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DISSECTIONS OF $p : q$ RECTANGLES

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ABSTRACT. We determine all simple perfect dissections of $p : q$ rectangles into at most twelve $p : q$ rectangular elements. A computer search shows there are only eight such dissections, two of order 10, three of order 11, and three of order 12.

1. INTRODUCTION

We consider a generalization of the problem of squaring the square. In that well-known problem, one wishes to dissect a square into smaller squares (the *elements* of the dissection). A dissection is *perfect* if no two elements have the same size. A dissection is *simple* if no proper subset of its elements forms a rectangle; otherwise the dissection is *compound*. The *order* of a dissection is its number of elements. The smallest order of a simple perfect dissection of a square (i.e., a *simple perfect squared square*) is 21 [4]. Recent work of Bouwkamp and Duijvestijn [2, 5, 6] has yielded a complete list of all simple perfect squared squares of order at most 25.

In [7], we investigated the problem of finding a simple perfect dissection of a $1 : 2$ rectangle (one whose ratio of width to height is $\frac{1}{2}$ or 2) into $1 : 2$ rectangular elements. Since these elements have two possible orientations, one would expect dissections of much smaller order than with square elements. Indeed, we found two simple perfect dissections of order 10. We further discovered that these are the only such dissections of order less than or equal to 12.

In this paper, we consider the general problem of finding a simple perfect dissection of a $p : q$ rectangle into $p : q$ rectangular elements. (This problem was posed by Müller in [8].) What relatively prime integers p and q give a dissection of smallest order? What are all such dissections of small order, say at most 12? We find:

1) There are two simple perfect order-10 dissections of $1 : 2$ rectangles into $1 : 2$ rectangular elements. (These are the dissections in [7].)
2) There are three simple perfect order-11 dissections. The ratios of the rectangles and their elements are $1 : 3$, $1 : 5$, and $1 : 7$.
3) There are three simple perfect order-12 dissections, of ratios $4 : 7$, $3 : 7$, and $1 : 9$.

This is a complete list of all such dissections of order ≤ 12.

Figures 2–9 below show these eight rectangles. The captions give a code analogous to the Bouwkamp code for dissections into squares. Each number is the shorter side of an element. A number with a prime denotes an element with horizontal orientation; no prime means vertical orientation. The codes and rectangles are also

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given in Table 1 below. Note the large size of the 3 : 7 rectangle compared to the other rectangles.

We generate dissections from c-nets, an idea developed in the basic paper of Brooks, Smith, Stone, and Tutte [3]. In §2, Theorem 2 shows how to produce all simple dissections. This theorem is a variation of the main result in [7], repeated below as Theorem 1. In §3, we describe the computation that yielded all simple dissections of order less than or equal to 12.

TABLE 1. Simple perfect dissections of $p : q$ rectangles

Order	Ratio Dimensions	Code
10	1 : 2 31 × 62	$(17', 10, 4, 7')$ $(1', 12)$ (6) $(14', 3')$ $(11')$
10	1 : 2 59 × 118	$(24, 28', 19')$ $(9', 20)$ $(12, 31')$ $(11', 2)$ $(7')$
11	1 : 3 64 × 192	$(37', 10', 17')$ $(9, 7')$ $(24')$ $(27', 13')$ $(3, 23')$ $(14')$
11	1 : 5 192 × 960	$(109', 78', 25)$ $(31', 47')$ $(83', 41', 16')$ $(5, 67')$ $(42')$
11	1 : 7 72 × 504	$(41', 30', 7)$ $(11', 19')$ $(31', 13', 8')$ $(5', 23')$ $(18')$
12	4 : 7 416 × 728	$(220', 196')$ $(92', 104')$ $(112, 88', 68')$ $(20', 128', 12')$ $(116')$ $(108')$
12	3 : 7 1440 × 3360	$(444', 807', 441)$ $(189, 363')$ $(78', 237', 633', 222')$ $(159')$ $(411')$ $(396')$
12	1 : 9 120 × 1080	$(67', 9, 52')$ $(29', 23')$ $(53', 14')$ $(6', 17')$ $(39', 11')$ $(28')$

2. THE MAIN THEOREM

According to the theory developed in [3], every simple order-n dissection of a rectangle into rectangular elements comes from a c-net (a 3-connected planar graph) having $n + 1$ edges. So our starting point is a list of c-nets with at most 13 edges. (Drawings of such c-nets can be found in [1].) Given a c-net with $n + 1$ edges and $m + 1$ vertices, removing an edge yields a p-net with n edges and $m + 1$ vertices; the p-net yields an order-n dissection of a rectangle. (Figure 1 shows an example with $n = 10$.) In [7], we developed a procedure for finding the dimensions of the

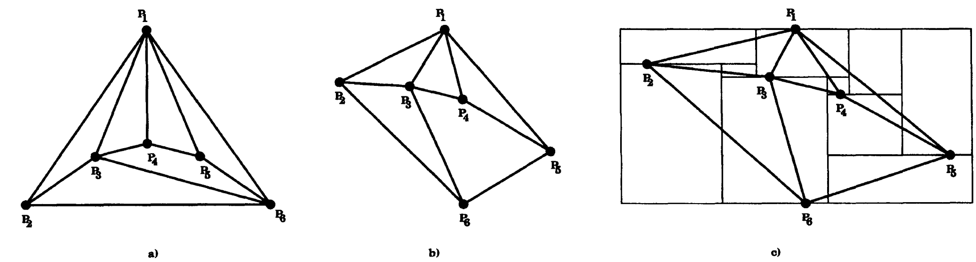

FIGURE 1. A c-net yields a simple dissection
(a) A c-net with 11 edges and 6 vertices.
(b) The p-net obtained by removing the edge joining P_1 and P_6.
(c) The dissection given by the p-net.

elements. We review the procedure and extend it to handle the general problem under investigation here.

Label the vertices of the p-net as $P_1, \ldots, P_{m+1}$ so that the horizontal line segment through P_i is above the line segment through P_j if $i < j$. Denote by E_{ij} the element in the dissection whose top and bottom edges lie along segments containing P_i and P_j. (The edges in the p-net are also labeled E_{ij}.) We encode the information given in the p-net as an $m \times n$ matrix B defined as follows. Label the n columns of B by the indices for the n elements. The entries in B are:

The entry in the ith row of column (ij) is 1.
If $j \leq m$, the entry in the jth row of column (ij) is -1.
All other entries are 0.

In the example of Figure 1, the matrix is:

$$\begin{array}{cc} & \begin{array}{cccccccccc} (12) & (13) & (14) & (15) & (23) & (26) & (34) & (36) & (45) & (56) \end{array} \\ B = & \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}. \end{array}$$

We encode the shapes of the n rectangular elements in a diagonal matrix C. Let h_{ij} and w_{ij} be the height and width of E_{ij}, let $c_{ij} = w_{ij}/h_{ij}$, and let C be the $n \times n$ diagonal matrix whose diagonal entries are the c_{ij}. Let $\vec{h}$ (resp. $\vec{w}$) be the column n-vector consisting of the h_{ij} (resp. w_{ij}). Then

$$\vec{w} = C\vec{h}. \tag{1}$$

Suppose, for definiteness, we fix the width of the dissected rectangle at 1 and let $\vec{e}$ be the column m-vector with first entry 1 and all other entries 0. Then

$$B\vec{w} = \vec{e}. \tag{2}$$

(*Proof.* The first entry in $B\vec{w}$ is $\sum_j w_{1j}$, the width of the dissected rectangle. For $i > 1$, the ith entry in $B\vec{w}$ is
$-$ ($\sum$ widths of the elements above the horizontal line segment through P_i)
$+$ ($\sum$ widths of the elements below the horizontal line segment through P_i)
$= 0$.)

Now introduce new variables. For $1 \leq i \leq m$, let y_i be the distance between the horizontal line segments through P_i and P_{m+1}. Let $\vec{y}$ be the column m-vector of the y_i. Then

$$\vec{h} = B^T\vec{y}. \tag{3}$$

(*Proof.* $h_{ij} = y_i - y_j$.)

Putting (1), (2), and (3) together, we have established all but the uniqueness in the following result.

Theorem 1. *The vectors $\vec{h}$ and $\vec{w}$ are uniquely determined, up to multiplication by a constant, by the matrices B and C. We find $\vec{h}$ and $\vec{w}$ by solving $(BCB^T)\vec{y} = \vec{e}$ for $\vec{y}$ and then computing $\vec{h} = B^T\vec{y}$ and $\vec{w} = C\vec{h}$.*

(The proof of uniqueness, a straightforward linear algebra argument, is given in [7].)

To apply this procedure for a given ratio $p : q$, the c_{ij} are all taken to be p/q or q/p. (In [7], each c_{ij} is either $\frac{1}{2}$ or 2.) Now suppose the ratio $p : q$ is to be determined. We modify the procedure as follows.

Let w be the width and h be the height of the dissected rectangle and let $x = w/h$, a variable to be determined. Then each ratio c_{ij} is either x or $1/x$. Let $\vec{w}_1$ be the column $(n+1)$-vector whose first n entries are the w_{ij} and whose last entry is w. Let B_1 be the $m \times (n+1)$ matrix whose first n columns are the columns of B and whose last column has first entry -1 and all other entries 0. From (2) we have

$$B_1\vec{w}_1 = \vec{0}. \tag{4}$$

Form 2^n $(n+1) \times (n+1)$ diagonal matrices where the first n diagonal entries are either x or $1/x$ and the last diagonal entry is $-x$. Denote any such matrix by $C_1(x)$. Let $\vec{h}_1$ be the column $(n+1)$-vector whose first n entries are the h_{ij} and whose last entry is $-h$. Then, with $\vec{y}$ defined as above (and noting that $y_1 = h$), from (3) we have

$$B_1^T\vec{y} = \vec{h}_1. \tag{5}$$

Further,

$$C_1(x)\vec{h}_1 = \vec{w}_1. \tag{6}$$

From (4), (5), and (6), we get

$$(B_1C_1(x)B_1^T)\vec{y} = \vec{0}. \tag{7}$$

Equation (7) says: A $p : q$ rectangle has a simple dissection into $p : q$ rectangular elements only if the symmetric $m \times m$ matrix $B_1C_1(x)B_1^T$ is singular, i.e., only if $\det(B_1C_1(x)B_1^T) = 0$. If we replace the matrix $B_1C_1(x)B_1^T$ by $B_1\big(xC_1(x)\big)B_1^T$, then $\det\big[B_1\big(xC_1(x)\big)B_1^T\big]$ is a polynomial in x of degree at most $2m$. The above argument establishes our main result.

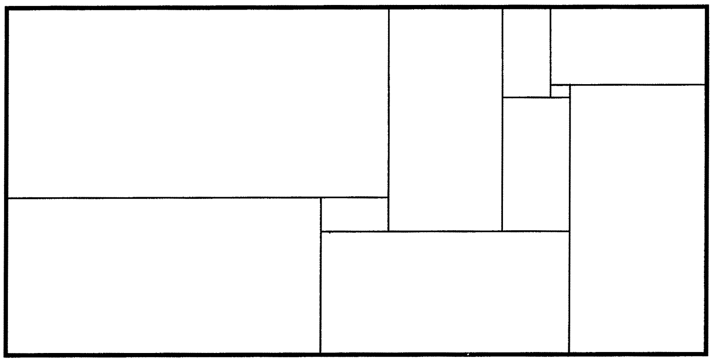

FIGURE 2. A simple order-10 dissection of a 31×62 rectangle (ratio 1:2) $(17', 10, 4, 7')(1', 12)(6)(14', 3')(11')$

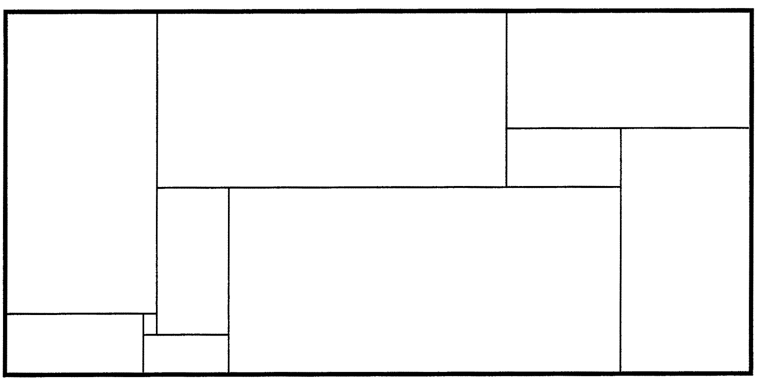

FIGURE 3. A simple order-10 dissection of a 59×118 rectangle (ratio 1:2) $(24, 28', 19')(9', 20)(12, 31')(11', 2)(7')$

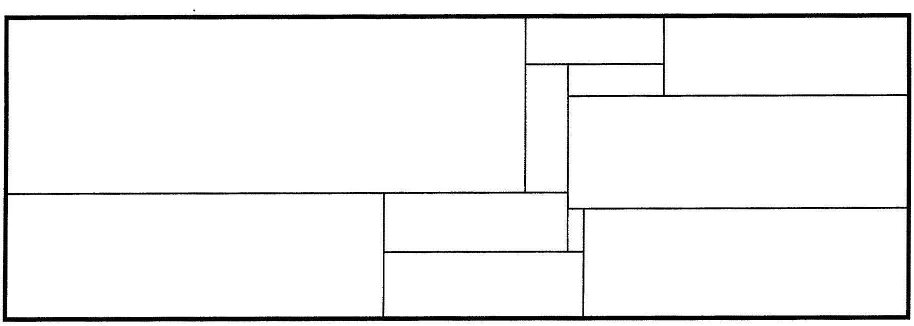

FIGURE 4. A simple order-11 dissection of a 64×192 rectangle (ratio 1:3) $(37', 10', 17')(9, 7')(24')(27', 13')(3, 23')(14')$

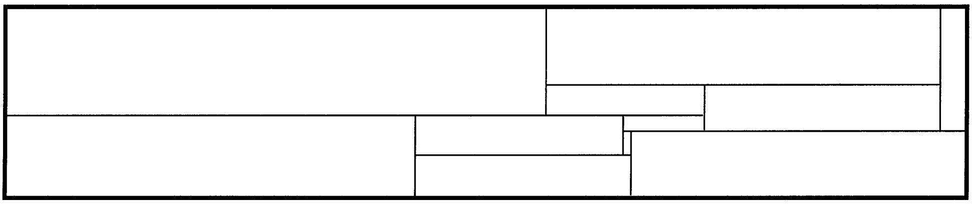

FIGURE 5. A simple order-11 dissection of a 192×960 rectangle (ratio 1:5) $(109', 78', 25)(31', 47')(83', 41', 16')(5, 67')(42')$

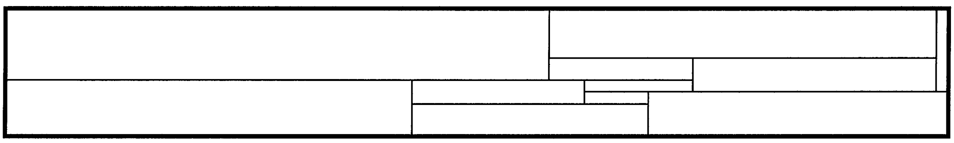

FIGURE 6. A simple order-11 dissection of a 72×504 rectangle (ratio 1:7) $(41', 30', 7)(11', 19')(31', 13', 8')(5', 23')(18')$

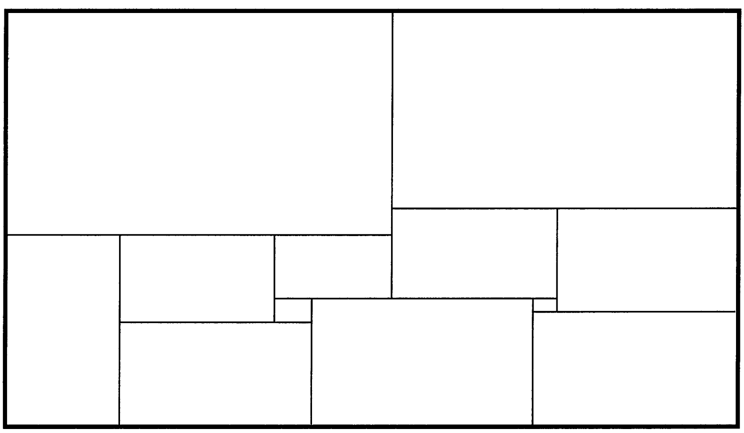

FIGURE 7. A simple order-12 dissection of a 416×728 rectangle (ratio 4:7) $(220', 196')(92', 104')(112, 88', 68')(20', 128', 12')(116')(108')$

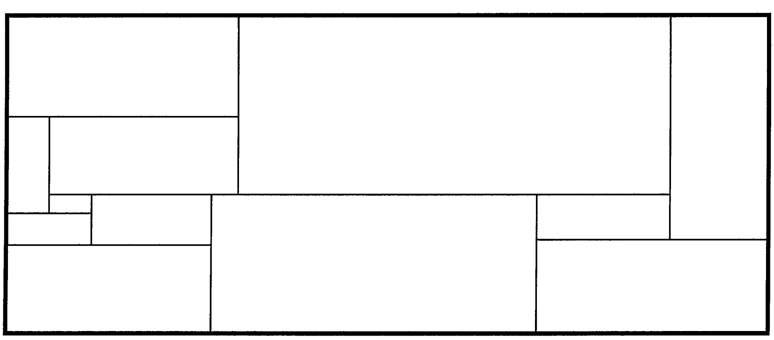

FIGURE 8. A simple order-12 dissection of a 1440×3360 rectangle (ratio 3:7) $(444', 807', 441)(189, 363')(78', 237', 633', 222')(159')(411')(396')$

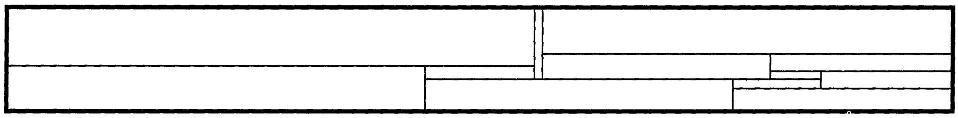

FIGURE 9. A simple order-12 dissection of a 120×1080 rectangle (ratio 1:9) $(67', 9, 52')(29', 23')(53', 14')(6', 17')(39', 11')(28')$

Theorem 2. *If there exists a simple dissection of a $p : q$ rectangle into $p : q$ rectangular elements, then either p/q or q/p is a positive rational zero of the polynomial* $\det\left[B_1\left(xC_1(x)\right)B_1^T\right]$.

3. Computation of the dissections

We now describe the scheme for computing all simple dissections of order ≤ 12. We used Maple to generate the dissections as follows. (The programming was done by my colleague Eugene A. Herman. His assistance is gratefully acknowledged.)

The input is the adjacency matrix for a p-net from which the matrix B_1 is computed. The diagonal matrix $C_1(x)$ is initialized to have diagonal entries $x, \ldots, x, -x$ and then is successively updated by a grey code, creating a main loop executed 2^n times. The program computes the $m \times m$ determinant $d(x) = \det\left[B_1\left(xC_1(x)\right)B_1^T\right]$ and then finds the zeros of the polynomial $d(x)$. A success is reported only if the polynomial has a positive rational zero not equal to 1.

The total number of successes reported for $n \leq 12$ was 87. Some results were duplicates because of symmetry in the p-net. Others contained elements of size 0. Still others had elements of the same size; i.e., the dissections were not perfect. After the answers were checked by hand, the trimmed list contained only the eight dissections given in the introduction and shown in Figures 2–9. We have not attempted to go beyond $n = 12$.

The computational complexity of the problem is interesting to note. The program was run on an HP 712/60 workstation with 32 meg of RAM. The computation for each p-net with $n = 12$ took 90 minutes. Further, the computation consumed so much memory that it was impossible to run two p-nets in succession. So a shell script was devised whereby Maple was entered, one p-net was run, Maple was exited, and the process was begun anew. The case $n = 12$ required $5\frac{1}{2}$ days of continuous running time. My guess is that the case $n = 13$ would take at least six times as long using the present program.

Where do we go from here? Certainly, we could find the results for larger values of n. We might also investigate compound dissections. I have done this for $1:2$ rectangles dissected into $1:2$ rectangular elements and have obtained a complete list of all such dissections for orders $n \leq 12$. This builds on [8] in which Müller shows, among other results, that the compound dissection of a $1:2$ rectangle into $1:2$ rectangular elements of smallest order is the dissection of an 18×36 rectangle into 8 elements. It would be interesting to know if this is the smallest order for a compound dissection of a $p:q$ rectangle into $p:q$ rectangular elements.

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