Corollary 2 leads to Theorem 2. □

Remark 6. Note that a solution u of (1.3) satisfies $\phi = \frac{\partial u}{\partial n} \in H_0^{-1/2}(\Gamma)$. Hence, we might be interested in determining $\phi \in H_0^{-1/2}(\Gamma)$ with

$$(5.15) \qquad \int_\Gamma (V\phi - f)\psi \, ds = 0 \qquad \text{for all} \quad \psi \in H_0^{-1/2}(\Gamma).$$

The corresponding numerical method is: *Find $\phi_h \in S_h := S_\pi^0(\Gamma) \cap L_0^2(\Gamma)$ with*

$$(5.16) \qquad \int_\Gamma (V\phi_h - f)\psi_h \, ds = 0 \qquad \text{for all} \quad \psi_h \in S_h.$$

Equations (5.15) and (5.16) have unique solutions according to Lax–Milgram's lemma. In this case, one also obtains Theorems 1 and 2 from Theorems 3 and 4.

6. Hypersingular integral equation

The Neumann problem for the Laplacian

$$\Delta u = 0 \quad \text{in } \Omega \quad , \qquad \frac{\partial u}{\partial n} = t_0 \quad \text{on } \Gamma,$$

is related to the hypersingular integral equation

$$(6.1) \qquad\qquad Wv(x) = f(x) \quad (x \in \Gamma).$$

Here, f can be computed from the Neumann data t_0 and $v = u|_\Gamma$ is the trace of u. The hypersingular operator

$$(6.2) \qquad Wv(x) := -\frac{1}{\pi} \frac{\partial}{\partial n_x} \int_\Gamma v(y) \frac{\partial}{\partial n_y} \log|x - y| ds_y$$

is linear, bounded, symmetric and a Fredholm operator of index zero [9]. Since $W = -\frac{\partial}{\partial s} V \frac{\partial}{\partial s}$ [22], W is positive definite between $H_0^{1/2}(\Gamma)$ and $H_0^{-1/2}(\Gamma)$. Furthermore, $A = W$ is a continuous mapping between $X_1 = H_0^1(\Gamma)$ and $Y_1 = L_0^2(\Gamma)$ as well as between $X_0 = L_0^2(\Gamma)$ and $Y_0 = H_0^{-1}(\Gamma)$. Let $L_0^2(\Gamma)$ be endowed with the usual norm in $L^2(\Gamma)$, but endow $H_0^1(\Gamma)$ with the equivalent $H^1(\Gamma)$–seminorm.

Fix $f \in L_0^2(\Gamma)$ and the solution $v \in H_0^1(\Gamma)$ of (6.1). Let $T_h = S_h = S_\pi^1(\Gamma) \cap H_0^1(\Gamma)$ as defined in (5.12) for a partition π of the polygon Γ with (1.7). Let $v_h \in S_h$ denote the Galerkin solution, i.e.,

$$R_h := f - W v_h \perp S_\pi^1(\Gamma),$$

where $\perp$ means orthogonality in $L^2(\Gamma)$.

Theorem 5 ([4]). *There exists a constant $c > 0$ (depending only on Γ) such that for any $s \in [0,1]$ there holds*

(6.3) $$\| v - v_h \|_{H^s(\Gamma)} \leq c \cdot \| R_h \|_{L^2(\Gamma)}^s \| h \cdot R_h \|_{L^2(\Gamma)}^{1-s}.$$

For quasi–uniform meshes, (6.3) is

(6.4) $$\| v - v_h \|_{H^s(\Gamma)} \leq c \cdot \max h(\pi)^{1-s} \| R_h \|_{L^2(\Gamma)}.$$

As Theorem 5 follows from Theorem 3, we infer its converse from Theorem 4.

Theorem 6. *Assume that f is smooth on each side of the polygon. Then there exist constants $c_0, h_0 > 0$ (depending only on Γ, f and c_u) such that for all partitions π of Γ with $\max h(\pi) < h_0$ and (1.7) and for $s \in [0,1]$, there holds*

(6.5) $$\max h(\pi)^{1-s} \cdot \| R_h \|_{L^2(\Gamma)} \leq c_0 \cdot \| v - v_h \|_{H^s(\Gamma)}.$$

Proof. Let Π_h^j denote the L^2–projection onto $S_\pi^j(\Gamma)$ for $j = 0, 1$. Since we endowed $H_0^1(\Gamma)$ with the norm

$$\| w \|_{H_0^1(\Gamma)} = \left(\int_\Gamma |w'(s)|^2 \, ds \right)^{1/2} \qquad (w \in H_0^1(\Gamma)),$$

the relation $E(v, S_h) = \| v - w_h \|_{H_0^1(\Gamma)}$ is equivalent to $v' - w_h' \perp S_\pi^0(\Gamma)$ for each $w_h \in S_h^1(\Gamma)$. Thus $w_h' = \Pi_h^0 v'$ and $E(v, S_h) = \| v' - \Pi_h^0 v' \|_{L^2(\Gamma)}$. Since the derivative v' of the solution u of the Neumann problem, $v = u|_\Gamma$, has a similar decomposition as ϕ in (5.4), this and the results of §5 verify the assumptions of Corollary 2. We omit the details.   □

Remark 7. A second proof of Theorem 6 is possible using the close relation between V and W and a modification of Theorem 2. Indeed, $\phi := v'$ and $\phi_h := v_h'$ satisfy $V\phi = F := I(f)$ and

$$\int_\Gamma V(\phi_h) \psi_h \, ds = \int_\Gamma F \psi_h \, ds \quad \text{for all } \psi_h \in S_\pi^0(\Gamma) \cap L_0^2(\Gamma).$$

Thus, $\phi \in L_0^2(\Gamma)$ and $\phi_h \in L_0^2(\Gamma)$ are solutions of problem (5.15) and (5.16), respectively. Then, arguing as in Remark 6, we can obtain Theorem 6 from Theorem 2.

7. INTEGRAL EQUATION FOR A TRANSMISSION PROBLEM

In the transmission problem we seek harmonic functions u_1 and u_2 in an interior and exterior domain Ω and $\mathbb{R}^2 \setminus \Omega$, respectively, with a prescribed jump (f, g) of their Cauchy data at the polygonal boundary $\Gamma = \partial\Omega$ (see, e.g., [11, 6] for details). This transmission problem is equivalently related to a boundary integral equation

(7.1) $$H \binom{v}{\phi} = \frac{1}{2}(1 + H)\binom{f}{g}$$

[11], where $(f, g) \in H^1(\Gamma) \times L^2(\Gamma)$ is given and

$$H := \begin{pmatrix} -K & V \\ W & K' \end{pmatrix} : H^s(\Gamma) \times H^{s-1}(\Gamma) \to H^s(\Gamma) \times H^{s-1}(\Gamma).$$

Here, V and W are given in (1.2) and (6.2), respectively, and the double-layer potential and its adjoint are defined by

$$Kv(x) \quad := \quad -\frac{1}{\pi} \int_\Gamma v(y) \frac{\partial}{\partial n_y} \log |x - y| \, ds_y,$$

$$K'\phi(x) \quad := \quad -\frac{1}{\pi} \int_\Gamma \phi(y) \frac{\partial}{\partial n_x} \log |x - y| \, ds_y.$$

The operator H is linear, bounded and bijective for $s \in [0, 1]$ and (7.1) has a unique solution $(v, \phi) \in H^1(\Gamma) \times L^2(\Gamma)$ (see [11, 6]). Define $S_h := S_\pi^1(\Gamma) \times S_\pi^0(\Gamma)$ and $T_h := S_\pi^0(\Gamma) \times S_\pi^1(\Gamma)$ for a partition π of the polygon Γ. The Galerkin BEM for (7.1) reads: *Find $(v_h, \phi_h) \in S_h$ such that the residual*

$$(R_1, R_2) := \frac{1}{2}(1 + H)\binom{f}{g} - H\binom{v_h}{\phi_h} \in H^1(\Gamma) \times L^2(\Gamma)$$

satisfies $0 = \int_\Gamma (R_1 \cdot \psi_h + R_2 \cdot w_h) \, ds$ for all $(\psi_h, w_h) \in T_h$.

Theorem 7 ([4])**.** *There exists a constant $C > 0$ such that, for $0 \le s \le 1$,*

$$\left\| \binom{v}{\phi} - \binom{v_h}{\phi_h} \right\|_{H^s(\Gamma) \times H^{s-1}(\Gamma)} \le C \cdot \left\| \binom{R_1'}{R_2} \right\|_{L^2(\Gamma)^2}^{1-s} \cdot \left\| h(\pi) \cdot \binom{R_1'}{R_2} \right\|_{L^2(\Gamma)^2}^s.$$

While Theorem 7 is a consequence of Theorem 3, its converse estimate is concluded from the complementary results presented in §§2 and 3.

Theorem 8. *If f and g are smooth, there exist constants $c_0, h_0 > 0$ (depending only on Γ, f, g and c_u) such that for all partitions π of Γ with $\max h(\pi) < h_0$ and (1.7), and for $s \in [0, 1]$, there holds*

$$\max h(\pi)^s \left\| \binom{R_1'}{R_2} \right\|_{L^2(\Gamma)^2} \le c_0 \cdot \left\| \binom{v}{\phi} - \binom{v_h}{\phi_h} \right\|_{H^s(\Gamma) \times H^{s-1}(\Gamma)}.$$

Proof. It can be checked that the assumptions of Corollary 2 are satisfied as in the previous cases. One combines arguments of §§5 and 6 for ϕ and v and uses regularity results from [11]. We omit the details. ☐

8. COMMENTS

8.1. Efficiency. Once the Galerkin solution $\phi_h \in S_\pi^0(\Gamma)$ is known, compute

$$a_j^2 = \int_{\Gamma_j} [(V\phi_h)'(x) - f'(x)]^2 \, ds_x,$$

where, e.g., $V\phi_h(x)$ is calculated analytically and the integral along Γ_j is approximated numerically; $h_j = |\Gamma_j|$. For $s = 1/2$, the upper bound in Theorem 1 is $C \cdot B_1$, where C is a constant and

$$B_1 := \left(a^2 \cdot \sum_{j=1}^n a_j^2 \cdot h_j^2 \right)^{1/4}, \qquad a^2 := \sum_{j=1}^n a_j^2.$$

Since $\int_\Gamma V(\cdot)(\cdot) \, ds$ defines an inner product which induces an equivalent norm in $H^{-1/2}(\Gamma)$, the quantity

$$\gamma_h := B_1^{-1} \cdot \left(\int_\Gamma V(\phi - \phi_h)(\phi - \phi_h) \, ds \right)^{1/2}$$

is called *efficiency quotient*. We say that an error estimate is *efficient* if its efficiency quotient γ_h satisfies $0 < c_0 \le \gamma_h \le c_1 < \infty$ for all discretizations under consideration. Using this concept we proved in this paper that the a posteriori estimates obtained in [4] are efficient for BEM on quasi–uniform meshes (in two dimensions) and gave a general guideline in §§2 and 3 on how to analyze this in other examples.

8.2. **Numerical tests.** The efficiency quotients γ_h are computed for various numerical examples reported in [5, 6, 7, 8]. From this, the efficiency results of this paper are confirmed; one observes efficiency in practice.

8.3. **Comparison with another estimate.** Another a posteriori error estimate is proved in [5] which bounds the error (in the $H^{-1/2}(\Gamma)$–norm) from above by $C \cdot B_2$, where C is a constant and

$$B_2 := \sum_{j=1}^{n} h_j^{1/2} \cdot a_j$$

(notation as in §8.1). Numerical computations reported in [5] show a slow decrease of the related efficiency quotient as n tends to infinity, even on quasi–uniform meshes. This "loss of efficiency" is indicated by the following example. Imagine that the residual is uniformly distributed on a uniform mesh π, i.e., suppose $a_1 = \cdots = a_n$ and $h_1 = \cdots = h_n$. Then, $B_1 \cdot \sqrt{n} = B_2$. According to the efficiency of B_1, this shows that B_2 overestimates the error by a factor $\sqrt{n}$, whence B_2 is not efficient in this model situation. However, if singularities occur, the estimate in [5] could be efficient.

8.4. **Restrictions on the mesh.** The condition of a quasi–uniform discretization (1.7) is used several times in the proofs. For an arbitrary mesh we have (1.6) and, according to (2.8) and (5.13),

$$\min h(\pi)^s \cdot \| R' \|_{L^2(\Gamma)} \le c_0 \cdot \| \phi - \phi_h \|_{H^{-s}(\Gamma)}$$

in Theorem 2 (assuming that δ is small). Hence, the methods of this paper used for the proof of efficiency give the impression that one needs (1.7). One way of overcoming this difficulty is to take local properties of the operator A into account. These appear to be unavailable in the literature for the operators studied here on polygons.

8.5. **Further applications.** The results in §§5–7 are stated for the simplest discretization only. It is known that Propositions 2–4 also hold for spline functions of higher degree, and this is expected for an (adapted version of) Proposition 1 as well (cf. Remark 5). Indeed, the essential additional condition is (3.5), which may be proven in many other situations where u is not an arbitrary function in X_1 but, e.g., is related to a solution of a partial differential equation and then inherits much more regularity. Therefore, it is expected that (3.5) is true for a larger class of discretizations, including problems in three dimensions, and is not restricted to solutions which are related to the Laplace equation.

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