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SIMULTANEOUS PELL EQUATIONS

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ABSTRACT. Let R and S be positive integers with $R < S$. We shall call the simultaneous Diophantine equations

$$x^2 - Ry^2 = 1,$$
$$z^2 - Sy^2 = 1$$

simultaneous Pell equations in R and S. Each such pair has the trivial solution $(1, 0, 1)$ but some pairs have nontrivial solutions too. For example, if $R = 11$ and $S = 56$, then $(199, 60, 449)$ is a solution. Using theorems due to Baker, Davenport, and Waldschmidt, it is possible to show that the number of solutions is always finite, and it is possible to give a complete list of them. In this paper we report on the solutions when $R < S \leq 200$.

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simultaneous Pell equations in R and S. Each such pair has the trivial solution $(1, 0, 1)$ but some pairs have nontrivial solutions too. For example, if $R = 11$ and $S = 56$, then $(199, 60, 449)$ is a solution. Indeed, there are infinitely many simultaneous Pell equations with nontrivial solutions, as can be seen by taking $y = 2$, $R = k^2 + k$, and $S = m^2 + m$. Using theorems due to Siegel [4, §1] and Baker [1], it is possible to show that the number of solutions is always finite, and it is possible to give a complete list of them. This is exactly what we have done, for all 19,900 simultaneous Pell equations with $R < S \leq 200$, and this paper, precisely, is a report on our method and results.

Note that the term 'simultaneous Pell equations' could be defined to apply to other pairs, such as

$$\begin{aligned} x^2 - Ry^2 &= 1, \\ z^2 - Sx^2 &= 1. \end{aligned}$$

There are other variants as well, including the 'simultaneous Pellian equations' solved in an article by R. G. E. Pinch [3] which overlaps this paper to some extent. All these 'simultaneous Pells' can be solved using methods similar to those described here.

Some of the simultaneous Pell equations under consideration in this paper can be solved simply by factoring. This is the case if R or S or RS is a square. In the

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latter case, if $RS = U^2$, we have

$$(Sx - Uz)(Sx + Uz) = S(S - R),$$

leading to finitely many solutions (since the right-hand side has only finitely many factorizations, each of which leads to a unique pair (x, z)). Indeed, in this case, we can put bounds on x and y as follows:

$$\begin{aligned} 0 &< |x| \leq \frac{S(S-R)+1}{2S} < \frac{S-R+1}{2}, \\ 0 &\leq |y| < \frac{x}{\sqrt{R}}. \end{aligned}$$

When $R < S \leq 200$, there are only two pairs of this sort with nontrivial solutions. When $R = 3$ and $S = 48$, we have a simultaneous solution with $y = 1$. When $R = 2$ and $S = 72$, we have a simultaneous solution with $y = 2$.

Note that if R and S have the same square factor n^2, any solutions can be obtained by solving the simultaneous Pell equations in R/n^2 and S/n^2.

In the general case, we proceed in accordance with the following theory. Let (a, b) be the least positive integer solution of $x^2 - Ry^2 = 1$. There is a standard simple continued fraction method for generating the integers a and b, and we used it, in connection with *Mathematica*, to find a and b for all $R \leq 200$ [2, §7.8]. Let $A = a + b\sqrt{R}$. The largest a for any $R \leq 200$ is

$$a = 2,469,645,423,824,185,801$$

for $R = 181$. Hence, if $R \leq 200$, then $a < 3 \times 10^{18}$. Since $a^2 - Rb^2 = 1$, it follows that $b\sqrt{R} < a$. Hence,

$$A = a + b\sqrt{R} < 6 \times 10^{18}.$$

Note that when $R = 3$, then $A = 2 + \sqrt{3}$, and this is the smallest value obtained by A, for any R.

Similarly, if (a', b') is the least positive integer solution of $x^2 - Sy^2 = 1$, and $A' = a' + b'\sqrt{S}$, then, assuming $S \leq 200$,

$$2 + \sqrt{3} \leq A' < 6 \times 10^{18}.$$

The *Néron height* of any algebraic number is defined as follows. If r is algebraic, there is a unique polynomial $P(x)$ in $\mathbf{Z}[x]$ with positive leading coefficient c and relatively prime coefficients such that r is a root of this polynomial. The roots of this 'minimal polynomial' for r are the 'conjugates' r' of r. The 'measure' $M(r)$ of r is

$$M(r) = c \prod_{\text{conj } r'} \max(1, |r'|),$$

and the *Néron height* of r is

$$\frac{\ln M(r)}{\deg r},$$

where $\deg r$ is the degree of $P(x)$. For example, the number $A = a + b\sqrt{R}$ has minimal polynomial $x^2 - 2ax + 1$ and measure A. Its Néron height H is thus $\frac{1}{2}\ln A < 22$. Similarly, the Néron height H' of A' is < 22, and, finally, if $E = \sqrt{S/R}$, then $H'' =$ Néron Height$(E) \leq \frac{1}{2}\ln S \leq 3$ (since $S \leq 200$). What is important for what follows is that the six numbers H, H', H'', $\frac{1}{4}\ln A$, $\frac{1}{4}\ln A'$, and $\frac{1}{4}\ln E$ are all bounded by $V = 22$.

Now all the nonnegative solutions of $x^2 - Ry^2 = 1$ are given by

$$x = u_m = \frac{A^m + A^{-m}}{2},$$

$$y = v_m = \frac{A^m - A^{-m}}{2\sqrt{R}},$$

for $m = 0, 1, 2, \ldots$. All the nonnegative solutions of $z^2 - Sy^2 = 1$ are given by

$$z = w_n = \frac{A'^n + A'^{-n}}{2},$$

$$y = t_n = \frac{A'^n - A'^{-n}}{2\sqrt{S}},$$

for $n = 0, 1, 2, \ldots$. To solve the simultaneous Pell equations, it suffices to find all (m, n) such that $v_m = t_n$. When $m = 0$ or $n = 0$, then $y = 0$, and we have only trivial solutions. When m and n are positive, but at least one of them is < 3, we have 407 nontrivial solutions. It is a routine matter to find them.

In order to find other nontrivial solutions, or to prove that we have not overlooked any nontrivial solutions, the following theorems are important.

Theorem 1. *Suppose none of the positive integers R, S, RS is a square. If $K = 2 + \sqrt{3}$ and m, $n > 2$ are such that $v_m = t_n$, then*

$$0 < |m \ln A - n \ln A' + \ln E| < K^{-\max(m,n)} \leq \frac{1}{K^m},$$

where $E = \sqrt{S/R}$.

Proof. If $\ln EA^mA'^{-n} = 0$, then

$$\sqrt{S}(a + b\sqrt{R})^m = \sqrt{R}(a' + b'\sqrt{S})^n,$$

so that, for some nonzero integers c, d, e, and f,

$$\sqrt{S}(c + d\sqrt{R}) = \sqrt{R}(e + f\sqrt{S})$$

and

$$\sqrt{S}(c + (d - f)\sqrt{R}) = e\sqrt{R}.$$

When we square this, we see that $d = f$ (since $\sqrt{R}$ is irrational) and hence $Sc^2 = Re^2$. Since c and e are nonzero, this means that RS is a square — contrary to assumption.

For the other inequalities, it suffices to show that

$$|EA^mA'^{-n} - 1| < \frac{1}{2}K^{-\max(m,n)}$$

— since the slope of the log function is < 2 when $x > 1/2$ and hence

$$|x - 1| < \epsilon \Longrightarrow |\ln x| < 2\epsilon.$$

Since $v_m = t_n$, we have

$$\frac{A^m - A^{-m}}{\sqrt{R}} = \frac{A'^n - A'^{-n}}{\sqrt{S}},$$

so that

$$E(A^m - A^{-m}) = A'^n - A'^{-n},$$

$$EA^m A'^{-n} - EA^{-m}A'^{-n} = 1 - A'^{-2n},$$

$$EA^m A'^{-n} - 1 = EA^{-m}A'^{-n} - A'^{-2n}.$$

If this is positive, then

$$|EA^m A'^{-n} - 1| < EA^{-m}A'^{-n} < EK^{-m-n} < \frac{1}{2}K^{-\max(m,n)}$$

with the last inequality following from the fact that $2E < 20 < K^3 \le K^{\min(m,n)}$.
And we get the same result if $EA^m A'^{-n} - 1$ is negative. □

Theorem 2 (Waldschmidt). *Let A, A', E be nonzero, nonnegative algebraic numbers, each of degree at least 2. Suppose 4 is the degree of $\mathbf{Q}(A, A', E)$ over $\mathbf{Q}$. Let m and n be integers > 2. Let $L = |m \ln A - n \ln A' + \ln E| > 0$. Let V be a positive integer greater than each of the Néron Heights of A, A', and E, and also greater than each of the absolute values of their natural logs divided by 4. Let $W = \max(\ln m, \ln n)$ (so that $e^W = \max(m,n)$). Then*

$$L > \exp(-2^{81}V^3(W + \ln(4eV))\ln(4eV)).$$

Proof. See [5]. □

In our case (with $R < S \le 200$), we can take $V = 22$ (see above), so that, by Theorem 1, if m and n both exceed 2, we have

$$K^{-\max(m,n)} > L > \exp(-2^{81}22^3(W+6)(6))$$

(since $\ln(88e) = 5.4773$), and hence

$$\exp(2^{81}22^3(W+6)6) > K^{e^W},$$

so that

$$2^{81}22^3(W+6)6 > e^W \ln K$$

and, dividing by $(W+6)\ln K$,

$$1.2 \times 10^{29} > \frac{e^W}{W+6}.$$

When $W > 0$, the function $e^W/(W+6)$ is increasing, so this allows us to conclude that $W < 72$ and hence $\max(m,n) < e^{72} < 10^{33}$.

This impractical bound can in practice be lowered using

Theorem 3 (A special case of Davenport's Lemma). *Let x_1 be a decimal approximation to $(\ln A)/\ln A'$, accurate to 80 decimal places. Let f/g and f'/g' be consecutive simple continued fraction convergents of x_1 such that $g \le 10^{39}$ but $g' > 10^{39}$. Suppose m and n are as above (and hence bounded by 10^{33}). Then, if the distance between*

$$\frac{g \ln E}{\ln A'} = g\frac{\ln S - \ln R}{2 \ln A'}$$

and the nearest integer exceeds $3/10^6$, then $m \le 80$. (If the distance between

$(g \ln E)/\ln A'$ and the nearest integer is less than $3/10^6$, then that nearest integer is $fm - gn$.)

Proof. See [1]. □

Let us call the ordered pair (R, S) *Davenportable* (at level $3/10^6$) if the distance between $(g \ln E)/\ln A'$ and the nearest integer exceeds $3/10^6$. If (R, S) is Davenportable, then m is bounded by 80. If m, $n > 2$, then n is the integer nearest

$$\frac{m \ln A + \ln E}{\ln A'}$$

(see Theorem 1 above) so it is easy to calculate the corresponding n's. From our computations — carried out using *Mathematica* on an IBM XP 486 — we found that there are, in fact, no non-Davenportable pairs with $R < S \leq 200$. Hence for all the simultaneous Pell equations under consideration here, $m \leq 80$.

For each triple (m, A, A') — with $2 < m \leq 80$ — we checked the inequality of Theorem 1,

$$0 < |m \ln A - n \ln A' + \frac{1}{2} \ln S - \frac{1}{2} \ln R| < \frac{1}{(2+\sqrt{3})^m},$$

to see if we might possibly have $v_m = t_n$. If the inequality held, we then calculated v_m and t_n to see if, indeed, they were equal. This, in fact, did not ever occur (with m, $n > 2$), so that we were able to conclude that the only simultaneous Pell equations with nontrivial solutions and $R < S \leq 200$ were the 407 pairs with m or $n \leq 2$ which we had already obtained. In conclusion, we have

Theorem 4. *With R and S in the range up to 200, there are no solutions to the simultaneous Pell equations with both m and $n > 2$. Indeed, the only case in which either of m and n is > 2 is the case with $R = 3$ and $S = 176$, when $m = 3$ and $n = 1$ give a solution. Finally, with R and S in the range up to 200, there are no simultaneous Pell solutions (x, y, z) with $y > 120$.*

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