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A CONTINUOUS SPACE-TIME FINITE ELEMENT METHOD FOR THE WAVE EQUATION

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ABSTRACT. We consider a finite element method for the nonhomogeneous second-order wave equation, which is formulated in terms of continuous approximation functions in both space and time, thereby giving a unified treatment of the spatial and temporal discretizations. Our analysis uses primarily energy arguments, which are quite common for spatial discretizations but not for time.

We present a priori nodal (in time) superconvergence error estimates without any special time step restrictions. Our method is based on tensor-product spaces for the full discretization.

1. INTRODUCTION

The continuous time Galerkin (CTG) method is a finite element technique which provides time discretizations for evolution problems using approximation spaces of continuous functions. This approach is particularly appropriate for wave problems as it retains discrete versions of the important energy conservation properties provided by the initial/boundary value problem being approximated (see [11]). Computations and analyses have shown this is especially useful in the approximation of solutions to nonlinear wave problems (see, for instance, [12], [13], or [18]). Recent work by DeFrutos and Sanz-Serna [7] indicates that the constants in long-time estimates may be smaller for such methods. Another advantage of the CTG approach is that CTG methods of any desired order of accuracy are easily formulated.

The main purpose of this paper is to demonstrate new variational techniques to analyze these high-order accurate space-time finite element methods. We will prove both global convergence and nodal in time superconvergence error estimates. The global error estimates we present have also been obtained by [4] (see also [5]), however, by nonvariational arguments, and in earlier work of the authors [10], but by different techniques, which required a time step restriction. The approximation of the heat equation by CTG methods was studied by Aziz and Monk [1]. Our report complements theirs; however, we note that the stability estimates for the wave equation are more complicated (see §3), and our proof of superconvergence is shorter and, we feel, more straightforward. The techniques we use would also apply to the heat equation.

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We remark that these CTG schemes in the homogeneous case ($f = 0$) are equivalent to Gauss-Legendre implicit Runge-Kutta (IRK) methods (see [11]). In this connection see also [3]. For three other nonclassical finite element treatments of the wave equation, see Babuška and Janik [2], Johnson [14], and Richter [17].

It may seem unusual to use an implicit method for approximation of the wave equation. However, for some wave problems, particularly nonlinear problems where there may be "blowup" or highly singular behavior, there is growing evidence of the advantages of implicit schemes. Bona et al. [6] use the Gauss-Legendre IRK methods to solve the generalized KdV equation efficiently. Strauss and Vazquez [18] note that certain explicit methods fail while an implicit energy-preserving scheme gives a sensible approximation to the generalized Klein-Gordon equation. In addition, the variational formulation of space-time finite element methods seems to facilitate the derivation of a posteriori error estimates, which may serve as the basis for rational adaptive grid refinement (see [14] for an example for the wave equation). Based on these observations, we consider the CTG method a viable approach to many wave problems, and hope that the analysis presented here for the linear wave equation will lay the foundation for future work on the sort of nonlinear problems mentioned above.

The outline of this paper is as follows. In §1 we specify our notations, collect important approximation results, and describe a useful reformulation of the wave problem. Our main estimate will involve the decomposition

$$y - Y = (y - P_x\tilde{y}) + (P_x\tilde{y} - \tilde{y}) + (\tilde{y} - Y) = \rho + \theta + \eta,$$

where y is the partial differential equation solution, Y is the fully discrete approximation, $\tilde{y}$ is a discrete in time and continuous in space approximation, and P_x is projection in the spatial variables. In §3 we present the fundamental arguments in an abstract setting from which the estimates of η for the wave equation will follow. We introduce $\tilde{y}$ and several necessary regularity results in §4, and in §5 we complete the estimate of the error $y - Y$, using the decomposition above and the theorems for $\tilde{y}$. Section 6 has the results of several numerical experiments with the scheme, where we explore the necessity of some of the assumptions on the initial data.

2. Preliminaries

Let Ω be a bounded region in $\mathbf{R}^d$ ($d = 1, 2, 3$) with a smooth boundary $\partial\Omega$, and let $[0, T]$ be a finite time interval. We consider the following initial/boundary value problem: find $U = U(x, t)$ such that

$$\begin{aligned} U_{tt} - \Delta U = f \quad &\text{in } \Omega \times [0, T], \\ U = 0 \quad &\text{on } \partial\Omega \times [0, T], \\ U(\cdot, 0) = U_0 \text{ and } U_t(\cdot, 0) = V_0 \quad &\text{in } \Omega. \end{aligned} \tag{1}$$

Our results easily generalize to the case where $-\Delta$ is replaced by any uniformly elliptic selfadjoint second-order operator which is independent of t; the time-dependent case will be the subject of future work.

For a domain $S \subset R^d$, we will use the Lebesgue spaces $L_2(S)$ and $L_\infty(S)$, and the Sobolev spaces $H^s(S)$ for s a positive integer, all defined in the usual way. We will also use $H_0^1(\Omega)$ and its dual $H^{-1}(\Omega)$. For $H_0^1(\Omega)$, we take the norm to be $\|v\|_{H_0^1(\Omega)} = \|\nabla v\|_{L_2(\Omega)}$. All of these spaces are Hilbert spaces except for $L_\infty(S)$. When $S = \Omega$, we will usually omit Ω from our notation. For functions depending on both space and time variables, given a time interval $[a, b]$ and H any of the above Hilbert spaces, we define the Hilbert space $L_2([a, b], H)$ by

$$\|v\|_{L_2([a,b],H)} = \left(\int_a^b \|v(\cdot, t)\|_H^2 dt \right)^{1/2}.$$

There is an analogous definition for $L_\infty([a, b], H)$. When $[a, b] = [0, T]$, these will be denoted simply by $L_2(H)$ and $L_\infty(H)$. We use C to denote a generic positive constant, not necessarily the same at different occurrences, but always independent of all discretization parameters, solutions, and of T.

We reformulate (2) as a first-order system by introducing the function $V = U_t$. Letting

$$Y = \begin{pmatrix} U \\ V \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad \text{and} \quad A = \begin{bmatrix} 0 & -I \\ -\Delta & 0 \end{bmatrix},$$

we then have

$$\begin{aligned} Y_t + AY &= F \quad \text{in } \Omega \times [0, T], \\ Y &= 0 \quad \text{on } \partial\Omega \times [0, T], \\ Y(\cdot, 0) &= \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} \quad \text{in } \Omega, \end{aligned} \tag{2}$$

where the domain of A is $D(A) = (H^2 \cap H_0^1) \times H_0^1$. It will also be convenient to define a mapping $T : H^{-1} \to H_0^1$ by

$$\begin{aligned} -\Delta(Tv) &= v \quad \text{in } \Omega, \\ Tv &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Finally, we define

$$B = \begin{pmatrix} 0 & T \\ -I & 0 \end{pmatrix},$$

with $D(B) = L_2 \times H^{-1}$. Notice that BA and AB are identities on the appropriate domains.

We next discuss the approximation spaces and their properties. Let S_p^h be a finite-dimensional subspace of H_0^1, depending on a discretization parameter $h > 0$. Define the L_2 projection $\pi_x : L_2 \to S_p^h$ by

$$(\pi_x u, \chi)_{L_2} = (u, \chi)_{L_2} \quad \forall \chi \in S_p^h,$$

and define the H_0^1 projection $P_x : H_0^1 \to S_p^h$ by

$$(\nabla P_x u, \nabla \chi)_{L_2} = (\nabla u, \nabla \chi)_{L_2} \quad \forall \chi \in S_p^h.$$

We assume for S_p^h the following properties:

$$\|u - \pi_x u\|_{L_2} \le C h^r \|u\|_{H^r}, \tag{3}$$

where $u \in H^r \cap H_0^1$ and $0 \le r \le p + 1$; and

$$\|u - P_x u\|_{H^s} \le C h^{r-s} \|u\|_{H^r}, \tag{4}$$

where $u \in H^r \cap H_0^1$, $0 \le s \le r \le p + 1$ and $s = 0, 1$. Define the discrete Laplace operator $-\Delta_h : S_p^h \to S_p^h$ by

$$(-\Delta_h \lambda, \chi)_{L_2} = (\nabla \lambda, \nabla \chi)_{L_2} \quad \forall \chi \in S_p^h,$$

and define $T_h : H^{-1} \to S_p^h$ by

$$(\nabla T_h \lambda, \nabla \chi)_{L_2} = (\lambda, \chi)_{L_2} \quad \forall \chi \in S_p^h.$$

Note that T_h restricted to S_p^h is the inverse of $-\Delta_h$. We also define the following two operators:

$$A_h = \begin{pmatrix} 0 & -I \\ -\Delta_h & 0 \end{pmatrix} \quad \text{and} \quad B_h = \begin{pmatrix} 0 & T_h \\ -I & 0 \end{pmatrix}.$$

Notice that on $S_p^h \times S_p^h$ these are inverses of each other.

Let $[0, T]$ be partitioned by $0 = t_0 < t_1 < \cdots < t_N = T$, and let

$$I_n = [t_{n-1}, t_n], \quad k_n = t_n - t_{n-1}, \quad k = \max\{k_n : 1 \le n \le N\}.$$

For functions ϕ which depend continuously on time, we will often use the notation $\phi_n = \phi(t_n)$. The space of polynomials of degree q on an interval $[a, b]$ is denoted by $P_q([a, b])$. We define S_q^k to be those continuous functions on $[0, T]$ whose restriction to any I_n belongs to $P_q(I_n)$. Define an operator $\pi_t : L^2(I_n) \to P_{q-1}(I_n)$ by the equation

$$(\pi_t u, \chi)_{L_2(I_n)} = (u, \chi)_{L_2(I_n)} \quad \forall \chi \in P_{q-1}(I_n),$$

and also define $P_t : H^1([0, T]) \to S_q^k$ by $P_t u(0) = u(0)$ and

$$(\partial_t (P_t u), \chi_t)_{L_2([0,T])} = (u_t, \chi_t)_{L_2([0,T])} \quad \forall \chi \in S_q^k.$$

Note that

$$P_t u(t_n) = u(t_n), \qquad n = 0, 1, \ldots, N,$$

and that there is no ambiguity if we talk of $P_t : H^1(I_n) \to P_q(I_n)$ (i.e., P_t may be computed locally). Also note that π_t and P_t are projections into different spaces. We have the following approximation properties:

$$\|u - \pi_t u\|_{L_2(I_n)} \le C k_n^r \|\partial_t^r u\|_{L_2(I_n)}, \tag{5}$$

where $u \in H^r(I_n)$ and $0 \le r \le q$; and

$$\|\partial_t^s (u - P_t u)\|_{L_2(I_n)} \le C k_n^{r-s} \|\partial_t^r u\|_{L_2(I_n)}, \tag{6}$$

where $u \in H^r(I_n)$ and $0 \le s \le r \le q+1$, $s = 0, 1$. Any function $\phi \in P_q(I_n)$ satisfies the following inverse properties:

$$\|\phi\|_{L_\infty(I_n)} \le C k_n^{-1/2} \|\phi\|_{L_2(I_n)}, \tag{7}$$

$$\|\phi_t\|_{L_2(I_n)} \le C k_n^{-1} \|\phi\|_{L_2(I_n)}, \tag{8}$$

$$\|\phi\|_{L_2(I_n)} \le C\{k_n^{1/2} |\phi(t_{n-1})| + \|\pi_t \phi\|_{L_2(I_n)}\}. \tag{9}$$

(See Lemma 1, p. 42, in [9] for a proof of (9).)

The space-time domains $Q = \Omega \times [0, T]$ and $S_n = \Omega \times I_n$ will be used in this paper. Our approximate solutions will be defined in the space $S_{pq}^{hk} = S_p^h \otimes S_q^k$. The operators and estimates we have introduced for S_q^k and S_p^h can be extended in obvious ways to the space S_{pq}^{hk}.

We now introduce the approximation scheme. The method is based on the formulation (2), so U, U_t are approximated separately by $u, v \in S_{pq}^{hk}$. These approximations are defined successively on each slab of space-time as follows:

$$(u_t - v, \chi)_{L_2(S_n)} = 0 \quad \forall \chi \in S_p^h \otimes P_{q-1}(I_n), \tag{10}$$

$$(11) \qquad (v_t, \lambda)_{L_2(S_n)} + (\nabla u, \nabla \lambda)_{L_2(S_n)} = (f, \lambda)_{L_2(S_n)} \quad \forall \lambda \in S_p^h \otimes P_{q-1}(I_n).$$

Also take $u(\cdot, 0) = u_0$ and $v(\cdot, 0) = v_0$, where $u_0, v_0 \in S_p^h$ are some suitable approximations of U_0, V_0. Note that the test functions χ, λ are one degree lower $(q-1)$ in time to account for the fact that u and v are fixed a priori by continuity at $t = t_{n-1}$. Letting $y = (u, v)$ and using the discrete Laplacian, we can reformulate this problem in the same way as was done for the partial differential equation. We obtain

$$(12) \qquad (y_t + A_h y, \phi)_{L_2(I_n, H_0^1 \times L_2)} = (F, \phi)_{L_2(I_n, H_0^1 \times L_2)} \quad \forall \phi \in [S_p^h \otimes P_{q-1}(I_n)]^2.$$

One of the most appealing properties of this scheme is that it conserves energy in the same way as the continuous problem. Letting $\chi = v_t$ in (10) and $\lambda = u_t$ in (11), we obtain

$$\mathcal{E}_n = \mathcal{E}_{n-1} + (f, u_t)_{S_n},$$

where

$$\mathcal{E}_n = \tfrac{1}{2}\|v_n\|_{L_2}^2 + \tfrac{1}{2}\|\nabla u_n\|_{L_2}^2,$$

and if $f = 0$, then the energy is observed.

3. CTG approximation of an abstract IVP

In this section we consider the discretization in time of an abstract initial value problem. Let H be a real Hilbert space, and let $\mathcal{A}$ be an operator defined on a dense domain $D(\mathcal{A}) \subset H$, which generates a strongly continuous semigroup, which we will denote by $e^{t\mathcal{A}}$. We assume that $(\mathcal{A}V, V) \geq 0$ and that $\|\mathcal{A}^* V\|_H \leq C\|\mathcal{A}V\|_H$ for all $V \in D(\mathcal{A})$. Then $\|e^{-t\mathcal{A}} V\|_H \leq \|V\|_H$ for all $V \in H$. In particular, these assumptions are satisfied if $\mathcal{A}$ is skew-symmetric, which is the case for the wave equation; however, our analysis is more general, and would apply also for example to the heat equation if we took Y to be a scalar representing the temperature and $\mathcal{A}$ the negative Laplacian operator. We consider the problem

$$(13) \qquad Y_t + \mathcal{A}Y = F, \qquad Y(0) = Y_0.$$

Precise assumptions on Y_0 and F will be stated below. In this section we will denote π_t and P_t simply by π and P, respectively.

The time-discrete CTG approximation to (13) is an element y of $D(\mathcal{A}) \otimes S_q^k$ which satisfies $y(0) = Y_0$ and for $1 \leq n \leq N$

$$(14) \qquad (y_t, \phi)_{L_2(I_n, H)} + (\mathcal{A}y, \phi)_{L_2(I_n, H)} = (F, \phi)_{L_2(I_n, H)} \quad \forall \phi \in H \otimes P_{q-1}(I_n).$$

We first derive a basic stability estimate.

Theorem 1. *If y satisfies* (14), *then*

(a) $\|y_t\|_{L_2(H)} + \|\mathcal{A}y\|_{L_2(H)} \leq C\{T^{1/2}\|\mathcal{A}Y_0\|_H + T\|\mathcal{A}F\|_{L_2(H)} + \|F\|_{L_2(H)}\}$,

and for $0 \leq t \leq T$

(b) $\|y_t(t)\|_H + \|\mathcal{A}y(t)\|_H \leq C\{\|\mathcal{A}Y_0\|_H + T^{1/2}\|\mathcal{A}F\|_{L_2(H)} + \|F\|_{L_\infty(H)}\}$.

Proof. On each subinterval, (14) is equivalent to $y_t = -\pi\mathcal{A}y + \pi F$. Therefore, by (9), we have

$$(15) \qquad \begin{aligned} \|\mathcal{A}y\|_{L_2(I_n, H)} &\leq C\{k_n^{1/2}\|\mathcal{A}y_{n-1}\|_H + \|\pi\mathcal{A}y\|_{L_2(I_n, H)}\} \\ &\leq C\{k_n^{1/2}\|\mathcal{A}y_{n-1}\|_H + \|y_t\|_{L_2(I_n, H)} + \|F\|_{L_2(I_n, H)}\}. \end{aligned}$$

Taking $\phi = \mathcal{A}y_t$ in (14) gives

$$(y_t, \mathcal{A}y_t)_{L_2(I_n,H)} + (\mathcal{A}y, \mathcal{A}y_t)_{L_2(I_n,H)} = (F, \mathcal{A}y_t)_{L_2(I_n,H)},$$
$$\tfrac{1}{2}\|\mathcal{A}y_n\|_H^2 - \tfrac{1}{2}\|\mathcal{A}y_{n-1}\|_H^2 \le -(\mathcal{A}F, y_t)_{L_2(I_n,H)}.$$

Summing over n gives

$$\tfrac{1}{2}\|\mathcal{A}y_n\|_H^2 \le \tfrac{1}{2}\|\mathcal{A}y_0\|_H^2 - (\mathcal{A}F, y_t)_{L_2([0,t_n],H)}. \tag{16}$$

On I_n let $w(t) = t - t_{n-1}$ and write $y = y_{n-1} + w\overline{y}$ with $\overline{y} \in H \otimes P_{q-1}(I_n)$. Choosing $\phi = \overline{y}$ in (14) gives

$$(y_t, \overline{y})_{L_2(I_n,H)} + (\mathcal{A}y, \overline{y})_{L_2(I_n,H)} = (F, \overline{y})_{L_2(I_n,H)},$$

$$(\overline{y}, \overline{y})_{L_2(I_n,H)} + (w\overline{y}_t\overline{y})_{L_2(I_n,H)} + (\mathcal{A}y_{n-1}, \overline{y})_{L_2(I_n,H)} + (w\mathcal{A}\overline{y}, \overline{y})_{L_2(I_n,H)} = (F, \overline{y})_{L_2(I_n,H)},$$

$$\|\overline{y}\|_{L_2(I_n,H)}^2 + (w\overline{y}_t, \overline{y})_{L_2(I_n,H)} \le -(\mathcal{A}y_{n-1}, \overline{y})_{L_2(I_n,H)} + (F, \overline{y})_{L_2(I_n,H)}.$$

Integration by parts in time establishes that

$$(w\overline{y}_t, \overline{y})_{L_2(I_n,H)} = -\tfrac{1}{2}\|\overline{y}\|_{L_2(I_n,H)}^2 + \tfrac{1}{2}k_n\|\overline{y}_n\|_H^2.$$

Thus,

$$\begin{aligned}\tfrac{1}{2}\|\overline{y}\|_{L_2(I_n,H)}^2 &\le \|\mathcal{A}y_{n-1}\|_{L_2(I_n,H)}\|\overline{y}\|_{L_2(I_n,H)} + \|F\|_{L_2(I_n,H)}\|\overline{y}\|_{L_2(I_n,H)} \\ &= k_n^{1/2}\|\mathcal{A}y_{n-1}\|_H\|\overline{y}\|_{L_2(I_n,H)} + \|F\|_{L_2(I_n,H)}\|\overline{y}\|_{L_2(I_n,H)},\end{aligned}$$

whence

$$\|\overline{y}\|_{L_2(I_n,H)} \le 2\{k_n^{1/2}\|\mathcal{A}y_{n-1}\|_H + \|F\|_{L_2(I_n,H)}\}. \tag{17}$$

By the inverse estimate (8) and properties of w, we have

$$\|y_t\|_{L_2(I_n,H)} = \|\overline{y} + w\overline{y}_t\|_{L_2(I_n,H)} \le \|\overline{y}\|_{L_2(I_n,H)} + \|w\overline{y}_t\|_{L_2(I_n,H)} \le C\|\overline{y}\|_{L_2(I_n,H)}. \tag{18}$$

Equations (15)–(18) combine to give

$$\begin{aligned}\|y_t\|_{L_2(I_n,H)}^2 + \|\mathcal{A}y\|_{L_2(I_n,H)}^2 &\le C\{k_n\|\mathcal{A}y_{n-1}\|_H^2 + \|F\|_{L_2(I_n,H)}^2\} \\ &\le C\{k_n\|\mathcal{A}y_0\|_H^2 + k_n\|\mathcal{A}F\|_{L_2([0,t_n],H)}\|y_t\|_{L_2([0,t_n],H)} + \|F\|_{L_2(I_n,H)}^2\}.\end{aligned} \tag{19}$$

Summing over n yields

$$\begin{aligned}&\|y_t\|_{L_2([0,t_n],H)}^2 + \|\mathcal{A}y\|_{L_2([0,t_n],H)}^2 \\ &\quad \le C\{t_n\|\mathcal{A}y_0\|_H^2 + t_n\|\mathcal{A}F\|_{L_2([0,t_n],H)}\|y_t\|_{L_2([0,t_n],H)} + \|F\|_{L_2([0,t_n],H)}^2\}.\end{aligned} \tag{20}$$

A simple kickback argument completes the proof of the first result.

To obtain the pointwise in time estimate, by (7) and (19),

$$\begin{aligned}\|y_t\|_{L_\infty(I_n,H)}^2 + \|\mathcal{A}y\|_{L_\infty(I_n,H)}^2 &\le Ck_n^{-1}\{\|y_t\|_{L_2(I_n,H)}^2 + \|\mathcal{A}y\|_{L_2(I_n,H)}^2\} \\ &\le C\{\|\mathcal{A}y_0\|_H^2 + \|\mathcal{A}F\|_{L_2([0,t_n],H)}\|y_t\|_{L_2([0,t_n],H)} + k_n^{-1}\|F\|_{L_2(I_n,H)}^2\} \\ &\le C\{\|\mathcal{A}y_0\|_H^2 + t_n\|\mathcal{A}F\|_{L_2([0,t_n],H)}^2 + t_n^{-1}\|y_t\|_{L_2([0,t_n],H)}^2 + k_n^{-1}\|F\|_{L_2(I_n,H)}^2\}.\end{aligned}$$

Now by (20) we have

$$\begin{aligned}&\|y_t\|_{L_\infty(I_n,H)}^2 + \|\mathcal{A}y\|_{L_\infty(I_n,H)}^2 \\ &\quad \le C\{\|\mathcal{A}y_0\|_H^2 + t_n\|\mathcal{A}F\|_{L_2([0,t_n],H)}^2 + \|F\|_{L_\infty([0,t_n],H)}^2\}.\end{aligned}$$

The desired result follows. □

If H is finite-dimensional, then existence and uniqueness of the CTG approximation follow at once from the preceding stability estimate. The next theorem shows that this holds true in general.

Theorem 2. *Given* $Y_0 \in D(\mathcal{A})$ *and* F *which satisfies* $\|\mathcal{A}F\|_{L_2(H)} + \|F\|_{L_2(H)} < \infty$, *there is a unique* $y \in D(\mathcal{A}) \otimes S_q^k$ *which satisfies* (14).

Proof. Let $\{\phi_n\}$ be an orthonormal basis for H, with $\phi_n \in D(\mathcal{A})$, and set $H_n = \operatorname{span}\{\phi_1, \ldots, \phi_n\}$. Let y_0^n be the orthogonal projection of Y_0 into H_n. It suffices to consider a generic time interval, such as $I = [0,1]$. For each n there exists a unique solution to the (finite-dimensional) problem: find $y^n \in H_n \otimes P_q(I)$ such that $y^n(0) = y_0^n$ and

$$(y_t^n + \mathcal{A}y^n, \chi)_{L_2(I,H)} = (F, \chi)_{L_2(I,H)} \quad \forall \chi \in H_n \otimes P_{q-1}(I).$$

By the previous theorem and the inequality

$$\|y\|_{L_2(I,H)} \le \|y(0)\|_H + \int_I \|y_t(t)\|_H \, dt \le \|y(0)\|_H + \|y_t\|_{L_2(I,H)}$$

it follows that $\{\|y^n\|_{L_2(I,H)} + \|\mathcal{A}y^n\|_{L_2(I,H)}\}$ is bounded. Because $\mathcal{A}$ is necessarily closed (being the generator of a strongly continuous semigroup), from this we can deduce that there is a subsequence, still denoted y^n, such that y^n converges weakly in $L_2(I,H)$ to some $y \in D(\mathcal{A})$, and further that $\mathcal{A}y^n$ converges weakly in $L_2(I,H)$ to $\mathcal{A}y$. Since

$$(y_t^n, \chi)_{L_2(I,H)} = -(y^n, \chi_t)_{L_2(I,H)} \to -(y, \chi_t)_{L_2(I,H)} = (y_t, \chi)_{L_2(I,H)},$$

we also have that y_t^n converges weakly to y_t. To show that y satisfies (14), given $\chi \in H \otimes P_{q-1}(I)$, let χ^m be the orthogonal projection of χ into $H_m \otimes P_{q-1}(I)$. Then for $n \ge m$

$$(y_t^n + \mathcal{A}y^n, \chi^m)_{L_2(I,H)} = (F, \chi^m)_{L_2(I,H)}.$$

Fix m, and let $n \to \infty$, and then let $m \to \infty$. It only remains to show that the initial condition is satisfied. But this is trivial: by construction $y^n(0)$ converges in H to Y_0, and it is also easy to deduce that $y^n(0)$ converges weakly in H to $y(0)$; it follows that $y(0) = Y_0$. This proves existence. Uniqueness follows immediately from the stability estimate. □

The previous theorem guarantees that $y(t) \in D(\mathcal{A})$; standard arguments show that $y(t)$ will have more regularity (i.e., lies in the domain of higher powers of $\mathcal{A}$) under the appropriate assumptions on Y_0 and F, and this fact will be tacitly used below. The stability estimate also allows us to prove the following error estimate.

Theorem 3. *Let* Y *be the solution of* (13), *and* y *the CTG approximation defined by* (14). *Then for* $0 \le t \le T$

$$\|\mathcal{A}y(t) - \mathcal{A}Y(t)\|_H \le Ck^{q+1}\{T^{1/2}\|\partial_t^{q+1}\mathcal{A}^2Y\|_{L_2(I,H)} + \|\partial_t^{q+1}\mathcal{A}Y\|_{L_\infty(H)}\}.$$

Proof. Write $y - Y = (y - PY) - (Y - PY) = \theta + \rho$. Note that $\theta \in H \otimes S_q^k$ and $\theta(0) = 0$. A short calculation establishes that θ satisfies, for any n and any $\phi \in H \otimes P_{q-1}(I_n)$,

$$(\theta_t + \mathcal{A}\theta, \phi)_{L_2(I_n,H)} = (\mathcal{A}\rho, \phi)_{L_2(I_n,H)}.$$

The stated estimate follows by applying Theorem 1b to θ, and estimating ρ using (6). □

Our next goal is to derive a higher-order estimate for the error at time nodes $t = t_n$. For this we will need the following stability result.

Lemma 1. *The solution y of* (14) *satisfies*

$$\|\partial_t^q \mathcal{A}^{q+1} y\|_{L_2(H)} \leq CT^{1/2}\|\mathcal{A}^{2q+1}Y_0\|_H + CT\|\mathcal{A}^{2q+1}F\|_{L_2(H)} + C\sum_{j=1}^{q}\|\partial_t^{q-j}\mathcal{A}^{q+j}F\|_{L_2(H)}.$$

Proof. Recall that on each subinterval I_n, y satisfies $y_t + \pi\mathcal{A}y = \pi F$. Operating on this identity with $\partial_t^{i-1}\mathcal{A}^j$ gives

$$\partial_t^i \mathcal{A}^j y = -\partial_t^{i-1}\mathcal{A}^{j+1}y + \partial_t^{i-1}(I-\pi)\mathcal{A}^{j+1}y + \partial_t^{i-1}\pi\mathcal{A}^j F. \tag{21}$$

For the second term on the right-hand side, we have by (5) and (8) that

$$\begin{aligned}\|\partial_t^{i-1}(I-\pi)\mathcal{A}^{j+1}y\|_{L_2(I_n,H)} &\leq Ck_n^{-(i-1)}\|(I-\pi)\mathcal{A}^{j+1}y\|_{L_2(I_n,H)}\\ &\leq Ck_n^{-(i-1)}k_n^{i-1}\|\partial_t^{i-1}\mathcal{A}^{j+1}y\|_{L_2(I_n,H)},\end{aligned}$$

and thus taking norms in (21) gives

$$\|\partial_t^i \mathcal{A}^j y\|_{L_2(I_n,H)} \leq C\|\partial_t^{i-1}\mathcal{A}^{j+1}y\|_{L_2(I_n,H)} + \|\partial_t^{i-1}\mathcal{A}^j\pi F\|_{L_2(I_n,H)}. \tag{22}$$

By summing over n and repeated use of (22) we obtain

$$\|\partial_t^q\mathcal{A}^{q+1}y\|_{L_2(H)} \leq C\|\partial_t\mathcal{A}^{2q}y\|_{L_2(H)} + C\sum_{j=1}^{q-1}\|\partial_t^{q-j}\mathcal{A}^{q+j}F\|_{L_2(H)}.$$

The proof is now completed by applying Theorem 1 to $\mathcal{A}^{2q}y$, which is just the CTG solution to the problem with initial data $\mathcal{A}^{2q}Y_0$ and nonhomogeneity $\mathcal{A}^{2q}F$. □

The following is the final result of this section.

Theorem 4. *Let Y be the solution of* (13), *and y the CTG approximation defined by* (14). *Then, assuming Y_0 and F have the indicated regularity, for $1 \leq n \leq N$*

$$\|y(t_n) - Y(t_n)\|_H \leq Ck^{2q}\left\{T\|\mathcal{A}^{2q+1}Y_0\|_H + T^{3/2}\|\mathcal{A}^{2q+1}F\|_{L_2(H)} + T^{1/2}\sum_{j=0}^{q}\|\partial_t^{q-j}\mathcal{A}^{q+j}F\|_{L_2(H)}\right\}.$$

Proof. Let $E = y - Y$. Then on I_n, E satisfies $E_t + \mathcal{A}E = (I-\pi)(\mathcal{A}y - F)$, so

$$E_n = e^{k_n\mathcal{A}}E_{n-1} + \int_{t_{n-1}}^{t_n} e^{(t-t_n)\mathcal{A}}(I-\pi)(\mathcal{A}y - F)\,dt.$$

The idea of the proof is to use Taylor's theorem to write

$$e^{(t-t_n)\mathcal{A}} = Q(t) + \frac{1}{(q-1)!}\int_{t_{n-1}}^{t}(t-s)^{q-1}\mathcal{A}^q e^{(s-t_n)\mathcal{A}}\,ds,$$

where Q is a polynomial of degree $q-1$ in time. In the case that $\mathcal{A}$ is unbounded, some care is required in interpreting this identity. The procedure can be made precise by, for example, use of the Yosida approximation to $\mathcal{A}$ (see Pazy [16, p. 9]). For the sake of clarity, we omit these details. We have

$$\begin{aligned} E_n &= e^{k_n\mathcal{A}}E_{n-1} \\ &\quad + \frac{1}{(q-1)!}\int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t}(t-s)^{q-1}\mathcal{A}^q e^{(s-t_n)\mathcal{A}}(I-\pi)(\mathcal{A}y-F)(t)\,ds\,dt, \end{aligned}$$

since the term involving Q is zero, by the definition of π and the fact that Q is degree $q-1$ in time. Since $\|e^{t\mathcal{A}}\| \le 1$, we obtain

$$\begin{aligned} \|E_n\|_H &\le \|E_{n-1}\|_H + C\int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t_n} k_n^{q-1}\|\mathcal{A}^q(I-\pi)(\mathcal{A}y-F)(t)\|)_H\,ds\,dt \\ &\le \|E_{n-1}\|_H + Ck_n^q\int_{t_{n-1}}^{t_n}\|\mathcal{A}^q(I-\pi)(\mathcal{A}y-F)(t)\|_H\,dt \\ &\le \|E_{n-1}\|_H + Ck_n^{q+1/2}\|\mathcal{A}^q(I-\pi)(\mathcal{A}y-F)\|_{L_2(I_n,H)}. \end{aligned}$$

Now by (5) we obtain

$$\|E_n\|_H \le \|E_{n-1}\|_H + Ck_n^{2q+1/2}\{\|\partial_t^q\mathcal{A}^{q+1}y\|_{L_2(I_n,H)} + \|\partial_t^q\mathcal{A}^qF\|_{L_2(I_n,H)}\}.$$

It follows in a straightforward way that

$$\|E_n\|_H \le Ct_n^{1/2}k^{2q}\{\|\partial_t^q\mathcal{A}^{q+1}y\|_{L_2([0,t_n],H)} + \|\partial_t^q\mathcal{A}^qF\|_{L_2([0,t_n],H)}\}.$$

The theorem now follows by the previous lemma. □

4. Applications

In this section we apply the results of the previous section to some specific examples. First let $H = H_0^1 \times L_2$, $\mathcal{A} = A$, as defined in §2, and $F = (0, f)$. Then the CTG approximation of the previous section is the time-discrete approximation for the wave equation, and will henceforth be denoted by $\tilde{y}$. An assumption such as $Y_0 \in D(A^{2q+1})$ implies not only assumptions about the regularity of U_0 and V_0, but also certain boundary conditions, also referred to as compatibility conditions, for these functions. These are most easily described by introducing the $\dot{H}^s$ spaces, defined by

$$\dot{H}^s = \{v \in H^s\colon \Delta^j v|_{\partial\Omega} = 0, j \in Z, 0 \le j < s/2\}.$$

These are Hilbert spaces, and on $\dot{H}^s$, $\|v\|_{\dot{H}^s} \equiv \|\Delta^{s/2}v\|_{L_2}$ and $\|v\|_{H^s}$ are equivalent norms. Notice that $\dot{H}^0 = L_2$ and $\dot{H}^1 = H_0^1$. It is easily checked that

$$D(A^j) = \dot{H}^{j+1} \times \dot{H}^j$$

for $j \ge 0$. We also have $D(A^{-1}) = L_2 \times H^{-1}$. We can now state the results that will be needed in the next section.

Theorem 5. *Let Y be the solution of (2), and $\tilde{y}$ the time-discrete CTG approximation. Then for $j = 0, 1, \ldots$*

$$\begin{aligned} \text{(a)}\quad & \|A^j\tilde{y}_t\|_{L_\infty(H_0^1\times L_2)} + \|A^{j+1}\tilde{y}\|_{L_\infty(H_0^1\times L_2)} \\ & \le C\{\|U_0\|_{\dot{H}^{j+2}} + \|V_0\|_{\dot{H}^{j+1}} + T^{1/2}\|f\|_{L_2(\dot{H}^{j+1})} + \|f\|_{L_\infty(\dot{H}^j)}\} \end{aligned}$$

and for $j = -1, 0, 1,$ *and* $0 \le n \le N$,

$$\text{(b)} \quad \|A^j(\tilde{y} - Y)(t_n)\|_{H_0^1 \times L_2} \le Ck^{2q}\{T\|U_0\|_{\dot{H}^{2q+2+j}} + T\|V_0\|_{\dot{H}^{2q+1+j}} + M_j(f, q, T)\},$$

where

$$(23) \qquad M_j(f, q, T) \equiv T^{3/2}\|f\|_{L_2(\dot{H}^{2q+1+j})} + T^{1/2}\sum_{i=0}^{q} \|\partial_t^i f\|_{L_2(\dot{H}^{2q-i+j})}.$$

Proof. The first result is the consequence of Theorem 1(b) applied to $A^j\tilde{y}$. Part (b) is obtained by applying Theorem 4 to $B\tilde{y}$, to $\tilde{y}$, and to $A\tilde{y}$. □

Note that part (b) of the above theorem gives L_2 and H_0^1 estimates for $U - \tilde{u}$ when $j = -1$ and $j = 0$, respectively, and gives such estimates for $V - \tilde{v}$ when $j = 0$ and $j = 1$.

The second specific case we consider is $H = S_p^h \times S_p^h$, with the $H_0^1 \times L_2$ inner product, and $\mathcal{A} = A_h$. Then the approximation defined in §3 is the fully discrete CTG approximation for the wave equation, and will henceforth be denoted by y. We will need the following results.

Theorem 6. *Let* $y = (u, v)$ *satisfy the fully discrete equation* (12). *Then*

$$\text{(a)} \quad \|u\|_{L_\infty(L_2)} + \|T_h v\|_{L_\infty(H_0^1)} \le C\{\|u_0\|_{L_2} + \|T_h v_0\|_{H_0^1} + T^{1/2}\|T_h f\|_{L_2(H_0^1)} + \|T_h f\|_{L_\infty(L_2)}\},$$

$$\text{(b)} \quad \|u\|_{L_\infty(H_0^1)} + \|v\|_{L_\infty(L_2)} \le C\{\|u_0\|_{H_0^1} + \|v_0\|_{L_2} + T^{1/2}\|f\|_{L_2(L_2)} + \|T_h f\|_{L_\infty(H_0^1)}\},$$

$$\text{(c)} \quad \|\Delta_h u\|_{L_\infty(L_2)} + \|v\|_{L_\infty(H_0^1)} \le C\{\|\Delta_h u_0\|_{L_2} + \|v_0\|_{H_0^1} + T^{1/2}\|f\|_{L_2(H_0^1)} + \|f\|_{L_\infty(L_2)}\}.$$

Proof. Apply Theorem 1(b) in turn to $B_h^2 y$, to $B_h y$, and to y. □

5. Error estimates for fully discrete CTG

Theorem 7. *Let* $Y = (U, V)$ *be the solution of* (2), *and* $y = (u, v)$ *the CTG approximation defined by* (12), *with* $(u_0, v_0) = (P_x U_0, P_x V_0)$. *Let* $\overline{p} = \max(p, 2)$. For $0 \le t \le T$ there holds

$$\text{(a)} \quad \|u(t) - U(t)\|_{L_2} \le C(T+1)k^{q+1}\{\|\partial_t^{q+2}U\|_{L_\infty(L_2)} + \|\partial_t^{q+1}U\|_{L_\infty(H_0^1)}\} + C(T+1)h^{p+1}\{\|U_{tt}\|_{L_\infty(H^{\overline{p}})} + \|U\|_{L_\infty(H^{p+1})}\},$$

$$\text{(b)} \quad \|u(t) - U(t)\|_{H_0^1} \le C(T+1)k^{q+1}\{\|\partial_t^{q+2}U\|_{L_\infty(H_0^1)} + \|\partial_t^{q+1}U\|_{L_\infty(H^2)}\} + C(T+1)h^{p}\{\|U_{tt}\|_{L_\infty(H^p)} + \|U\|_{L_\infty(H^{p+1})}\},$$

$$\text{(c)} \quad \|v(t) - V(t)\|_{L_2} \le C(T+1)k^{q+1}\{\|\partial_t^{q+2}U\|_{L_\infty(H_0^1)} + \|\partial_t^{q+1}U\|_{L_\infty(H^2)}\} + C(T+1)h^{p+1}\{\|U_{tt}\|_{L_\infty(H^{p+1})} + \|U_t\|_{L_\infty(H^{p+1})}\},$$

$$\begin{aligned}\text{(d)}\quad &\|v(t)-V(t)\|_{H_0^1}\\ &\le C(T+1)k^{q+1}\{\|\partial_t^{q+2}U\|_{L_\infty(H^2)}+\|\partial_t^{q+1}U\|_{L_\infty(H^3)}\}\\ &\quad + C(T+1)h^p\{\|U_{tt}\|_{L_\infty(H^{p+1})}+\|U_t\|_{L_\infty(H^{p+1})}\}.\end{aligned}$$

Proof. We write

$$y-Y=(y-P_xP_tY)+(P_xP_tY-Y)=\theta+\rho.$$

Then $u-U=\theta_1+\rho_1$ and $v-V=\theta_2+\rho_2$. Note that $\theta\in[S_{pq}^{hk}]^2$, $\theta(0)=0$, and θ satisfies, for $1\le n\le N$,

$$(\theta_t+A_h\theta,\phi)_{L_2(I_n,H_0^1\times L_2)}=(G,\phi)_{L_2(I_n,H_0^1\times L_2)}\quad \forall\phi\in[P_{q-1}(I_n)\otimes S_p^h]^2$$

with $G=((P_t-I)U_t,(I-P_x)U_{tt}-(I-P_t)\Delta U)$. To derive the L_2-estimate for u, begin by applying Theorem 1(b) with $\mathcal{A}=A_h$ to $B_h^2\theta$ to obtain

$$\begin{aligned}\|\theta_1(t)\|_{L_2}&\le\|B_h\theta(t)\|_{H_0^1\times L_2}\\ &\le C\{T^{1/2}\|B_hG\|_{L_2(H_0^1\times L_2)}+\|B_h^2G\|_{L_\infty(H_0^1\times L_2)}\}\\ &\le C(T+1)\|B_hG\|_{L_\infty(H_0^1\times L_2)}\\ &\le C(T+1)\{\|G_1\|_{L_\infty(L_2)}+\|T_hG_2\|_{L_\infty(H_0^1)}\}.\end{aligned}\tag{24}$$

For G_1, we have by (6),

$$\|G_1\|_{L_2}=\|(I-P_t)U_t\|_{L_2}\le Ck^{q+1}\|\partial_t^{q+2}U\|_{L_2}.\tag{25}$$

For G_2, we have

$$\begin{aligned}\|T_hG_2\|_{H_0^1}&\le\|G_2\|_{H^{-1}}\\ &\le\|(I-P_x)U_{tt}\|_{H^{-1}}+\|(I-P_t)\Delta U\|_{H^{-1}}\\ &\le Ch^{p+1}\|U_{tt}\|_{H^{\overline{p}}}+Ck^{q+1}\|\partial_t^{q+1}U\|_{H_0^1}.\end{aligned}\tag{26}$$

In the last inequality we have used a negative norm estimate for P_x when $p>1$. Combining (24), (25) and (26) gives an estimate for $\|\theta_1\|_{L_2}$. By writing $\rho_1=(I-P_x)U+P_x(I-P_t)U$, and using the approximation properties of P_x and P_t, we obtain the estimate

$$\|\rho_1\|_{L_2}\le Ch^{p+1}\|U\|_{H^{p+1}}+Ck^{q+1}\|\partial_t^{q+1}U\|_{H_0^1}.$$

The first of the four results now follows.

Next we apply Theorem 1(b) to $B_h\theta$ to obtain

$$\begin{aligned}\|\theta(t)\|_{L_\infty(H_0^1\times L_2)}&\le C\{T^{1/2}\|G\|_{L_2(H_0^1\times L_2)}+\|B_hG\|_{L_\infty(H_0^1\times L_2)}\}\\ &\le C(T+1)\|G\|_{L_\infty(H_0^1\times L_2)}\\ &\le C(T+1)\{\|G_1\|_{L_\infty(H_0^1)}+\|G_2\|_{L_\infty(L_2)}\}.\end{aligned}\tag{27}$$

We can estimate these terms by

$$\|G_1\|_{H_0^1}=\|(I-P_t)U_t\|_{H_0^1}\le Ck^{q+1}\|\partial_t^{q+2}U\|_{H_0^1},\tag{28}$$

and

$$\begin{aligned}\|G_2\|_{L_2}&\le\|(I-P_x)U_{tt}\|_{L_2}+\|(I-P_t)\Delta U\|_{L_2}\\ &\le Ch^{p+s}\|U_{tt}\|_{H^{p+s}}+Ck^{q+1}\|\partial_t^{q+1}\Delta U\|_{L_2},\end{aligned}\tag{29}$$

for $s=0,1$. Inequalities (27), (28), and (29) with $s=0$ give an estimate for $\|\theta_1\|_{H_0^1}$, which, when combined with the appropriate estimate for $\|\rho_1\|_{H_0^1}$, yields part (b) of the theorem.

To derive the third result, we use (27), (28), and (29) with $s = 1$ to obtain an estimate for $\|\theta_2\|_{L_2}$. And ρ_2 can be bounded as was ρ_1, but with U_t in the place of U.

The final result follows in a similar way from applying Theorem 1(b) to θ. $\square$

Theorem 8. *Let $Y = (U, V)$ be the solution of (2), and $y = (u, v)$ the CTG approximation defined by (12), with $(u_0, v_0) = (P_x U_0, P_x V_0)$. Let $\overline{p} = \max(p, 2)$, and let M be as defined in (23). For $1 \le n \le N$ there holds*

$$\begin{aligned}
\text{(a)}\quad & \|u(t_n) - U(t_n)\|_{L_2} \\
&\le C(T+1)(h^{p+1} + k^{2q})\{\|U_0\|_{\dot{H}^{\max(\overline{p}+2,2q+1)}} + \|V_0\|_{\dot{H}^{\max(\overline{p}+1,2q)}}\} \\
&\quad + C(T+1)^2 h^{p+1}\|f\|_{L_\infty(\dot{H}^{\overline{p}+1})} + Ck^{2q}M_{-1}(f,q,T),
\end{aligned}$$

$$\begin{aligned}
\text{(b)}\quad & \|u(t_n) - U(t_n)\|_{H_0^1} \\
&\le C(T+1)(h^{p} + k^{2q})\{\|U_0\|_{\dot{H}^{\max(p+2,2q+2)}} + \|V_0\|_{\dot{H}^{\max(p+1,2q+1)}}\} \\
&\quad + C(T+1)^2 h^{p}\|f\|_{L_\infty(\dot{H}^{p+1})} + Ck^{2q}M_{0}(f,q,T),
\end{aligned}$$

$$\begin{aligned}
\text{(c)}\quad & \|v(t_n) - V(t_n)\|_{L_2} \\
&\le C(T+1)(h^{p+1} + k^{2q})\{\|U_0\|_{\dot{H}^{\max(p+3,2q+2)}} + \|V_0\|_{\dot{H}^{\max(p+2,2q+1)}}\} \\
&\quad + C(T+1)^2 h^{p+1}\|f\|_{L_\infty(\dot{H}^{p+2})} + Ck^{2q}M_{0}(f,q,T),
\end{aligned}$$

$$\begin{aligned}
\text{(d)}\quad & \|v(t_n) - V(t_n)\|_{H_0^1} \\
&\le C(T+1)(h^{p} + k^{2q})\{\|U_0\|_{\dot{H}^{\max(p+3,2q+3)}} + \|V_0\|_{\dot{H}^{\max(p+2,2q+2)}}\} \\
&\quad + C(T+1)^2 h^{p}\|f\|_{L_\infty(\dot{H}^{p+2})} + Ck^{2q}M_{1}(f,q,T).
\end{aligned}$$

Proof. Let $\tilde{y} = (\tilde{u}, \tilde{v})$ be the time-discrete CTG approximation with $\tilde{y}_0 = (U_0, V_0)$, and write

$$\begin{aligned}
y - Y &= (y - P_x\tilde{y}) + (P_x\tilde{y} - \tilde{y}) + (\tilde{y} - Y) \\
&= \theta + \rho + \eta,
\end{aligned} \tag{30}$$

where $P_x\tilde{y} = (P_x\tilde{u}, P_x\tilde{v})$. Note that $\theta \in [S_{pq}^{hk}]^2$, $\theta(0) = 0$, and a short calculation shows that θ satisfies, for $1 \le n \le N$,

$$(\theta_t + A_h\theta, \phi)_{L_2(I_n, H_0^1 \times L_2)} = (G, \phi)_{L_2(I_n, H_0^1 \times L_2)} \quad \forall \phi \in [P_{q-1}(I_n) \otimes S_p^h]^2,$$

where $G = (0, g)$, with $g = (I - P_x)\tilde{v}_t$. We will prove in detail only the L_2 estimate for $U - u = \theta_1 + \rho_1 + \eta_1$, and begin by looking at the first component θ_1 of θ. By Theorem 6(a) applied to θ,

$$\begin{aligned}
\|\theta_1\|_{L_\infty(L_2)} &\le C\{T^{1/2}\|T_h g\|_{L_2(H_0^1)} + \|T_h g\|_{L_\infty(L_2)}\} \\
&\le C(T+1)\|g\|_{L_\infty(H^{-1})} \\
&\le C(T+1)h^{p+1}\|\tilde{v}_t\|_{L_\infty(H^{\overline{p}})} \\
&\le C(T+1)h^{p+1}\|A^{\overline{p}}\tilde{y}_t\|_{L_\infty(H_0^1 \times L_2)}.
\end{aligned}$$

In the second-to-last inequality, we have again used a negative norm estimate for P_x when $p > 1$. For the first component of ρ_1 of ρ, we have

$$\begin{aligned}\|\rho_1\|_{L_\infty(L_2)} &= \|(I-P_x)\tilde{u}\|_{L_\infty(L_2)} \\ &\le Ch^{p+1}\|\tilde{u}\|_{L_\infty(H^{p+1})} \\ &\le Ch^{p+1}\|A^p\tilde{y}\|_{L_\infty(H_0^1\times L_2)}.\end{aligned}$$

Together, we have

$$\begin{aligned}\|\theta_1\|_{L_\infty(L_2)} + \|\rho_1\|_{L_\infty(L_2)} &\le Ch^{p+1}\{(T+1)\|A^{\overline{p}}\tilde{y}_t\|_{L_\infty(H_0^1\times L_2)} + \|A^p\tilde{y}\|_{L_\infty(H_0^1\times L_2)}\} \\ &\le Ch^{p+1}(T+1)\{\|U_0\|_{\dot{H}^{\overline{p}+2}} + \|V_0\|_{\dot{H}^{\overline{p}+1}} + (T+1)\|f\|_{L_\infty(\dot{H}^{\overline{p}+1})}\},\end{aligned}$$

where the last inequality is obtained by applying Theorem 5(a). Combining this estimate with an estimate for $\eta_1 = \tilde{u} - U$ from Theorem 5(b) gives the desired result. The proofs of parts (b)–(d) follow the same pattern. □

Remark 1. For other optimal-order choices of the discrete initial data, the above estimates for $u - U$ remain valid, while the derivation of the estimates for $v - V$ require $u_0 = P_x U_0$. Numerical examples indicate that this restriction is necessary in practice.

Remark 2. Global (in time) bounds of order k^{q+1} as in Theorem 7 could also be obtained from the splitting (30), by estimating θ and ρ as in the proof of Theorem 8, and using Theorem 3 with $\mathcal{A} = A$ to estimate η.

Remark 3. In place of the splitting (30) used in the derivation of the order k^{2q} estimates, we could use the simpler splitting

$$Y^{hk} - Y = (Y^{hk} - Y^h) + (Y^h - Y) = \theta + \rho.$$

Here, for clarity, we have used superscripts to indicate space and time discretizations. Estimates for ρ are well known, and θ can be estimated by applying Theorem 4 with $\mathcal{A} = A_h$. However, this results in the appearance of discrete norms of the data. For example, in the case $f = 0$ one would have

$$\begin{aligned}\|\theta(t_n)\|_{H_0^1\times L_2} &\le CTk^{2q}\|A_h^{2q+1}y_0\|_{H_0^1\times L_2} \\ &\le CTk^{2q}\{\|\Delta_h^{q+1}u_0\|_{L_2} + \|\Delta_h^q v_0\|_{H_0^1}\}.\end{aligned}$$

Bounding the quantities on the right-hand side in terms of continuous Sobolev norms of the data U_0, V_0 can be done in some cases, but apparently not in all.

6. Numerical results

In this section we present some numerical results for $\Omega = (0,1) \subset R$, with S_p^h based on a uniform mesh, and uniform time steps. The estimated rates of convergence reported in the tables are all with respect to the parameter h.

First, we investigate how the choice of the discrete initial data affects the approximation. In Example 1 the exact solution is smooth, and we take $p = 3$, $q = 2$, and $k = O(h)$, so that L_2 errors should be fourth-order with respect to h, and H_0^1 errors should be third-order. The results are consistent with Remark 2: the approximation of U is insensitive to the choice of u_0 and v_0, whereas for the approximation of V to be of optimal order it is necessary that u_0 be the elliptic projection of U_0 (but v_0 is still free to be any reasonable choice). However, we have also observed in practice that when $p = q = 1$, any optimal-order choice of the discrete initial data results in all quantities being of optimal order, so that in this case the assumption $u_0 = P_x U_0$ required for our analysis may be unnecessary.

Example 1. Choice of discrete initial data

$$U_{tt} - U_{xx} = 0$$
$$U(x,t) = \sin(\pi x)\cos(\pi t + 1)$$
$$p = 3,\ q = 2,\ T = 1.0,\ k = h$$

(1a) $u_0 = P_x U_0, v_0 = \pi_x V_0$

$1/h$	$1/k$	$\|(U-u)(T)\|_{L_2}$	*rate*	$\|(U-u)(T)\|_{H_0^1}$	*rate*
16	16	$0.3845e-5$		$0.3106e-4$	
32	32	$0.2413e-6$	4.00	$0.3656e-5$	3.09
64	64	$0.1509e-7$	4.00	$0.4496e-6$	3.02
128	128	$0.9412e-9$	4.00	$0.5597e-7$	3.00

$1/h$	$1/k$	$\|(V-v)(T)\|_{L_2}$	*rate*	$\|(V-v)(T)\|_{H_0^1}$	*rate*
16	16	$0.7883e-5$		$0.1561e-3$	
32	32	$0.4916e-6$	4.00	$0.1813e-4$	3.11
64	64	$0.3051e-7$	4.00	$0.2320e-5$	2.97
128	128	$0.1905e-8$	4.00	$0.2891e-6$	3.00

(1b) $u_0 = \pi_x U_0, v_0 = P_x V_0$

$1/h$	$1/k$	$\|(U-u)(T)\|_{L_2}$	*rate*	$\|(U-u)(T)\|_{H_0^1}$	*rate*
16	16	$0.3859e-5$		$0.3367e-4$	
32	32	$0.2415e-6$	4.00	$0.3765e-5$	3.16
64	64	$0.1508e-7$	4.00	$0.4764e-6$	2.98
128	128	$0.9418e-9$	4.00	$0.5910e-7$	3.01

$1/h$	$1/k$	$\|(V-v)(T)\|_{L_2}$	*rate*	$\|(V-v)(T)\|_{H_0^1}$	*rate*
16	16	$0.1627e-4$		$0.1688e-2$	
32	32	$0.2261e-5$	2.85	$0.5373e-3$	1.65
64	64	$0.2515e-6$	3.16	$0.1224e-3$	2.13
128	128	$0.3116e-7$	3.01	$0.3102e-4$	1.98

Next, we consider the compatibility conditions. Suppose f is identically zero and $p > 1$. Then for $\|(U-u)(t_n)\|_{L_2}$ to be of optimal order, Theorem 8 requires that

$$U_0 \in \dot{H}^{\max(p+2,2q+1)}, \qquad V_0 \in \dot{H}^{\max(p+1,2q)}.$$

If, for example, $2q = p+1$, a reasonable choice if $k = O(h)$, then these assumptions are no more than those required for the standard time-continuous space-discrete finite element approximation to be of optimal order. For another example, suppose $p = q > 1$. Then the assumptions are stronger than those required for the space-discretization alone. In Example 2, $p = q = 2$ and $k = O(h^{3/4})$, so that for L_2 errors $O(h^3)$ would be optimal. We have set $V_0 = 0$ and chosen U_0 to be a smooth function, so that the only remaining issue is whether U_0 satisfies the appropriate compatibility conditions. For $\|U-u\|_{L_2}$ to be $O(h^3)$, our analysis requires that

$$U_0 \in \dot{H}^5 \to U_0 = U_{0xx} = U_{0xxxx} = 0 \quad \text{on } \partial\Omega.$$

The numerical results indicate that this assumption is necessary.

Example 2. Compatibility conditions for U_0

$$U_{tt} - U_{xx} = 0$$

$$U_0(x) = [x(1-x)]^a, V_0 = 0$$

$$p = 2,\ q = 2,\ T = 0.9,\ k = O(h^{3/4}),\ u_0 = P_x U_0$$

(2a) $a = 4, U_0 = U_{0xx} = 0$ on $\partial\Omega$

$1/h$	$1/k$	$\|(U-\hat{u})(T)\|_{L_2}$	*rate*
32	13	$0.1937E-05$	
64	22	$0.3431E-06$	2.50
128	38	$0.5486E-07$	2.64
256	64	$0.9164E-08$	2.58
512	107	$0.1526E-08$	2.59

(2b) $a = 5, U_0 = U_{0xx} = U_{0xxxx} = 0$ on $\partial\Omega$

$1/h$	$1/k$	$\|(U-u)(T)\|_{L_2}$	*rate*
32	13	$0.7700E-06$	
64	22	$0.1018E-06$	2.92
128	38	$0.1181E-07$	3.11
256	64	$0.1487E-08$	2.99
512	107	$0.1913E-09$	2.96

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