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A CAPITULATION PROBLEM AND GREENBERG'S CONJECTURE ON REAL QUADRATIC FIELDS

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ABSTRACT. We give a sufficient condition in order that an ideal of a real quadratic field F capitulates in the cyclotomic $\mathbb{Z}_3$-extension of F by using a unit of an intermediate field. Moreover, we give new examples of F's for which Greenberg's conjecture holds by calculating units of fields of degree 6, 18, 54 and 162.

1. INTRODUCTION

Let p be a prime number, F a totally real number field, F_∞ the cyclotomic $\mathbb{Z}_p$-extension of F and F_n the nth layer of F_∞/F. Let A_n be the p-part of the ideal class group of F_n. In [1], Greenberg showed the following:

Proposition . *Assume that only one prime of F lies over p and that this prime is totally ramified in F_∞/F. Then the following two statements are equivalent.*

(1) *Every ideal class of A_0 becomes trivial in A_n for some n.*
(2) *The order of A_n is bounded as $n \to \infty$.*

In this paper, we treat the case that F is a real quadratic field and $p = 3$. In §2 we give a sufficient condition for (1) by using a unit in F_n. In §3 we give a method of finding the above unit.

2. THEOREM

We put $\zeta_{3^n} = e^{2\pi\sqrt{-1}/3^n}$ for a positive integer n. Our main purpose of this section is to prove the following theorem which plays a fundamental role in the next section.

Theorem . *Let F be a real quadratic field. Let $F_n = F(\zeta_{3^{n+1}}) \cap \mathbb{R}$, $G(F_n/\mathbb{Q}) = \langle\sigma\rangle$ the Galois group F_n over $\mathbb{Q}$, ε a fundamental unit of F and A_n the 3-part of the ideal class group of F_n. We assume that 3 divides the class number h_F of F and that 3 does not split in $F/\mathbb{Q}$. If there exists a unit η of F_n such that $\eta^{1+\sigma}$ is a cube of an element of F_n and that neither η nor $\eta\varepsilon$ nor $\eta\varepsilon^2$ is a cube of an element of F_n, then the natural mapping of A_0 to A_n is not injective.*

Let $F_n^* = F(\zeta_{3^{n+1}})$ and F' be the imaginary quadratic field contained in F_0^* such that $F' \cap \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}$. Let M be the maximal abelian 3-extension of F_0^* unramified outside 3, $X = G(M/F')$ and ρ the complex conjugation. We put

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$X^+ = \{ x \in X \mid \rho^{-1} x \rho = x \}$. Let M^- be the intermediate field between F_0^* and M corresponding to X^+. For a real number α, we denote by $\sqrt[3]{\alpha}$ the real number whose cube is α. Even though the following Lemma 2.1 is well known, for completeness we give a proof.

Lemma 2.1. *Let α be an element of F. If $F_0^*(\sqrt[3]{\alpha}) \subset M$, then $F_0^*(\sqrt[3]{\alpha}) \subset M^-$.*

Proof. Let σ be an element of X^+ with $\sqrt[3]{\alpha}^{\sigma} = \sqrt[3]{\alpha}\zeta$, where ζ is a cubic root of unity. Then we have $\sqrt[3]{\alpha}^{\rho\sigma\rho^{-1}} = (\sqrt[3]{\alpha}\zeta)^{\rho^{-1}} = \sqrt[3]{\alpha}\zeta^{-1} = \sqrt[3]{\alpha}^{\sigma} = \sqrt[3]{\alpha}\zeta$. Hence we have $\zeta = 1$. This shows $\sqrt[3]{\alpha} \in M^-$. □

For an ideal $\mathfrak{A}$ of F, we denote by $\bar{\mathfrak{A}}$ the ideal class of F which contains $\mathfrak{A}$. Let $\bar{\mathfrak{A}}_1, \dots, \bar{\mathfrak{A}}_r$ be a basis of $\{ a \in A_0 \mid a^3 = 1 \}$, $\mathfrak{A}_i^3 = (\alpha_i)$ and k the intermediate field between F_0^* and M corresponding to $X^3 = \{ x^3 \mid x \in X \}$. Then under the assumption that 3 does not split in $F/\mathbb{Q}$ we have by Lemma 2.1 the following result.

Lemma 2.2 (cf. [1, p. 281]). *Let k^- be the field $k \cap M^-$. Then we have $k^- = F_0^*(\sqrt[3]{3}, \sqrt[3]{\varepsilon}, \sqrt[3]{\alpha_1}, \dots, \sqrt[3]{\alpha_r})$.*

The following is well known (cf. [1, p. 280]):

Lemma 2.3. *Let σ be a generator of the Galois group $G(F_n^*/F')$ and α be a non-zero element of F_n^* such that there exists an element β with $\alpha^{\sigma} = \alpha^{-1}\beta^3$. Then $F_n^*(\sqrt[3]{\alpha})$ is an abelian extension of F'.*

Proof of the Theorem. Since $\eta^{1-\sigma^2} = (\eta^{1+\sigma})^{1-\sigma}$, there exists an element β of F_n with $\eta^{1-\sigma^2} = \beta^3$. Hence we have $N_{F_n/F_0}(\beta^3) = 1$, which means $N_{F_n/F_0}(\beta) = 1$. Hence there exists an element γ of F_n with $\beta = \gamma^{1-\sigma^2}$, which shows $\eta\gamma^{-3} \in F_0$. This shows $F_n^*(\sqrt[3]{\eta}) = F_n^*(\sqrt[3]{\eta\gamma^{-3}}) = F_n^* F_0^*(\sqrt[3]{\eta\gamma^{-3}})$. Since $F_n^*(\sqrt[3]{\eta})$ is an abelian 3-extension of F_0^* unramified outside 3 by Lemma 2.3 and since $\eta\gamma^{-3} \in F_0^*$, we have $F_0^*(\sqrt[3]{\eta\gamma^{-3}}) \subset k^- = F_0^*(\sqrt[3]{3}, \sqrt[3]{\varepsilon}, \sqrt[3]{\alpha_1}, \dots, \sqrt[3]{\alpha_r})$ by Lemmas 2.1 and 2.2. Hence there exist integers $n_1, n_2, \dots, n_r, n$ and an element δ of F_0 with $\eta\gamma^{-3} = \alpha_1^{n_1} \cdots \alpha_r^{n_r} \varepsilon^n \delta^3$ by Lemma 2.2. This shows by the assumption on η that $\mathfrak{A}_1^{n_1} \cdots \mathfrak{A}_r^{n_r}$ is not principal in F_0 but principal in F_n. □

3. Method of finding η

In this section, we explain how to compute and find a unit η in the theorem. Let E_n be the unit group of F_n and $r = 2 \cdot 3^n - 1$. If a basis $\{\varepsilon_1 E_n^3, \dots, \varepsilon_r E_n^3\}$ of E_n/E_n^3 is obtained, without loss of generality, η can be written in the form $\eta = \varepsilon_1^{e_1} \cdots \varepsilon_r^{e_r}$ with $0 \le e_i \le 2$. Therefore, we can decide whether or not such an η exists by examining all the combinations of $\{e_1, \dots, e_r\}$. If $n = 1$, we can obtain fundamental units of F_1 (cf. [3]) and can use this direct algorithm. But it is hard to obtain a basis of E_n/E_n^3 for $n \ge 2$. So we proceed as follows.

For an element ξ of F_n, we denote ξ^{σ^i} by ξ_i. Let C_n be the cyclotomic unit group of F_n. First we assume that there exists an element $\xi \in C_n$ such that $C_n = \langle -1, \xi_0, \dots, \xi_{r-1} \rangle$. Moreover, we assume that the 3-Sylow subgroup $(E_n/C_n)_3$ of E_n/C_n is cyclic of order 3^n. Under these assumptions, we determine the form of $\alpha \in E_n$ which satisfies $(E_n/C_n)_3 = \langle \alpha C_n \rangle$ and $\alpha^{1+\sigma} \in E_n^3$. From the assumption $A_0 \ne 1$, there exists $\gamma \in E_0$ such that

$$\gamma^3 = \prod_{i=0}^{3^n-1} \xi_{2i} \,.$$

Assume that $(E_n/C_n)_3 = \langle \alpha C_n \rangle$ and $\alpha^{1+\sigma} = \beta^3$ for some $\beta \in E_n$. Since the order of $(E_n/C_n)_3$ is 3^n, we see that $\alpha^{3^{n-1}} = \gamma u,\ \beta = \alpha^e v$ for some $u, v \in C_n$ and $e \in \mathbb{N}$. Then

$$u^{1+\sigma} = \pm(\alpha^{3^{n-1}})^{1+\sigma} = \pm\beta^{3^n} = \pm\alpha^{e3^n} v^{3^n} \equiv (\gamma u)^{3e} = \prod_{i=0}^{3^n-1} \xi_{2i}^{e} u^{3e} \pmod{C_n^{3^n}}.$$

We write $u = \xi_0^{e_0} \cdots \xi_{r-1}^{e_{r-1}}$ with $e_i \in \mathbb{Z}$ and substitute this in both sides of the above congruence relation. Since $\xi_r = \pm(\xi_0 \cdots \xi_{r-1})^{-1}$, we obtain the following system of simultaneous equations:

$$e_{i-1} + e_i - e_{r-1} \equiv \begin{cases} e + 3ee_i & \text{if } i \text{ is even,} \\ 3ee_i & \text{if } i \text{ is odd.} \end{cases}$$

Here the congruence is modulo 3^n and $e_{-1} = 0$. This equation is easily solved. In fact, if we put $x = e_{r-1}$ and $y = e$, then we can represent all e_i by x and y. Now, we fix x to be 0 and vary y from 0 to $3^n - 1$. If we find that γu is contained in $E_n^{3^{n-1}}$ for some y, then we put $\eta = (\gamma u)^{1/3^{n-1}}$. It is easy to check whether η, $\eta\varepsilon$ or $\eta\varepsilon^2$ is a cube in E_n.

A Galois generator ξ of C_n is hard to find. But we know the cyclotomic unit of Hasse (cf. [2]) which generates a fairly large subgroup of C_n. So, we execute the above procedure by letting ξ to be Hasse's unit. We will be able to find η by this method with some luck.

4. Examples

Let $F = \mathbb{Q}(\sqrt{m})$ where m is a positive square-free integer congruent to 2 modulo 3. There are 207 m's less than 10000 which satisfy $|A_0| = 3$. We denote $\mathrm{Ker}(A_0 \longrightarrow A_n)$ by H_n. We used a computer to implement the above method for these F's and fortunately found η and conclude that $H_n \neq 1$ for many F's. We show the results of our computation in Table 1 (next page). The proposition in §1 implies that if $m \equiv 2 (\mathrm{mod}\ 3)$, $|A_0| = 3$, and $H_n \neq 1$ for some $n \geq 1$, then the order of A_n is bounded, namely, Greenberg's conjecture is valid for F, and the Iwasawa invariant $\lambda_3(F)$ is zero. A question mark in the table means that we do not know the value. For example, we got $|H_1| = 1$ when $m = 899$ (cf. the remark below). So we searched $\eta \in F_2$ with the method of §3 but could not find it. We cannot conclude whether $|H_2|$ is 1 or 3. Next we pursued a calculation in F_3 and found $\eta \in F_3$. Therefore $|H_3| = 3$ and $\lambda_3(F) = 0$.

Remark . Since $|H_1| = (E_0 : N_{F_1/F_0}(E_1))$, we can obtain the exact value of $|H_1|$ by computing E_1 (cf. [3]). We note that $|H_1| = 1$ for all m's in Table 1 for which we could not find $\eta \in E_1$.

TABLE 1. All m's satisfying $m \equiv 2 \pmod 3$ and $|A_0| = 3$ ($m < 10000$)

m	$\lvert H_1\rvert$	$\lvert H_2\rvert$	$\lvert H_3\rvert$	$\lvert H_4\rvert$	$\lambda_3(F)$	m	$\lvert H_1\rvert$	$\lvert H_2\rvert$	$\lvert H_3\rvert$	$\lvert H_4\rvert$	$\lambda_3(F)$
254	1	?	?	?	?	3221	3	3	3	3	0
257	3	3	3	3	0	3281	3	3	3	3	0
326	3	3	3	3	0	3287	3	3	3	3	0
359	3	3	3	3	0	3305	1	?	?	?	?
443	1	3	3	3	0	3419	3	3	3	3	0
473	1	?	?	?	?	3422	1	3	3	3	0
506	3	3	3	3	0	3482	3	3	3	3	0
659	3	3	3	3	0	3569	1	?	3	3	0
761	3	3	3	3	0	3590	3	3	3	3	0
785	1	?	3	3	0	3602	3	3	3	3	0
839	3	3	3	3	0	3803	3	3	3	3	0
842	3	3	3	3	0	3941	3	3	3	3	0
899	1	?	3	3	0	3962	3	3	3	3	0
1091	3	3	3	3	0	4001	3	3	3	3	0
1211	3	3	3	3	0	4094	3	3	3	3	0
1223	3	3	3	3	0	4106	3	3	3	3	0
1229	3	3	3	3	0	4151	3	3	3	3	0
1367	3	3	3	3	0	4193	3	3	3	3	0
1373	3	3	3	3	0	4238	1	3	3	3	0
1406	3	3	3	3	0	4283	3	3	3	3	0
1478	3	3	3	3	0	4286	1	?	3	3	0
1523	3	3	3	3	0	4355	3	3	3	3	0
1646	1	?	?	?	?	4367	3	3	3	3	0
1787	3	3	3	3	0	4481	1	3	3	3	0
1811	1	3	3	3	0	4493	3	3	3	3	0
1847	3	3	3	3	0	4511	1	3	3	3	0
1901	3	3	3	3	0	4649	3	3	3	3	0
1907	3	3	3	3	0	4670	3	3	3	3	0
1937	1	?	?	?	?	4706	3	3	3	3	0
2021	1	?	3	3	0	4778	3	3	3	3	0
2099	1	3	3	3	0	4841	3	3	3	3	0
2177	3	3	3	3	0	4853	3	3	3	3	0
2207	3	3	3	3	0	4886	3	3	3	3	0
2213	3	3	3	3	0	4907	1	3	3	3	0
2429	1	?	3	3	0	4910	3	3	3	3	0
2459	3	3	3	3	0	4934	3	3	3	3	0
2495	3	3	3	3	0	4970	3	3	3	3	0
2510	1	?	3	3	0	4982	3	3	3	3	0
2543	3	3	3	3	0	4994	3	3	3	3	0
2666	1	?	?	3	0	5042	3	3	3	3	0
2678	1	3	3	3	0	5063	1	?	?	?	?
2711	3	3	3	3	0	5081	1	?	?	3	0
2726	3	3	3	3	0	5099	3	3	3	3	0
2777	1	3	3	3	0	5102	3	3	3	3	0
2831	3	3	3	3	0	5255	3	3	3	3	0
2894	3	3	3	3	0	5261	3	3	3	3	0
2918	1	?	3	3	0	5297	1	?	?	3	0
2981	3	3	3	3	0	5303	3	3	3	3	0
2993	3	3	3	3	0	5327	3	3	3	3	0
3023	3	3	3	3	0	5333	3	3	3	3	0
3035	3	3	3	3	0	5369	3	3	3	3	0
3047	1	?	?	3	0	5477	3	3	3	3	0
3062	3	3	3	3	0	5621	3	3	3	3	0
3071	3	3	3	3	0	5738	3	3	3	3	0
3158	1	?	3	3	0	5741	3	3	3	3	0
3173	3	3	3	3	0	5798	3	3	3	3	0

TABLE 1 (continued)

m	$\|H_1\|$	$\|H_2\|$	$\|H_3\|$	$\|H_4\|$	$\lambda_3(F)$	m	$\|H_1\|$	$\|H_2\|$	$\|H_3\|$	$\|H_4\|$	$\lambda_3(F)$
5903	3	3	3	3	0	8282	1	?	3	3	0
5918	3	3	3	3	0	8285	3	3	3	3	0
5930	3	3	3	3	0	8306	3	3	3	3	0
5954	1	?	3	3	0	8339	1	?	?	3	0
6026	3	3	3	3	0	8363	1	3	3	3	0
6053	3	3	3	3	0	8399	3	3	3	3	0
6185	3	3	3	3	0	8426	3	3	3	3	0
6209	3	3	3	3	0	8438	3	3	3	3	0
6311	3	3	3	3	0	8447	3	3	3	3	0
6401	3	3	3	3	0	8519	3	3	3	3	0
6515	3	3	3	3	0	8543	3	3	3	3	0
6557	3	3	3	3	0	8597	3	3	3	3	0
6623	3	3	3	3	0	8603	3	3	3	3	0
6686	3	3	3	3	0	8711	1	?	?	?	?
6770	3	3	3	3	0	8735	3	3	3	3	0
6782	3	3	3	3	0	8789	3	3	3	3	0
6791	1	3	3	3	0	8837	1	3	3	3	0
6806	1	?	?	?	?	8909	3	3	3	3	0
6887	3	3	3	3	0	8930	3	3	3	3	0
6995	1	?	?	?	?	8999	3	3	3	3	0
7019	3	3	3	3	0	9062	3	3	3	3	0
7055	3	3	3	3	0	9086	3	3	3	3	0
7058	3	3	3	3	0	9149	3	3	3	3	0
7235	3	3	3	3	0	9155	3	3	3	3	0
7259	3	3	3	3	0	9215	3	3	3	3	0
7262	3	3	3	3	0	9218	3	3	3	3	0
7310	3	3	3	3	0	9278	3	3	3	3	0
7319	3	3	3	3	0	9281	3	3	3	3	0
7415	3	3	3	3	0	9293	3	3	3	3	0
7481	3	3	3	3	0	9323	3	3	3	3	0
7598	1	?	3	3	0	9413	3	3	3	3	0
7601	1	?	3	3	0	9419	3	3	3	3	0
7643	1	3	3	3	0	9467	3	3	3	3	0
7655	3	3	3	3	0	9479	3	3	3	3	0
7658	1	?	?	3	0	9551	3	3	3	3	0
7673	3	3	3	3	0	9578	1	3	3	3	0
7694	3	3	3	3	0	9590	1	?	?	3	0
7709	1	3	3	3	0	9659	1	3	3	3	0
7721	3	3	3	3	0	9710	3	3	3	3	0
7745	3	3	3	3	0	9749	3	3	3	3	0
7883	1	3	3	3	0	9830	3	3	3	3	0
7994	3	3	3	3	0	9833	3	3	3	3	0
8051	3	3	3	3	0	9869	3	3	3	3	0
8057	3	3	3	3	0	9902	3	3	3	3	0
8069	1	3	3	3	0	9905	3	3	3	3	0
8255	3	3	3	3	0	9926	1	?	?	3	0
8267	3	3	3	3	0	9995	1	?	3	3	0
8279	1	3	3	3	0						

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