TWO NEW POINTS IN THE SPECTRUM
OF THE ABSOLUTE MAHLER MEASURE
OF TOTALLY POSITIVE ALGEBRAIC INTEGERS

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ABSTRACT. For totally positive algebraic integers $\alpha \neq 0, 1$ of degree $d(\alpha)$, we consider the set $\mathcal{L}$ of values of $M(\alpha)^{\frac{1}{d(\alpha)}} = \Omega(\alpha)$, where $M(\alpha)$ is the Mahler measure of α. C. J. Smyth has found the four smallest values of $\mathcal{L}$ and conjectured that the fifth point is $\Omega((2\cos\frac{2\pi}{60})^2)$. We prove that this is so and, moreover, we give the sixth point of $\mathcal{L}$.

1. INTRODUCTION

Let $P = a_0 x^d + \cdots + a_d = a_0(x - \alpha_1) \cdots (x - \alpha_d)$, $a_0 a_d \neq 0$, be a polynomial of degree $d \geq 1$ with complex coefficients.

The *relative Mahler measure* of P is given by $M(P) = |a_0| \prod_{i=1}^{d} \max(1, |\alpha_i|)$.

We denote by $\Omega(P) = M(P)^{\frac{1}{\deg(P)}}$ the *absolute Mahler measure* of P and by $\mathcal{L}$ the set of $\Omega(P)$ where P is a monic polynomial with integer coefficients, different from x, $x - 1$, irreducible over $\mathbb{Q}$ and totally positive (i.e., all its roots are real and positive). The *absolute Mahler measure* of a nonzero algebraic integer α is the *absolute Mahler measure* of its minimal polynomial in $\mathbb{Z}[x]$.

We define $\alpha_n = 2\cos(2\pi/n)$ and the real numbers $(\beta_n)_{n \geq 0}$ by

$$\begin{cases} \beta_0 = 1, \\ \beta_{n+1} > 0 \text{ and } H\beta_{n+1} = \beta_n \quad \text{for } n \geq 0, \end{cases}$$

where

$$Hx = x - \frac{1}{x}.$$

In 1981, C.J. Smyth [4] found the four smallest elements of $\mathcal{L}$, which are

$$\begin{array}{ll} \Omega(\beta_1^2) \approx 1.618033, & \Omega(\beta_2^2) \approx 1.685389, \\ \Omega(\beta_3^2) \approx 1.710197, & \Omega(\alpha_7^2) \approx 1.715534. \end{array}$$

He had conjectured that the fifth point of $\mathcal{L}$ was $\Omega(\alpha_{60}^2) \approx 1.719388$.

In this paper, we show

Theorem. *Each monic polynomial P of degree d, with integer coefficients and totally positive satisfies*

$$\Omega(P) \geq 1.720566 \cdot \prod_{j=0}^{9} |\text{resultant}(P, R_j)|^{\frac{c_j}{d}}.$$

In particular, if P is not divisible by any of the R_j, $0 \leq j \leq 9$, then

$$\Omega(P) \geq 1.720566.$$

Here,

$R_0(x) = x - 1$,
$R_1(x) = x^2 - 3x + 1$,
$R_2(x) = x^4 - 7x^3 + 13x^2 - 7x + 1$,
$R_3(x) = x^8 - 15x^7 + 83x^6 - 220x^5 + 303x^4 - 220x^3 + 83x^2 - 15x + 1$,
$R_4(x) = x^{16} - 31x^{15} + 413x^{14} - 3141x^{13} + 15261x^{12} - 50187x^{11} + 115410x^{10}$
$\quad -189036x^9 + 222621x^8 - 189036x^7 + 115410x^6 - 50187x^5 + 15261x^4 - 3141x^3$
$\quad +413x^2 - 31x + 1$,
$R_5(x) = x^4 - 8x^3 + 15x^2 - 8x + 1$,
$R_6(x) = x$,
$R_7 = (x^3 - 5x^2 + 6x - 1).(x^3 - 6x^2 + 5x - 1)$,
$R_8 = (x^4 - 7x^3 + 14x^2 - 8x + 1).(x^4 - 8x^3 + 14x^2 - 7x + 1)$,
$R_9 = x^6 - 12x^5 + 44x^4 - 67x^3 + 44x^2 - 12x + 1$

and

$$
\begin{array}{llll}
c_0 = & 0.32506177, & c_1 = & 0.04536243, \\
c_2 = & 0.01688943, & c_3 = & 0.00569441, \\
c_4 = & 0.00195846, & c_5 = & 0.00118560, \\
c_6 = & 0.20333607, & c_7 = & 0.00292360, \\
c_8 = & 0.00069160, & c_9 = & 0.00087933.
\end{array}
$$

Note that α_7^2 and α_7^{-2} are roots of R_7 and α_{60}^2 and α_{60}^{-2} are roots of R_8. Note also that $H^3(\alpha_7) = -\alpha_7$ and $H^4(\alpha_{60}) = \alpha_{60}$.

Corollary. *The first six elements of $\mathcal{L}$ are*

$$\Omega(\beta_1^2), \ \Omega(\beta_2^2), \ \Omega(\beta_3^2), \ \Omega(\alpha_7^2), \ \Omega(\alpha_{60}^2), \ \Omega(\beta_4^2).$$

These are the only points of $\mathcal{L}$ in the interval $(1, 1.720556]$.

Remark. C.J. Smyth [3] showed that $\mathcal{L}$ is everywhere dense in (l, ∞), where $l \approx 1.7273$ is the first accumulation point known.

The proof follows Smyth's [4]. In the second paragraph, we recall briefly the methods. In paragraph 3, we will describe the arguments which permit us to get a general result on $\Omega(P)$ improving Smyth's theorem.

2. PRINCIPLE OF AUXILIARY FUNCTIONS

For $x > 0$, put $\log_+(x) = \max(0, \log(x))$. For $n \geq 1$, let $c = (c_j)_{1 \leq j \leq n}$ be nonnegative real numbers and R_j, $0 \leq j \leq n$, polynomials in $\mathbb{Z}[x]$.

For $x > 0$, we consider the auxiliary function

$$f(x, c) = \log_+(x) - \sum_{j=0}^{n} c_j \log |R_j(x)|$$

and $m(c) = \min_{x>0} f(x, c)$.

The polynomial P with integer coefficients, which is irreducible over $\mathbb{Q}$, has positive zeros $\alpha_1, \ldots, \alpha_d$ and is not dividing any of the R_j, satisfies

$$|\text{resultant}(P, R_j)| = \prod_{i=1}^{d} |R_j(\alpha_i)| \geq 1, \qquad j = 0, \ldots, n,$$

which implies

$$\log_+(\alpha_i) \geq m(c), \qquad i = 1, \ldots, d,$$

and therefore,

$$\Omega(P) \geq e^{m(c)}.$$

We then have to solve the following optimization problem: determine

$$m = \max_c m(c) = \max_{(c_j)} \min_{x>0} f(x, c).$$

A semi-infinite linear programming method gives good numerical values for the c_j. For more details, one can refer to [4] or [2].

3. NUMERICAL RESULTS

The proof of the theorem is based on four main points : first, a better choice of polynomials in the auxiliary function, and then three numerical improvements in the semi-infinite linear programming methods.

3.1. The auxiliary function.
C.J. Smyth [3] has used a transformation T which is essential in his paper : to a totally positive polynomial P of degree d, one associates the reciprocal polynomial $T(P)$ (i.e., $T(P)(x) = x^{deg(T(P))} \cdot T(P)(\frac{1}{x})$) of degree $2d$ and also totally positive defined by

$$T(P)(x) = x^d \cdot P(x + \frac{1}{x} - 2).$$

We define the nth *descendant* of a polynomial P the polynomial $T^n(P)$ defined by

$$T^n(P)(x) = x^{\deg(T^{n-1}(P))} \cdot T^{n-1}(P)(x + \frac{1}{x} - 2).$$

The minimal polynomial P_n of β_n^2 is the nth descendant of $P_0(x) = x - 1$.

The R_j, $j = 0, \ldots, 5$, belong to two particular families of polynomials. The first consists of iterates by T of $P_0 = x - 1$. Smyth [4] already used $R_j = P_j = T^j(P_0)$, $j = 0, \ldots, 3$. We add $R_4 = P_4$, of degree 16. The second family appears here for the first time. It is composed of iterates by T of $Q_0 = x - 2$. We need

$$R_5 = T^2(Q_0) = x^4 - 8x^3 + 15x^2 - 8x + 1.$$

Although this new family occurs only with one of its polynomials, this one is absolutely necessary to get the result.

Remark. Let $G = H^2$ be defined for $x > 0$ by

$$Gx = x + \frac{1}{x} - 2.$$

The fixed point of G is the root of the polynomial $Q_0' = 1 - 2x$ whose reciprocal polynomial is Q_0.

Note that R_9 is used here for the first time and is the first descendant of the polynomial $x^3 - 6x^2 + 5x - 1$.

For $x > 0$ such that $R_j(x) \neq 0$, $0 \leq j \leq 9$, the function $f(x, c)$ becomes

$$f(x, c) = \log_+(x) - \sum_{j=0}^{9} c_j \log |R_j(x)|.$$

As it is composed of x, $x - 1$ and reciprocal polynomials, we can use the remark of [2, §2] to impose $f(x) = f(1/x)$ and limit the study of f to the interval $(0, 1)$.

This condition implies that :

$$2c_6 + \sum_{j=0, j \neq 6}^{9} c_j \cdot \deg(R_j) = 1.$$

3.2. Numerical improvements.
Semi-infinite linear programming is an iterative method. At each step, we have to :

(a) determine $\min_{x>0} f(x, c)$ for a particular family of coefficients $c = (c_j)$;

(b) increase the set of points on which the function is controlled;

(c) evaluate $\log |P_j(x_i)|$ at control points x_i.

For (a), we use a computational algorithm (AMOEBA [5]) giving the minimum of a function without calculating its derivative. This technique reduces the risk of computational errors that can occur when the numerator of the derivative has high degree (here, 55).

For (b), there are two common procedures: either use a method similar to the Remes algorithm [1], or add systematically all points where the derivative f' of f vanishes. We choose an intermediate method consisting in adding points where not only $f' = 0$ but also f is small. This avoids saturation and guarantees fast convergence. For example, at the ith iteration, for a particular family of coefficients c, we decide to remove points $y > 0$ not satisfying

$$f(y, c) < \min_{x>0} f(x, c) + \epsilon$$

for suitable $\epsilon > 0$.

(c) Let x_0 be a control point of f. We consider the polynomials P_1, P_2, P_3, P_4 derived from P_0. Instead of evaluating directly $\log |P_j(x_0)|$, $j = 1, \ldots, 4$, we take advantage of the definition of these P_j. Remember that $P_j = T^j(P_0)$. By recurrence, we can easily show that

$$\log |P_j(x_0)| = \sum_{k=1}^{j} \log |x_{j-k}| + \log |x_j - 1|$$

with

$$x_{k+1} = x_k + \frac{1}{x_k} - 2.$$

This procedure is easy to use and permits to limit the computational errors when the degree of polynomials becomes high.

Using semi-infinite linear programming, after 15 iterations we obtain

$$1.72056628 \le e^m \le 1.7205664.$$

But the accuracy of the calculations is not guaranteed, especially for the zeros of f' whose numerator has degree 55. They have been checked by the formal computational system PARI with great precision and we so find the constant 1.720566 announced in the theorem.

4. CONCLUSION

All the above results seem to show that the elements of $\mathcal{L}$ in the interval $\left(\frac{1+\sqrt{5}}{2}, l \right)$ are only of two forms : absolute measures of β_n^2 or, as suggests C.J.Smyth, of fixed points of iterates of the transformation T. But we believe that this conjecture cannot be resolved with the methods used in this paper.

ACKNOWLEDGMENTS

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