MATHEMATICS OF COMPUTATION
Volume 65, Number 215
July 1996, Pages 897–921

TWO-LEVEL ADDITIVE SCHWARZ PRECONDITIONERS FOR NONCONFORMING FINITE ELEMENT METHODS

SUSANNE C. BRENNER

ABSTRACT. Two-level additive Schwarz preconditioners are developed for the nonconforming P1 finite element approximation of scalar second-order symmetric positive definite elliptic boundary value problems, the Morley finite element approximation of the biharmonic equation, and the divergence-free nonconforming P1 finite element approximation of the stationary Stokes equations. The condition numbers of the preconditioned systems are shown to be bounded independent of mesh sizes and the number of subdomains in the case of generous overlap.

1. INTRODUCTION

In this paper we develop two-level additive Schwarz preconditioners for the systems of linear equations resulting from nonconforming finite element approximations of elliptic boundary value problems. We obtain results with optimal convergence rate (i.e., the condition numbers of the preconditioned systems are uniformly bounded) when the overlap between subdomains is generous for the following three cases: (I) the P1 nonconforming finite element for the Laplace equation, (II) the Morley finite element for the biharmonic equation, and (III) the divergence-free P1 nonconforming finite element for the stationary Stokes equations. Our preconditioner is a variant of Dryja and Widlund's (cf. [10]) preconditioner for conforming finite element methods (cf. also [14]).

There is some recent work in this area for scalar second-order equations. Sarkis (cf. [15]) has developed a two-level additive Schwarz method using P1 nonconforming finite elements on both grids, which is insensitive to the jumps in coefficients but converges in a suboptimal rate. Cowsar (cf. [8]) has obtained the optimal convergence rate for a two-level additive Schwarz method using P1 nonconforming finite elements on the fine grid, but P1 conforming finite elements on the coarser grid.

In our approach, both the fine-grid and the coarse-grid spaces are nonconforming. The critical step is therefore the construction of intergrid transfer operators with certain properties. Our construction is based on the connection between the nonconforming finite element space and an appropriate conforming finite element

Received by the editor July 6, 1993 and, in revised form, November 18, 1993 and August 1, 1994.

1991 *Mathematics Subject Classification.* Primary 65F10, 65N30, 65N55.

Key words and phrases. Domain decomposition, additive Schwarz preconditioner, nonconforming finite elements, Laplace equation, biharmonic equation, stationary Stokes equations.

This work was supported in part by the National Science Foundation under Grant No. DMS-92-09332.

space. For problems (I) and (II) we use the P2 conforming Lagrange finite element and the P5 Argyris finite element, respectively. Problem (III) is treated through the connection between the Morley finite element and the divergence-free P1 nonconforming finite element. The results in this paper were first announced in [3].

The rest of this paper is organized as follows. The abstract theory for scalar elliptic problems is developed in §2. We show in §§3 and 4 that the abstract theory is applicable to problems (I) and (II) by constructing the intergrid transfer operators and verifying the assumptions of the abstract theory. In §5 the theory for scalar problems is modified and applied to the elliptic system of stationary Stokes equations.

Throughout the paper we use the following conventions for Sobolev norms and semi-norms of a function v defined on an open set G:

$$\|v\|_{H^m(G)} := \left(\int_G \sum_{|\alpha| \le m} |\partial^\alpha v| \, dx \right)^{1/2}$$

and

$$|v|_{H^m(G)} := \left(\int_G \sum_{|\alpha| = m} |\partial^\alpha v| \, dx \right)^{1/2} .$$

We shall also denote the space of polynomials of degree less than or equal to ℓ on G by $\mathcal{P}_\ell(G)$.

2. Abstract theory

Here we will develop a theory for scalar elliptic equations which satisfy homogeneous Dirichlet boundary conditions. Let Ω be a bounded polygonal domain in $\mathbb{R}^2$. We assume that $\Omega = \bigcup_{j=1}^J \Omega_j$, where Ω_j are open subdomains of Ω. Let $\mathcal{T}_H$ be a quasi-uniform triangulation of Ω and $\mathcal{T}_h$ be a subdivision of $\mathcal{T}_H$ such that $\mathcal{T}_h$ is aligned with each $\partial\Omega_j$. The parameters H and h represent the mesh sizes. We assume that there exist nonnegative C^∞ functions $\theta_1, \theta_2, \cdots, \theta_J$ in $\mathbb{R}^2$ such that

$$\theta_j = 0 \quad \text{on } \Omega \backslash \Omega_j, \tag{2.1}$$

$$\sum_{j=1}^{J} \theta_j = 1 \quad \text{on } \overline{\Omega}, \tag{2.2}$$

$$\|\nabla \theta_j\|_{L^\infty} \le \frac{C}{\delta}, \quad \|\nabla^2 \theta_j\|_{L^\infty} \le \frac{C}{\delta^2}, \tag{2.3}$$

where $\nabla^2\theta$ is the Hessian, C is a universal constant and δ is a parameter, $0 < h \le C_1\delta$, $0 < \delta \le C_2 H$. The constructions of Ω_j and θ_j are standard (cf. [10]). The parameter δ measures the amount of overlap among the subdomains Ω_j. From now on, C (with or without subscripts) will denote a generic positive constant independent of h, H, δ, and J. We assume that there exists an integer N_c independent of h, H, δ, and J such that any point in Ω can belong to at most N_c subregions. We shall also assign the value 1 to the parameter k for second-order problems, and the value 2 for fourth-order problems.

Let V_h be a finite element space associated with $\mathcal{T}_h$ whose members vanish at the boundary nodes, and V_j be the subspace of V_h whose members vanish at all nodes that are not interior to Ω_j. The existence of the partition of unity $\theta_1, \theta_2, \cdots, \theta_J$ implies that

$$V_h = \sum_{j=1}^{J} V_j. \tag{2.4}$$

Also let V_H be a finite element space associated with the triangulation $\mathcal{T}_H$ whose members vanish at the boundary nodes of Ω. The members of V_h and V_H are piecewise polynomials of degree less than or equal to k.

The discretized problem is:
Find $u \in V_h$ such that

$$a_h(u,v) = F(v) \quad \forall\, v \in V_h, \tag{2.5}$$

where $a_h(\cdot,\cdot)$ is a positive definite symmetric bilinear form on V_h and $F \in V_h'$. We assume that there is also a related positive definite bilinear form $a_H(\cdot,\cdot)$ defined on V_H.

For the description of the preconditioner we adopt the notation in [18]. Let $(\cdot,\cdot)_h$ and $(\cdot,\cdot)_H$ be two inner products on V_h and V_H respectively.

We define $A_h : V_h \longrightarrow V_h$, $A_j : V_j \longrightarrow V_j$ and $A_H : V_H \longrightarrow V_H$ by

$$(A_h v, w)_h = a_h(v,w) \quad \forall\, v,w \in V_h, \tag{2.6}$$
$$(A_j v, w)_h = a_h(v,w) \quad \forall\, v,w \in V_j, \tag{2.7}$$
$$(A_H v, w)_H = a_H(v,w) \quad \forall\, v,w \in V_H. \tag{2.8}$$

The operators $Q_j : V_h \longrightarrow V_j$, $1 \le j \le J$, are defined by

$$(Q_j v, w)_h = (v,w)_h \quad \forall\, v \in V_h, w \in V_j. \tag{2.9}$$

The operators $P_j : V_h \longrightarrow V_j$, $1 \le j \le J$, are defined by

$$a_h(P_j v, w) = a_h(v,w) \quad \forall\, v \in V_h, w \in V_j. \tag{2.10}$$

It can be easily proved that

$$A_j P_j = Q_j A_h, \quad 1 \le j \le J. \tag{2.11}$$

We assume that there is an intergrid transfer operator $I_H^h : V_H \longrightarrow V_h$ such that

$$|I_H^h v|_{H^k(\mathcal{T}_h)} \le C\, |v|_{H^k(\mathcal{T}_H)} \quad \forall\, v \in V_H, \tag{A.1a}$$
$$|I_H^h v - v|_{H^\ell(\mathcal{T}_h)} \le C\, H^{k-\ell}\, |v|_{H^k(\mathcal{T}_H)} \quad \forall\, v \in V_H,\ 0 \le \ell \le k-1, \tag{A.1b}$$

where the (possibly) nonconforming norms $|\cdot|_{H^m(\mathcal{T}_h)}$ and $|\cdot|_{H^m(\mathcal{T}_H)}$ are defined by

$$|v|_{H^m(\mathcal{T}_h)} := \sqrt{\sum_{T\in\mathcal{T}_h} |v|^2_{H^m(T)}} \quad \forall v \in V_h,$$

$$|v|_{H^m(\mathcal{T}_H)} := \sqrt{\sum_{T\in\mathcal{T}_H} |v|^2_{H^m(T)}} \quad \forall v \in V_H.$$

We also assume that

(A.2a) $\sqrt{a_h(v,v)}$ (resp.,$\sqrt{a_H(v,v)}$) is equivalent to $|v|_{H^k(\mathcal{T}_h)}$ (resp.,$|v|_{H^k(\mathcal{T}_H)}$)

for $v \in V_h$ (resp., $v \in V_H$), and

(A.2b) $$a_h(v,w) \le C\,|v|_{H^k(\mathcal{T}_{h,j})}\,|w|_{H^k(\mathcal{T}_h)}$$

for all $v \in V_h$, $w \in V_j$, where

$$|v|_{H^k(\mathcal{T}_{h,j})} = \left(\sum_{\substack{T\subset\Omega_j\\ T\in\mathcal{T}_h}} |v|^2_{H^k(T)}\right)^{1/2}.$$

The operators I_h^H, $P_h^H : V_h \longrightarrow V_H$ are defined by

(2.12) $$(I_h^H v, w)_H = (v, I_H^h w)_h \quad \forall v \in V_h, w \in V_H,$$

(2.13) $$a_H(P_h^H v, w) = a_h(v, I_H^h w) \quad \forall v \in V_h, w \in V_H.$$

In terms of operators, we can also express (2.13) as

(2.14) $$A_H P_h^H = I_h^H A_h.$$

Also, (A.1a), (A.2a) and (2.13) imply that

(2.15) $$|P_h^H v|_{H^k(\mathcal{T}_H)} \le C|v|_{H^k(\mathcal{T}_h)} \quad \forall v \in V_h.$$

The two-level additive Schwarz preconditioner $B : V_h \longrightarrow V_h$ is defined by

(2.16) $$B := I_H^h R_H I_h^H + \sum_{j=1}^{J} R_j Q_j,$$

where R_H (resp., R_j) is an approximate solver of A_H (resp., A_j) which is symmetric positive definite with respect to $(\cdot,\cdot)_H$ (resp., $(\cdot,\cdot)_h$).

The discretized problem (2.5) can be written as:

(2.17) $$A_h u = f,$$

where $F(v) = (f, v)_h \quad \forall v \in V_h$.

The preconditioned system is:

$$BA_h u = Bf. \tag{2.18}$$

The operator BA_h is symmetric positive definite with respect to $a_h(\cdot,\cdot)$ because of the defining properties of the various operators and (2.4), and hence has positive eigenvalues $0 < \lambda_{\min}(BA_h) \leq \cdots \leq \lambda_{\max}(BA_h)$. Our goal is to show that

$$\frac{\lambda_{\max}(BA_h)}{\lambda_{\min}(BA_h)} \leq C. \tag{2.19}$$

By (2.11) and (2.14) we have

$$\begin{aligned} BA_h &= I_H^h R_H I_h^H A_h + \sum_{j=1}^{J} R_j Q_j A_h \\ &= I_H^h R_H A_H P_h^H + \sum_{j=1}^{J} R_j A_j P_j. \end{aligned} \tag{2.20}$$

Note that in the case where $R_H = A_H^{-1}$ and $R_j = A_j^{-1}$, (2.20) can be simplified to

$$BA_h = I_H^h P_h^H + \sum_{j=1}^{J} P_j = \left(I_H^h A_H^{-1} I_h^H\right) A_h + \sum_{j=1}^{J} P_j. \tag{2.21}$$

Comparing (2.21) with equation (13) in [15], we see that the two preconditioners are slightly different.

The techniques we use to bound the eigenvalues are based on the ideas of Dryja and Widlund in [10] and [11] (see also [18]). Their theory has also been extended by Zhang (cf. [19]) to fourth-order problems in the case of conforming finite elements. We begin with the upper bound for the eigenvalues of BA_h.

Lemma 2.1. *The following inequality holds:*

$$\sum_{j=1}^{J} a_h(P_j v, P_j v) \leq C\, N_c\, a_h(v,v) \quad \forall\, v \in V_h. \tag{2.22}$$

Proof. By (A.2) and (2.10) we have

$$\begin{aligned} a_h(P_j v, P_j v) &= a_h(v, P_j v) \\ &\leq C\,|v|_{H^k(\mathcal{T}_{h,j})}\,|P_j v|_{H^k(\mathcal{T}_h)} \\ &\leq C\,|v|_{H^k(\mathcal{T}_{h,j})}\sqrt{a_h(P_j v, P_j v)}. \end{aligned}$$

Therefore,

$$a_h(P_j v, P_j v) \leq C\,|v|^2_{H^k(\mathcal{T}_{h,j})}.$$

Summing over $1 \leq j \leq J$, we find by (A.2a) and the definition of N_c that

$$\sum_{j=1}^{J} a_h(P_j v, P_j v) \leq C\, N_c\, |v|^2_{H^k(\mathcal{T}_h)} \leq C\, N_c\, a_h(v,v). \quad \square$$

Lemma 2.2. *The following upper bound for the eigenvalues of BA_h holds:*

$$\lambda_{\max}(BA_h) \leq C\, \omega_1\, N_c,$$

where $\omega_1 := \max(\rho(R_H A_H), \rho(R_1 A_1), \cdots, \rho(R_J A_J))$ and $\rho(\cdot)$ denotes the spectral radius.

Proof. By (2.20), (2.13), (2.15), (2.22) and (A.2a) we have

$$\begin{aligned} a_h(BA_h v, v) &= a_h(I_H^h R_H A_H P_h^H v, v) + \sum_{j=1}^{J} a_h(R_j A_j P_j v, v) \\ &= a_H(R_H A_H P_h^H v, P_h^H v) + \sum_{j=1}^{J} a_h(R_j A_j P_j v, P_j v) \\ &\leq \omega_1 \left[a_H(P_h^H v, P_h^H v) + \sum_{j=1}^{J} a_h(P_j v, P_j v) \right] \\ &\leq \omega_1 [C\, a_h(v,v) + C\, N_c\, a_h(v,v)] \\ &\leq C\, \omega_1\, N_c\, a_h(v,v). \end{aligned} \tag{2.23}$$

In this derivation we have also used the fact that $R_H A_H$ (resp., $R_j A_j$) is symmetric positive definite with respect to $a_H(\cdot,\cdot)$ (resp., $a_h(\cdot,\cdot)|_{V_j}$). The lemma follows immediately from (2.23). $\square$

We now turn our attention to the lower bound for the eigenvalues of BA_h. We assume that there exists an operator $J_h^H : V_h \longrightarrow V_H$ with the following properties:

$$|J_h^H v|_{H^k(\mathcal{T}_H)} \leq C\, |v|_{H^k(\mathcal{T}_h)} \quad \forall\, v \in V_h, \tag{A.3a}$$

$$|J_h^H v - v|_{H^\ell(\mathcal{T}_h)} \leq C\, H^{k-\ell}\, |v|_{H^k(\mathcal{T}_h)} \quad \forall\, v \in V_h,\ 0 \leq \ell \leq k-1. \tag{A.3b}$$

Note that in our theory, the finite element spaces V_h and V_H are connected by the operators I_H^h, J_h^H, (A.1) and (A.3), but only I_H^h appears in the preconditioner.

Let Π_h be the nodal interpolation operator associated with $\mathcal{T}_h$. We assume that

$$|\Pi_h(\lambda v)|_{H^k(T)} \leq C\, |\lambda v|_{H^k(T)} \quad \forall\, T \in \mathcal{T}_h,\ v \in \mathcal{P}_k(T),\ \lambda \in \mathcal{P}_{k-1}(T) \tag{A.4a}$$

and

$$\|\Pi_h(gv)\|_{L^2(T)} \leq C\, \left(\|g\|_{L^\infty(T)} + (k-1)\, h\, \|\nabla g\|_{L^\infty(T)} \right) \|v\|_{L^2(T)} \tag{A.4b}$$

$\forall\, T \in \mathcal{T}_h,\ v \in \mathcal{P}_k(T),\ g \in C^\infty(\bar{T})$, where C only depends on the minimum angle in $\mathcal{T}_h$.

Lemma 2.3. *Given any $v \in V_h$, there exists $v_0 \in V_H$, $v_j \in V_j$ $(1 \le j \le J)$ such that*

$$v = I_H^h v_0 + \sum_{j=1}^{J} v_j \tag{2.24}$$

and

$$a_H(v_0, v_0) + \sum_{j=1}^{J} a_h(v_j, v_j) \le C\, N_c \left(1 + \left(\frac{H}{\delta}\right)^{2k}\right) a_h(v, v). \tag{2.25}$$

Proof. Let $v_0 = J_h^H v$ and $v_j = \Pi_h\left(\theta_j(v - I_H^h v_0)\right)$, where Π_h is the nodal variable interpolation operator associated with V_h. Clearly, (2.24) holds.

We treat the cases $k = 1$ and $k = 2$ separately. For $k = 1$, let $\bar{\theta}_{j,T} = \frac{1}{|T|}\int_T \theta_j\, dx$ for all $T \in \mathcal{T}_h$. Then we have by a straightforward computation that

$$\|\theta_j - \bar{\theta}_{j,T}\|_{L^\infty(T)} \le h\, \|\nabla\theta_j\|_{L^\infty(T)}. \tag{2.26}$$

Let $w = v - I_H^h v_0$. Then by the triangle inequality, a standard inverse estimate (cf. [7, 6]), (A.4), (2.26), (2.2) and (2.3) we have

$$\begin{aligned} |v_j|_{H^1(T)} &= |\Pi_h(\theta_j w)|_{H^1(T)} \\ &\le |\bar{\theta}_{j,T} w|_{H^1(T)} + |\Pi_h[(\theta_j - \bar{\theta}_{j,T})w]|_{H^1(T)} \\ &\le |w|_{H^1(T)} + C\, h^{-1}\|\Pi_h[(\theta_j - \bar{\theta}_{j,T})w]\|_{L^2(T)} \\ &\le |w|_{H^1(T)} + C\, h^{-1}\, \|\theta_j - \bar{\theta}_{j,T}\|_{L^\infty(T)}\, \|w\|_{L^2(T)} \\ &\le |w|_{H^1(T)} + \frac{C}{\delta}\|w\|_{L^2(T)}. \end{aligned} \tag{2.27}$$

Summing the square of (2.27) over T in Ω_j, we find by (A.2b) that

$$a_h(v_j, v_j) \le C\left(|w|^2_{H^1(\mathcal{T}_{h,j})} + \frac{1}{\delta^2}\|w\|^2_{L^2(\Omega_j)}\right). \tag{2.28}$$

Summing (2.28) for $1 \le j \le J$, we obtain

$$\sum_{j=1}^{J} a_h(v_j, v_j) \le C\, N_c\left(a_h(w, w) + \frac{1}{\delta^2}\|w\|^2_{L^2(\Omega)}\right). \tag{2.29}$$

By (A.1a) and (A.3a) we have

$$\begin{aligned} |w|_{H^1(\mathcal{T}_h)} &= |v - I_H^h v_0|_{H^1(\mathcal{T}_h)} \\ &\le |v|_{H^1(\mathcal{T}_h)} + |I_H^h v_0|_{H^1(\mathcal{T}_h)} \\ &\le |v|_{H^1(\mathcal{T}_h)} + C\, |v_0|_{H^1(\mathcal{T}_H)} \\ &= |v|_{H^1(\mathcal{T}_h)} + C\, |J_h^H v|_{H^1(\mathcal{T}_H)} \\ &\le C\, |v|_{H^1(\mathcal{T}_h)}, \end{aligned}$$

which together with (A.2a) imply that

$$a_h(w,w) \le C\, a_h(v,v). \tag{2.30}$$

On the other hand, by (A.3b), (A.1b), (A.3a) and (A.2a),

$$\begin{aligned}\|w\|_{L^2(\Omega)} &= \|v - I_H^h J_h^H v\|_{L^2(\Omega)} \\ &\le \|v - J_h^H v\|_{L^2(\Omega)} + \|J_h^H v - I_H^h J_h^H v\|_{L^2(\Omega)} \\ &\le C\,H\,|v|_{H^1(\mathcal{T}_h)} + C\,H\,|J_h^H v|_{H^1(\mathcal{T}_H)} \\ &\le C\,H\,|v|_{H^1(\mathcal{T}_h)} \\ &\le C\,H\,\sqrt{a_h(v,v)}.\end{aligned} \tag{2.31}$$

Similarly, by (A.2a) and (A.3a),

$$a_H(v_0,v_0) \le C\, a_h(v,v). \tag{2.32}$$

Inequality (2.25) now follows from (2.29)–(2.32).

For $k=2$, let $\tilde{\theta}_{j,T}$ be the linear interpolant of θ_j on T, i.e., $\tilde{\theta}_{j,T} \in \mathcal{P}_1(T)$ and $\tilde{\theta}_{j,T} = \theta_j$ at the vertices of T. It is clear that

$$\|\tilde{\theta}_{j,T}\|_{L^\infty(T)} \le \|\theta_j\|_{L^\infty(T)} \quad \text{and} \quad \|\nabla\tilde{\theta}_{j,T}\|_{L^\infty(T)} \le C\|\nabla\theta_j\|_{L^\infty(T)}, \tag{2.33}$$

where C depends only on the minimum angle of the triangulation $\mathcal{T}_h$.

By a simple homogeneity (scaling) argument we also have

$$\begin{aligned}&\|\theta_j - \tilde{\theta}_{j,T}\|_{L^\infty(T)} + h\,\|\nabla(\theta_j - \tilde{\theta}_{j,T})\|_{L^\infty(T)} \\ &\quad \le C\,h^2\,\|\nabla^2\theta_j\|_{L^\infty(T)},\end{aligned} \tag{2.34}$$

where C again depends only on the minimum angle of the triangulation $\mathcal{T}_h$.

Let $w = v - I_H^h v_0$. Then by the triangle inequality, a standard inverse estimate, (A.4), (2.33), (2.34) and (2.3) we have

(2.35)

$$\begin{aligned}|v_j|_{H^2(T)} &= |\Pi_h(\theta_j w)|_{H^2(T)} \\ &\le |\Pi_h(\tilde{\theta}_{j,T} w)|_{H^2(T)} + |\Pi_h[(\theta_j - \tilde{\theta}_{j,T})w]|_{H^2(T)} \\ &\le C\,|\tilde{\theta}_{j,T} w|_{H^2(T)} + C\,h^{-2}\,\|\Pi_h[(\theta_j - \tilde{\theta}_{j,T})w]\|_{L^2(T)} \\ &\le C\left(\|\tilde{\theta}_{j,T}\|_{L^\infty(T)}\,|w|_{H^2(T)} + \|\nabla\tilde{\theta}_{j,T}\|_{L^\infty(T)}\,|w|_{H^1(T)}\right) \\ &\quad + C\,h^{-2}\left(\|\theta_j - \tilde{\theta}_{j,T}\|_{L^\infty(T)} + h\,\|\nabla(\theta_j - \tilde{\theta}_{j,T})\|_{L^\infty(T)}\right)\|w\|_{L^2(T)} \\ &\le C\left(|w|_{H^2(T)} + \frac{1}{\delta}|w|_{H^1(T)} + \frac{1}{\delta^2}\|w\|_{L^2(T)}\right).\end{aligned}$$

Summing up the square of (2.35) over T in Ω_j, we find by (A.2b) that

$$a_h(v_j,v_j) \le C\left(|w|^2_{H^2(\mathcal{T}_{h,j})} + \frac{1}{\delta^2}|w|^2_{H^1(\mathcal{T}_{h,j})} + \frac{1}{\delta^4}\|w\|^2_{L^2(\Omega_j)}\right). \tag{2.36}$$

Summing up (2.36) for $1 \leq j \leq J$, we obtain

$$\sum_{j=1}^{J} a_h(v_j, v_j) \leq C\, N_c \left(a_h(w,w) + \frac{1}{\delta^2} |w|^2_{H^1(\mathcal{T}_h)} + \frac{1}{\delta^4} \|w\|^2_{L^2(\Omega)} \right). \tag{2.37}$$

As in the previous case we deduce from (A.1a), (A.2), and (A.3a) that

$$a_h(w,w) \leq C\, a_h(v,v) \tag{2.38}$$

and

$$a_H(v_0, v_0) \leq C\, a_h(v,v). \tag{2.39}$$

Also, analogous to (2.31), we have by (A.1), (A.2) and (A.3) that

$$\|w\|_{L^2(\Omega)} + H\, |w|_{H^1(\mathcal{T}_h)} \leq C\, H^2 \sqrt{a_h(v,v)}. \tag{2.40}$$

Inequality (2.25) now follows by combining (2.37)–(2.40). □

Lemma 2.4. *The following lower bound for the eigenvalues of BA_h holds:*

$$\lambda_{\min}(BA_h) \geq C \frac{\omega_0}{N_c(1 + (\frac{H}{\delta})^{2k})}, \tag{2.41}$$

where $\omega_0 := \min(\lambda_{\min}(R_H A_H), \lambda_{\min}(R_1 A_1), \cdots, \lambda_{\min}(R_J A_J))$.

Proof. Let $\beta := N_c(1 + (\frac{H}{\delta})^{2k})$. Given any $v \in V_h$, by Lemma 2.3 there exists $v_0 \in V_H$, $v_j \in V_j$ $(1 \leq j \leq J)$ such that (2.24) and (2.25) hold. It follows from (2.25) that

$$\begin{aligned} C\,\beta\, a_h(v,v) &\geq a_H(v_0, v_0) + \sum_{j=1}^{J} a_h(v_j, v_j) \\ &= (R_H^{-1} R_H A_H v_0, v_0)_H + \sum_{j=1}^{J} (R_j^{-1} R_j A_j v_j, v_j)_h \\ &\geq \omega_0 \left[(R_H^{-1} v_0, v_0)_H + \sum_{j=1}^{J} (R_j^{-1} v_j, v_j)_h \right], \end{aligned} \tag{2.42}$$

where we have used the fact that $R_H A_H : V_H \longrightarrow V_H$ (resp., $R_j A_j : V_j \longrightarrow V_j$) is positive definite with respect to $(R_H^{-1}\cdot, \cdot)_H$ (resp., $(R_j^{-1}\cdot, \cdot)_h$).

Let $T_H := I_H^h R_H A_H P_h^H$ and $T_j := R_j A_j P_j$. In other words, we can rewrite (2.20) as

$$BA_h = T_H + \sum_{j=1}^{J} T_j. \tag{2.43}$$

Using (2.24), (2.13), (2.10), the Cauchy-Schwarz inequality, (2.42) and (2.43), we have

$$
\begin{aligned}
a_h(v,v) &= a_h(I_H^h v_0, v) + \sum_{j=1}^{J} a_h(v_j, v) \\
&= a_H(v_0, P_h^H v) + \sum_{j=1}^{J} a_h(v_j, P_j v) \\
&= (R_H^{-1/2} v_0, R_H^{1/2} A_H P_h^H v)_H + \sum_{j=1}^{J} (R_j^{-1/2} v_j, R_j^{1/2} A_j P_j v)_h \\
&\le (R_H^{-1} v_0, v_0)_H^{1/2} (A_H P_h^H v, R_H A_H P_h^H v)_H^{1/2} \\
&\quad + \sum_{j=1}^{J} (R_j^{-1} v_j, v_j)_h^{1/2} (A_j P_j v, R_j A_j P_j v)_h^{1/2} \\
&= (R_H^{-1} v_0, v_0)_H^{1/2} a_H(P_h^H v, R_H A_H P_h^H v)^{1/2} \\
&\quad + \sum_{j=1}^{J} (R_j^{-1} v_j, v_j)_h^{1/2} a_h(P_j v, R_j A_j P_j v)^{1/2} \\
&= (R_H^{-1} v_0, v_0)_H^{1/2} a_h(v, T_H v)^{1/2} + \sum_{j=1}^{J} (R_j^{-1} v_j, v_j)_h^{1/2} a_h(v, T_j v)^{1/2} \\
&\le \left((R_H^{-1} v_0, v_0)_H + \sum_{j=1}^{J} (R_j^{-1} v_j, v_j)_h \right)^{1/2} \left(a_h(v, T_H v) + \sum_{j=1}^{J} a_h(v, T_j v) \right)^{1/2} \\
&\le \frac{C\beta^{1/2}}{\omega_0^{1/2}}\, a_h(v,v)^{1/2}\, a_h(v, BA_h v)^{1/2},
\end{aligned}
$$

which implies that

$$
a_h(v,v) \le \frac{C\beta}{\omega_0} a_h(v, BA_h v). \tag{2.44}
$$

Inequality (2.41) now follows immediately from (2.44). □

In summary, we have the following theorem.

Theorem 2.1. *Under the geometric assumptions* (2.1)–(2.3) *and the assumptions* (A.1)–(A.4) *on the finite element spaces we have*

$$
\frac{\lambda_{\max}(BA_h)}{\lambda_{\min}(BA_h)} \le C\,\frac{\omega_1}{\omega_0}\, N_c^2 \left(1 + \left(\frac{H}{\delta} \right)^{2k} \right).
$$

Therefore, if the approximate solvers R_H and R_j are accurate enough so that ω_1 is bounded and ω_0 is bounded away from zero, and if the overlap between subregions is generous enough so that $\frac{H}{\delta}$ is bounded, then the condition number of the preconditioned system is bounded independent of h, δ, H and J.

Remark. If we are more careful about the definition of Ω_j and θ_j, then the factor $1 + \left(\frac{H}{\delta}\right)^{2k}$ of Theorem 2.1 can be reduced to $1 + \left(\frac{H}{\delta}\right)^{2k-1}$. This is done by using a trace theorem type argument in [17], which can be applied to nonconforming finite elements after a slight modification (cf. [4]).

3. Scalar P1 nonconforming finite element

In this section we apply the abstract theory to the P1 nonconforming finite element (cf. [9]) approximation of the Laplace equation. The finite element space V_h is defined by

$$V_h := \{v \in L^2(\Omega) : v \in \mathcal{P}_1(T)\ \forall T \in \mathcal{T}_h, v \text{ is continuous at the midpoints of interelement boundaries, and } v \text{ vanishes at the midpoints along } \partial\Omega\}.$$

V_H is defined the same way with respect to $\mathcal{T}_H$. Members of V_h (resp., V_H) are completely determined by their values at the midpoints of $\mathcal{T}_h$ (resp., $\mathcal{T}_H$).

The symmetric positive definite bilinear forms $a_h(\cdot,\cdot)$ and $a_H(\cdot,\cdot)$ are given by

$$a_h(v_1, v_2) := \sum_{T\in\mathcal{T}_h} \int_T \nabla v_1 \cdot \nabla v_2 \, dx \quad \forall v_1, v_2 \in V_h, \tag{3.1}$$

and

$$a_H(v_1, v_2) := \sum_{T\in\mathcal{T}_H} \int_T \nabla v_1 \cdot \nabla v_2 \, dx \quad \forall v_1, v_2 \in V_H. \tag{3.2}$$

The inner products $(\cdot,\cdot)_h$ and $(\cdot,\cdot)_H$ are just the L^2-inner products restricted to V_h and V_H, respectively. Note that the natural nodal basis functions in V_h are L^2-orthogonal, so that the constructions of the Q_j are trivial.

Assumptions (A.2) and (A.4a) are trivially satisfied, while (A.4b) follows from the following quadrature formula:

$$\int_T v^2 dx = \frac{|T|}{3}\left[(v(m_1))^2 + (v(m_2))^2 + (v(m_3))^2\right] \quad \forall v \in \mathcal{P}_1(T),$$

where m_1, m_2, and m_3 are the midpoints of the three sides of the triangle T.

It only remains to define the operators I_H^h and J_h^H, and to verify assumptions (A.1a), (A.1b), (A.3a), and (A.3b). We introduce two other finite element spaces W_h and W_H, where

$$W_h := \{w \in C(\bar{\Omega}) : w|_T \in \mathcal{P}_2(T) \quad \forall T \in \mathcal{T}_h \text{ and } w = 0 \text{ on } \partial\Omega\},$$

and W_H is defined similarly with respect to $\mathcal{T}_H$. The members of W_h (resp., W_H) are completely determined by their values at the vertices and midpoints of $\mathcal{T}_h$ (resp., $\mathcal{T}_H$). Note that $W_H \subset W_h$ since $\mathcal{T}_h$ is a subdivision of $\mathcal{T}_H$.

We define $E_h : V_h \longrightarrow W_h$ and $F_h : W_h \longrightarrow V_h$ by

$$\begin{cases} (E_h v)(m) = v(m) & \text{for all internal midpoints } m \in \mathcal{T}_h, \\ (E_h v)(p) = \text{average of } v_i(p) & \text{for all internal vertices } p \in \mathcal{T}_h, \end{cases} \tag{3.3}$$

where $v_i = v|_{T_i}$ and $T_i \in \mathcal{T}_h$ contains p as a vertex, and

$$(F_h w)(m) = w(m) \quad \text{for all midpoints } m \in \mathcal{T}_h. \tag{3.4}$$

The operators $E_H : V_H \longrightarrow W_H$ and $F_H : W_H \longrightarrow V_H$ are defined similarly with respect to $\mathcal{T}_H$.

The intergrid transfer operator $I_H^h : V_H \longrightarrow V_h$ is given by

$$I_H^h := F_h \circ E_H. \tag{3.5}$$

The operator $J_h^H : V_h \longrightarrow V_H$ is given by

$$J_h^H := F_H \circ Q_h^H \circ E_h, \tag{3.6}$$

where $Q_h^H : W_h \longrightarrow W_H$ is the L^2-orthogonal projection operator. The relations of these operators are illustrated by the commutative diagrams in Fig. 1 (where i stands for natural injection).

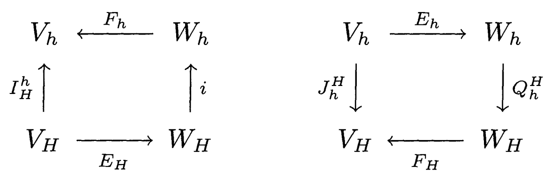

FIGURE 1

Note that I_H^h is represented by a sparse, banded matrix with respect to the natural nodal bases of V_h and V_H.

The estimates (A.1) and (A.3) are established through a sequence of lemmas. The next lemma follows from the result of Bramble and Xu in [2].

Lemma 3.1. *The following estimates on Q_h^H hold:*

$$|Q_h^H w|_{H^1(\Omega)} \le C\,|w|_{H^1(\Omega)} \quad \forall\, w \in W_h, \tag{3.7}$$

$$\|w - Q_h^H w\|_{L^2(\Omega)} \le C\,H\,|w|_{H^1(\Omega)} \quad \forall\, w \in W_h. \tag{3.8}$$

Lemma 3.2. *The following estimates on F_h and F_H hold:*

$$|F_h w|_{H^1(\mathcal{T}_h)} \le C\,|w|_{H^1(\Omega)} \quad \forall\, w \in W_h, \tag{3.9a}$$

$$|F_H w|_{H^1(\mathcal{T}_H)} \le C\,|w|_{H^1(\Omega)} \quad \forall\, w \in W_H, \tag{3.9b}$$

$$\|w - F_h w\|_{L^2(\Omega)} \le C\,h\,|w|_{H^1(\Omega)} \quad \forall\, w \in W_h, \tag{3.10a}$$

$$\|w - F_H w\|_{L^2(\Omega)} \le C\,H\,|w|_{H^1(\Omega)} \quad \forall\, w \in W_H. \tag{3.10b}$$

Proof. It suffices to establish (3.9a) and (3.10a). On a reference triangle $\hat{T}$, $|\cdot|_{H^1(\hat{T})}$ defines a norm on the quotient space $\mathcal{P}_2(\hat{T})/\mathcal{P}_0(\hat{T})$. Given any $w \in \mathcal{P}_2(\hat{T})$, let $w' \in \mathcal{P}_1(\hat{T})$ be defined by $w'(m_i) = w(m_i)$ at the midpoints m_i $(i = 1, 2, 3)$ of $\hat{T}$. Since $w' = w$ if $w \in \mathcal{P}_0(\hat{T})$, $w \longrightarrow w - w'$ is a well-defined linear map from $\mathcal{P}_2(\hat{T})/\mathcal{P}_0(\hat{T})$ into $\mathcal{P}_2(\hat{T})$. Therefore we have

$$\|w - w'\|_{L^2(\hat{T})} \le C\,|w|_{H^1(\hat{T})} \quad \forall\, w \in \mathcal{P}_2(\hat{T}). \tag{3.11}$$

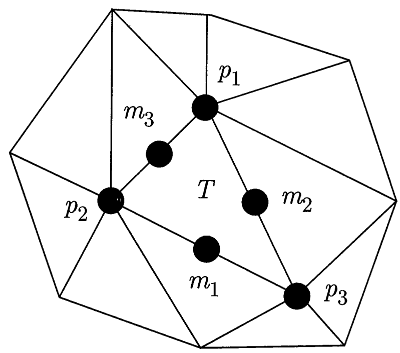

FIGURE 2

The estimate (3.11) together with a homogeneity argument yields

$$\|w - F_h w\|_{L^2(\Omega)} \leq C\, h\, |w|_{H^1(\Omega)} \quad \forall\, w \in W_h, \tag{3.12}$$

where C depends only on the minimum angle of $\mathcal{T}_h$.

By a standard inverse estimate and (3.12), we obtain

$$\begin{aligned} |F_h w|_{H^1(\mathcal{T}_h)} &\leq |w - F_h w|_{H^1(\mathcal{T}_h)} + |w|_{H^1(\mathcal{T}_h)} \\ &\leq C\, h^{-1}\, \|w - F_h w\|_{L^2(\mathcal{T}_h)} + |w|_{H^1(\mathcal{T}_h)} \\ &\leq C\, |w|_{H^1(\Omega)}. \qquad \square \end{aligned}$$

Lemma 3.3. *The following estimates on E_h and E_H hold:*

$$|E_h v|_{H^1(\Omega)} \leq C\, |v|_{H^1(\mathcal{T}_h)} \quad \forall\, v \in V_h, \tag{3.13a}$$

$$|E_H v|_{H^1(\Omega)} \leq C\, |v|_{H^1(\mathcal{T}_H)} \quad \forall\, v \in V_H, \tag{3.13b}$$

$$\|v - E_h v\|_{L^2(\Omega)} \leq C\, h\, |v|_{H^1(\mathcal{T}_h)} \quad \forall\, v \in V_h, \tag{3.14a}$$

$$\|v - E_H v\|_{L^2(\Omega)} \leq C\, H\, |v|_{H^1(\mathcal{T}_H)} \quad \forall\, v \in V_H. \tag{3.14b}$$

Proof. It suffices to establish (3.13a) and (3.14a). Observe that $F_h \circ E_h = \mathrm{Id}$. Hence by (3.9a) we have

$$\begin{aligned} \|v - E_h v\|_{L^2(\Omega)} &= \|F_h(E_h v) - E_h v\|_{L^2(\Omega)} \\ &\leq C\, h\, |E_h v|_{H^1(\Omega)}. \end{aligned} \tag{3.15}$$

In view of (3.15), the whole problem is reduced to proving (3.13a).

Let $T \in \mathcal{T}_h$ be a triangle away from $\partial\Omega$, and G be the union of all triangles in $\mathcal{T}_h$ sharing a vertex with T (cf. Fig. 2). (The triangle T is itself in G.) Let $V_G = \{v \in L^2(G) : v|_T \in \mathcal{P}_1(T) \quad \forall T \subset G, v$ is continuous at the midpoints of the interelement boundaries$\}$.

Given $v \in V_G$, let $v' \in \mathcal{P}_2(T)$ be defined by

$$\begin{cases} v'(m_i) = v(m_i) & \text{for } i = 1, 2, 3, \\ v'(p_i) = \text{average of } v_{ij} \text{ at } p_i & \text{for } i = 1, 2, 3, \end{cases}$$

where $v_{ij} = v|_{T_j}$ and $T_j \subset G$ contains p_i as a vertex.

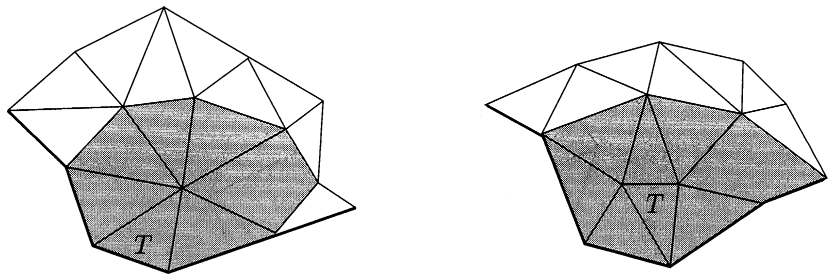

FIGURE 3

Let $|\cdot|_{H^1_*(G)}$ be defined by

$$|v|_{H^1_*(G)} = \left(\sum_{T\subset G} |v|^2_{H^1(T)}\right)^{1/2} \qquad \forall\, v \in V_G.$$

Observe that

$$\begin{aligned}(3.16)\qquad |v|_{H^1_*(G)} = 0 &\Longrightarrow v \in \mathcal{P}_0(G)\\ &\Longrightarrow v' = v \quad \text{on } T\\ &\Longrightarrow |v'|_{H^1(T)} = 0.\end{aligned}$$

It follows from (3.16) and a homogeneity argument using reference triangles that

$$(3.17)\qquad |v'|_{H^1(T)} \le C_1 |v|_{H^1_*(G)} \quad \forall\, v \in V_G,$$

where C depends on the number of triangles in G and the shape of the triangles in G. Since $\mathcal{T}_h$ is quasi-uniform, C_1 ultimately depends on the minimum angle in $\mathcal{T}_h$.

The same estimate holds if the triangle T is close to $\partial\Omega$, in which case the members of V_G will vanish at certain midpoints (cf. Fig. 3).

Summing up the square of (3.17) over all triangles $T \in \mathcal{T}_h$, we obtain (3.13a). □

Proposition 3.1. *Assumptions* (*A*.1) *and* (*A*.3) *hold for* I_H^h *and* J_h^H *defined by* (3.5) *and* (3.6), *respectively.*

Proof. The estimates (A.1a) and (A.3a) follow immediately from the estimates (3.7), (3.9) and (3.13).

Using (3.10a), (3.14a), and (3.13b), we have

$$\begin{aligned}\|I_H^h v - v\|_{L^2(\Omega)} &= \|F_h(E_H v) - v\|_{L^2(\Omega)}\\ &\le \|F_h(E_H v) - E_H v\|_{L^2(\Omega)} + \|E_H v - v\|_{L^2(\Omega)}\\ &\le C\, h\, |E_H v|_{H^1(\Omega)} + C\, H\, |v|_{H^1(\mathcal{T}_H)}\\ &\le C\, H\, |v|_{H^1(\mathcal{T}_H)}.\end{aligned}$$

Similarly, using (3.10b), (3.8), (3.14a), (3.7), and (3.13a), we obtain

$$\begin{aligned}\|J_h^H v - v\|_{L^2(\Omega)} &= \|F_H(Q_h^H(E_h v)) - v\|_{L^2(\Omega)} \\ &\le \|F_H(Q_h^H(E_h v)) - Q_h^H(E_h v)\|_{L^2(\Omega)} \\ &\quad + \|Q_h^H(E_h v) - E_h v\|_{L^2(\Omega)} + \|E_h v - v\|_{L^2(\Omega)} \\ &\le C\,H\,|Q_h^H(E_h v)|_{H^1(\Omega)} + C\,H\,|E_h v|_{H^1(\Omega)} + C\,h\,|v|_{H^1(\mathcal{T}_H)} \\ &\le C\,H\,|v|_{H^1(\mathcal{T}_H)}. \quad \square\end{aligned}$$

Therefore, the abstract theory in §2 is applicable to the case of the scalar P1 nonconforming finite element approximation of the Laplace equation. The generalization to more general symmetric positive definite second-order scalar elliptic problems is straightforward.

Remark. We can also use the P1 conforming finite element space on the coarser grid. Let

$$\tilde{V}_H := \{v \in C(\bar{\Omega}) : v|_T \in \mathcal{P}_1(T) \quad \forall T \in \mathcal{T}_H \text{ and } v|_{\partial\Omega} = 0\}.$$

Since $\tilde{V}_H \subset V_H$ we can define $\tilde{I}_H^h : \tilde{V}_H \longrightarrow V_h$ to be the natural injection. Assumptions (A.1a) and (A.1b) then become trivial for $\tilde{I}_H^h$.

Let the operator $\tilde{F}_H : W_H \longrightarrow \tilde{V}_H$ be defined by

$$(\tilde{F}_H w)(p) = w(p) \quad \text{for all vertices } p \in \mathcal{T}_H.$$

The estimates (3.9b) and (3.10b) with F_H replaced by $\tilde{F}_H$ can be established by arguments analogous to those in the proof of Lemma 3.2. Hence, if we define $\tilde{J}_h^H : V_h \longrightarrow \tilde{V}_H$ by

$$\tilde{J}_h^H := \tilde{F}_H \circ Q_h^H \circ E_h,$$

then assumptions (A.3a) and (A.3b) hold for $\tilde{J}_h^H$. Therefore, our theory is also applicable for these choices and we recover the results in [8].

4. The Morley finite element

In this section we apply the abstract theory to the Morley finite element approximation of the biharmonic equation. Let V_h be the Morley finite element space associated with $\mathcal{T}_h$. Then $v \in V_h$ if and only if it has the following three properties:

(i) $v|_T$ is quadratic for all $T \in \mathcal{T}_h$

(ii) v is continuous at the vertices and vanishes at the vertices along $\partial\Omega$

(iii) $\frac{\partial v}{\partial n}$ is continuous at the midpoints of interelement boundaries and vanishes at the midpoints along $\partial\Omega$.

V_H is defined the same way with respect to $\mathcal{T}_H$. Members of V_h (resp., V_H) are completely determined by their values at vertices of $\mathcal{T}_h$ (resp., $\mathcal{T}_H$) and the values of their normal derivatives at the midpoints of $\mathcal{T}_h$ (resp., $\mathcal{T}_H$). The symmetric positive definite bilinear forms $a_h(\cdot,\cdot)$ and $a_H(\cdot,\cdot)$ are defined by

$$a_h(v_1, v_2) := \sum_{T \in \mathcal{T}_h} \int_T \sum_{i,j=1}^{2} \frac{\partial^2 v_1}{\partial x_i \partial x_j} \frac{\partial^2 v_2}{\partial x_i \partial x_j}\, dx \quad \forall v_1, v_2 \in V_h \tag{4.1}$$

and

$$(4.2)\qquad a_H(v_1,v_2) := \sum_{T\in\mathcal{T}_H}\int_T \sum_{i,j=1}^{2} \frac{\partial^2 v_1}{\partial x_i \partial x_j}\frac{\partial^2 v_2}{\partial x_i \partial x_j}\,dx \quad \forall\, v_1, v_2 \in V_H.$$

The inner product $(\cdot,\cdot)_h$ is defined by

$$(4.3)\qquad (v_1,v_2)_h := h^2\sum_p v_1(p)v_2(p) + h^4\sum_m \frac{\partial v_1}{\partial n}(m)\frac{\partial v_2}{\partial n}(m) \quad \forall\, v_1, v_2 \in V_h,$$

where the summation is over all vertices p and midpoints m of the triangulation $\mathcal{T}_h$. The inner product $(\cdot,\cdot)_H$ is defined analogously with respect to $\mathcal{T}_H$. It follows from a standard calculation using reference elements and a homogeneity argument for almost affine elements (cf. [7]) that

$$(4.4)\qquad C_1\,\|v\|^2_{L^2(T)} \le |T|\sum_{i=1}^{3}(v(p_i))^2 + |T|^2\sum_{i=1}^{3}\left(\frac{\partial v}{\partial n}(m_i)\right)^2 \le C_2\,\|v\|^2_{L^2(T)}$$

for all $v \in \mathcal{P}_2(T)$, where C_1, C_2 depend on the shape of T. Hence we have

$$(4.5\text{a})\qquad C\,\|v\|^2_{L^2(\Omega)} \le (v,v)_h \le C\,\|v\|^2_{L^2(\Omega)} \quad \forall\, v\in V_h$$

and

$$(4.5\text{b})\qquad C\,\|v\|^2_{L^2(\Omega)} \le (v,v)_H \le C\,\|v\|^2_{L^2(\Omega)} \quad \forall\, v\in V_H.$$

Assumption (A.2) is trivially satisfied. Let $T \in \mathcal{T}_h$ and Π be the Morley nodal variable interpolation operator from $C^1(\bar{T})$ into $\mathcal{P}_2(T)$. For $g \in C^1(\bar{T})$ and $v \in \mathcal{P}_2(T)$ we have by (4.4)

(4.6)

$$\begin{aligned}\|\Pi(gv)\|^2_{L^2(T)} &\le C\left(|T|\sum_{i=1}^{3}[g(p_i)v(p_i)]^2 + |T|^2\sum_{i=1}^{3}\left(\frac{\partial(gv)}{\partial n}(m_i)\right)^2\right)\\ &\le C\left(\|g\|^2_{L^\infty(T)}\|v\|^2_{L^2(T)} + |T|^2\|\nabla g\|^2_{L^\infty(T)}\sum_{i=1}^{3}v(m_i)^2\right).\end{aligned}$$

A homogeneity argument shows that

$$(4.7)\qquad |T|\sum_{i=1}^{3}v(m_i)^2 \le C_3\,\|v\|^2_{L^2(T)} \quad \forall\, v \in \mathcal{P}_2(T),$$

where C_3 depends only on the shape of T. Assumption (A.4b) now follows from (4.6) and (4.7).

Next, we verify assumption (A.4a). Let $\hat{T}$ be a reference triangle. Since $|\cdot|_{H^2(\hat{T})}$ is a norm on the space $\mathcal{P}_3(\hat{T})/\mathcal{P}_1(\hat{T})$, and $\zeta \longrightarrow \Pi\zeta - \zeta$ is a well-defined linear map from $\mathcal{P}_3(\hat{T})/\mathcal{P}_1(\hat{T})$ into $\mathcal{P}_3(\hat{T})$, we have

$$|\Pi\zeta - \zeta|_{H^2(\hat{T})} \le C_4\,|\zeta|_{H^2(\hat{T})} \quad \forall\, \zeta \in \mathcal{P}_3(\hat{T}),$$

where C_4 depends only on the shape of the triangle $\hat{T}$. Therefore,

$$|\Pi\zeta|_{H^2(\hat{T})} \leq (1 + C_4)\,|\zeta|_{H^2(\hat{T})} \quad \forall\,\zeta \in \mathcal{P}_3(\hat{T}). \tag{4.8}$$

Assumption (A.4a) now follows from (4.8) and a homogeneity argument for almost affine elements.

It remains to define the operators I_H^h and J_h^H, and to verify assumptions (A.1) and (A.3). As in the case of P1 nonconforming finite elements, we introduce two spaces W_h and W_H, where

$$W_h := \left\{ w \in C^1(\bar{\Omega}) : w|_T \in \mathcal{P}_5(T) \quad \forall\, T \in \mathcal{T}_h \text{ and } w = \frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega \right\}$$

and W_H is defined similarly with respect to $\mathcal{T}_H$. Note that W_h (resp., W_H) contains the Argyris finite element space $\tilde{W}_h$ (resp., $\tilde{W}_H$) whose members are completely determined by the values of their derivatives up to second order at the vertices of $\mathcal{T}_h$ (resp., $\mathcal{T}_H$) and their normal derivatives at the midpoints of $\mathcal{T}_h$ (resp., $\mathcal{T}_H$) (cf. [1]). Note also that $W_H \subset W_h$ since $\mathcal{T}_h$ is a subdivision of $\mathcal{T}_h$, but $\tilde{W}_H \not\subset \tilde{W}_h$.

As in the case of P1 nonconforming finite elements, the operators I_H^h and J_h^H are defined through the commutative diagrams in Fig. 1, where $E_h : V_h \longrightarrow \tilde{W}_h \subseteq W_h$ (resp., $E_H : V_H \longrightarrow \tilde{W}_H \subseteq W_H$) and $F_h : W_h \longrightarrow V_h$ (resp., $F_H : W_H \longrightarrow V_H$) are defined as follows:

$$\begin{cases} (E_h v)(p) = v(p) & \text{for all internal vertices } p \in \mathcal{T}_h, \\ (\partial^\alpha E_h v)(p) = \text{average of } (\partial^\alpha v_i)(p), |\alpha| = 1 & \text{for all internal vertices } p \in \mathcal{T}_h, \\ (\partial^\alpha E_h v)(p) = 0, |\alpha| = 2 & \text{for all internal vertices } p \in \mathcal{T}_h, \\ \left(\frac{\partial}{\partial n} E_h v\right)(m) = \frac{\partial v}{\partial n}(m) & \text{for all internal midpoints } m \in \mathcal{T}_h, \end{cases} \tag{4.9}$$

where $v_i = v|_{T_i}$ and T_i contains p as a vertex, and

$$\begin{cases} (F_h w)(p) = w(p) & \text{for all internal vertices } p \in \mathcal{T}_h, \\ \left(\frac{\partial}{\partial n}(F_h w)\right)(m) = \frac{\partial w}{\partial n}(m) & \text{for all internal midpoints } m \in \mathcal{T}_h. \end{cases} \tag{4.10}$$

The nodal values of $E_h v$ are zero along $\partial\Omega$. The definitions of E_H and F_h are the same with respect to $\mathcal{T}_H$.

The operators $I_H^h : V_H \longrightarrow V_h$ and $J_h^H : V_h \longrightarrow V_H$ are then defined by

$$I_H^h := F_h \circ E_H \tag{4.11}$$

and

$$J_h^H := F_H \circ Q_h^H \circ E_h, \tag{4.12}$$

where $Q_h^H : W_h \longrightarrow W_H$ is the L^2-orthogonal projection operator.

Again, note that I_H^h is represented by a sparse, banded matrix with respect to the natural nodal bases of V_h and V_H.

Lemma 4.1. *The following estimates on Q_h^H hold:*

$$|Q_h^H w|_{H^2(\Omega)} \leq C\,|w|_{H^2(\Omega)}, \tag{4.13}$$

$$\|w - Q_h^H w\|_{L^2(\Omega)} + H\,|w - Q_h^H w|_{H^1(\Omega)} \leq C\,H^2\,|w|_{H^2(\Omega)}, \tag{4.14}$$

for all $w \in W_h$.

Proof. Let $w \in W_h$. By Theorem 4.1.2 in [19], there exits $w' \in W_H$ such that

$$|w - w'|_{H^s(\Omega)} \leq C\,H^{2-s}\,|w|_{H^2(\Omega)}, \quad s = 0, 1, 2. \tag{4.15}$$

Let $Q_H : L^2(\Omega) \longrightarrow W_H$ be the L^2-orthogonal projection operator (hence $Q_h^H = Q_H\big|_{W_h}$). By (4.15) and standard inverse estimates, we have for $s = 0, 1, 2$,

$$\begin{aligned} |w - Q_h^H|_{H^s(\Omega)} &\leq |w - w'| + |Q_H(w' - w)|_{H^s(\Omega)} \\ &\leq C H^{2-s}|w|_{H^2(\Omega)} + C H^{-s}\|Q_H(w' - w)\|_{L^2(\Omega)} \\ &\leq C H^{2-s}|w|_{H^2(\Omega)} + C H^{-s}\|(w' - w)\|_{L^2(\Omega)} \\ &\leq C H^{2-s}|w|_{H^2(\Omega)}\,. \end{aligned} \tag{4.16}$$

The estimates (4.13) and (4.14) follow immediately from (4.16). □

The proof of the following lemma is similar to the proof of Lemma 3.2 and is therefore omitted.

Lemma 4.2. *The following estimates on F_h and F_H hold:*

$$|F_h w|_{H^2(\mathcal{T}_h)} \leq C\,|w|_{H^2(\Omega)}, \tag{4.17a}$$

$$|F_H \tilde{w}|_{H^2(\mathcal{T}_H)} \leq C\,|\tilde{w}|_{H^2(\Omega)}, \tag{4.17b}$$

$$\|w - F_h w\|_{L^2(\Omega)} + h\,|w - F_h w|_{H^1(\mathcal{T}_h)} \leq C\,h^2\,|w|_{H^2(\Omega)}, \tag{4.18a}$$

$$\|\tilde{w} - F_H \tilde{w}\|_{L^2(\Omega)} + H\,|\tilde{w} - F_H \tilde{w}|_{H^1(\mathcal{T}_H)} \leq C\,H^2\,|\tilde{w}|_{H^2(\Omega)}, \tag{4.18b}$$

for all $w \in W_h$ and $\tilde{w} \in W_H$.

Lemma 4.3. *The following estimates on E_h and E_H hold:*

$$|E_h v|_{H^2(\Omega)} \leq C\,|v|_{H^2(\mathcal{T}_h)}, \tag{4.19a}$$

$$|E_H \tilde{v}|_{H^2(\Omega)} \leq C\,|\tilde{v}|_{H^2(\mathcal{T}_H)}, \tag{4.19b}$$

$$\|v - E_h v\|_{L^2(\Omega)} + h\,|v - E_h v|_{H^1(\mathcal{T}_h)} \leq C\,h^2\,|v|_{H^2(\mathcal{T}_h)}, \tag{4.20a}$$

$$\|\tilde{v} - E_H \tilde{v}\|_{L^2(\Omega)} + H\,|\tilde{v} - E_H \tilde{v}|_{H^1(\mathcal{T}_H)} \leq C\,H^2\,|\tilde{v}|_{H^2(\mathcal{T}_H)}, \tag{4.20b}$$

for all $v \in V_h$ and $\tilde{v} \in V_H$.

Proof. It suffices to establish (4.19a) and (4.20a). Let $v \in V_h$, $T \in \mathcal{T}_h$, $w = v|_T$ and $\tilde{w} = (E_h v)|_T$. The two functions $w, \tilde{w} \in \mathcal{P}_5(T)$ are related by

$$w - \tilde{w} = \sum_{i=1}^{3} \sum_{|\alpha|=1,2} \partial_\alpha (w - \tilde{w})(p_i) r_{\alpha,i}, \tag{4.21}$$

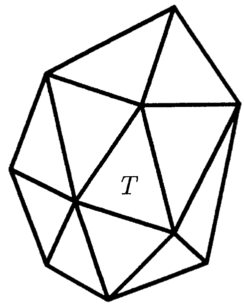

FIGURE 4

where the points p_i are the vertices of T, and the functions $r_{\alpha,i}$ are the nodal basis functions corresponding to the nodal variables $(\partial_\alpha v)(p_i)$ of the Argyris finite element. The following estimates are obtained by the standard techniques of almost affine family of finite elements (cf. [7]):

$$\|r_{\alpha,i}\|_{L^2(T)} \leq C(T)h_T^2 \quad \text{for } |\alpha| = 1, \tag{4.22}$$

$$\|r_{\alpha,i}\|_{L^2(T)} \leq C(T)h_T^3 \quad \text{for } |\alpha| = 2, \tag{4.23}$$

where $h_T = \operatorname{diam} T$, and $C(T)$ represents a generic positive constant which depends continuously on the minimum angle of the triangle T. By a standard inverse estimate and (4.9) we have

$$\begin{aligned} |\partial_\alpha(w - \tilde{w})(p_i)| &= |\partial_\alpha w(p_i)| \\ &\leq |v|_{W^2_\infty(T)} \\ &\leq C(T)h_T^{-1}|v|_{H^2(T)}, \end{aligned} \tag{4.24}$$

for $|\alpha| = 2$.

Recall from (4.9) that for $|\alpha| = 1$, $[\partial_\alpha(E_h v)](p) =$ average of $\partial_\alpha v_j(p)$, where $v_j = v|_{T_j}$ and T_j contains p as a vertex. Suppose T_1 and T_2 are two triangles in $\mathcal{T}_h$ sharing the common edge e which contains p as an endpoint. Since $v|_{T_1}$ and $v|_{T_2}$ agree at the two endpoints of e, the difference of $\partial_s(v|_{T_1})$ and $\partial_s(v|_{T_2})$ (s is the arc length along e) at p is bounded by $(|e|/2)[|v|_{W^2_\infty(T_1)} + |v|_{W^2_\infty(T_2)}]$.

Similarly, since the normal derivatives of $v|_{T_1}$ and $v|_{T_2}$ at the midpoint m of e agree, the difference of $\partial_n(v|_{T_1})$ and $\partial_n(v|_{T_2})$ (n is a normal of e) at p is bounded by $(|e|/2)[|v|_{W^2_\infty(T_1)} + |v|_{W^2_\infty(T_2)}]$.

Therefore, we have the following estimate:

$$\begin{aligned} \sum_{i=1}^{3} \sum_{|\alpha|=1} |\partial_\alpha(w - \tilde{w})(p_i)| &\leq k_T \sum_{T'} h_{T'} |(v|_{T'})|_{W^2_\infty(T')} \\ &\leq k_T \sum_{T'} C(T')|(v|_{T'})|_{H^2(T')}, \end{aligned} \tag{4.25}$$

where the summation is over all the triangles T' which share at least one vertex with T (cf. Fig. 4), and k_T is a constant which depends only on the total number of such T'.

Combining (4.21)–(4.25) and using the quasi-uniformity of $\mathcal{T}_h$, we have

$$\|v - E_h v\|_{L^2(T)} \leq Ch^2 \sum_{T'} |v|_{H^2(T')}. \tag{4.26}$$

Note that estimate (4.26) also holds if some of the vertices of T belong to $\partial\Omega$. Summing up the square of (4.26) over all the triangles T in $\mathcal{T}_h$, we obtain

$$\|v - E_h v\|_{L^2(\Omega)} \leq Ch^2 |v|_{H^2(\mathcal{T}_h)}. \tag{4.27}$$

The rest of the estimates in (4.19a) and (4.20a) now follow from standard inverse estimates and the triangle inequality. □

The following proposition follows from Lemmas 4.1–4.3, just as Proposition 3.1 followed from Lemmas 3.1–3.3.

Proposition 4.1. *Assumptions* (*A*.1) *and* (*A*.3) *hold for* I_H^h *and* J_h^H *defined by* (4.11) *and* (4.12), *respectively.*

5. Divergence-free P1 nonconforming finite element

In this section we adapt the abstract theory to the divergence-free P1 nonconforming finite element approximation of the stationary Stokes equations. We assume that Ω is simply connected (i.e., flow without obstacle). The case where there are obstacles is more complicated and is discussed elsewhere (cf. [5]). Throughout this section we use undertildes to denote vector-valued functions and operators. The operators $\underset{\sim}{\text{curl}}$ and div are given by

$$\underset{\sim}{\text{curl}}\, p = \begin{pmatrix} \partial p/\partial x_2 \\ -\partial p/\partial x_1 \end{pmatrix},$$
$$\text{div}\, \underset{\sim}{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}.$$

The finite element space V_h is defined by

$$V_h := \{\underset{\sim}{v} \in \underset{\sim}{L}^2(\Omega) : \underset{\sim}{v} \in \underset{\sim}{\mathcal{P}}_1(T) \quad \forall T \in \mathcal{T}_h, \underset{\sim}{v} \text{ is continuous at the midpoints of interelement boundaries, } \underset{\sim}{v} \text{ vanishes at the midpoints along } \partial\Omega, \text{ and } \text{div}(\underset{\sim}{v}|_T) = 0 \quad \forall T \in \mathcal{T}_h\}.$$

V_H is defined the same way with respect to $\mathcal{T}_H$.

The symmetric positive definite bilinear forms $a_h(\cdot,\cdot)$ and $a_H(\cdot,\cdot)$ are given by

$$a_h(\underset{\sim}{v}, \underset{\sim}{w}) := \sum_{T\in\mathcal{T}_h} \int_T \sum_{i,j=1}^{2} \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j}\, dx \quad \forall \underset{\sim}{v}, \underset{\sim}{w} \in V_h \tag{5.1}$$

and

$$a_H(\underset{\sim}{v}, \underset{\sim}{w}) := \sum_{T\in\mathcal{T}_H} \int_T \sum_{i,j=1}^{2} \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j}\, dx \quad \forall \underset{\sim}{v}, \underset{\sim}{w} \in V_H. \tag{5.2}$$

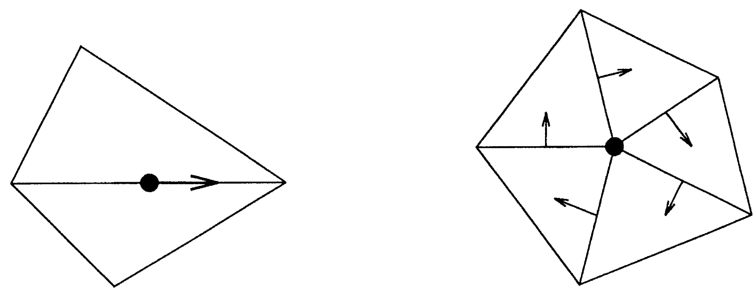

FIGURE 5

In order to define the inner products $(\cdot,\cdot)_h$ and $(\cdot,\cdot)_H$, we must first describe the bases of V_h and V_H.

Let e be an edge in $\mathcal{T}_h$. Denote by ϕ_e the piecewise linear function on Ω (with respect to $\mathcal{T}_h$) that takes the value 1 at the midpoint of the edge e and 0 at all other midpoints.

The first kind of basis function is associated with internal edges. Let

$$\underset{\sim}{\mu}_e := \phi_e \underset{\sim}{t}_e, \tag{5.3}$$

where e is an internal edge and $\underset{\sim}{t}_e$ is a unit vector tangential to e (cf. Fig. 5).

The second kind of basis function is associated with internal vertices. Let p be an internal vertex and let $e_1, e_2, \cdots, e_\ell$ be the edges in $\mathcal{T}_h$ that have p as an endpoint. Let

$$\underset{\sim}{\nu}_p := \sum_{i=1}^{\ell} |e_i|^{-1}\, \phi_{e_i} \underset{\sim}{n}_{e_i}, \tag{5.4}$$

where $\underset{\sim}{n}_{e_i}$ is a unit vector normal to e_i pointing in the clockwise direction (cf. Fig. 5). Then $B_h := \{\underset{\sim}{\mu}_e\colon e \text{ is an internal edge of } \mathcal{T}_h\} \cup \{\underset{\sim}{\nu}_p\colon p \text{ is an internal vertex of } \mathcal{T}_h\}$ is a basis of V_h (cf. [16]). The basis B_H (resp., B_j) of V_H (resp., V_j) is defined similarly.

Let $\underset{\sim}{v} = \sum a_i \underset{\sim}{\mu}_{e_i} + \sum b_j \underset{\sim}{\nu}_{p_j}$ and $\underset{\sim}{w} = \sum \alpha_i \underset{\sim}{\mu}_{e_i} + \sum \beta_j \underset{\sim}{\nu}_{p_j}$ be two members of V_h, where the summations are taken over all internal edges and all internal vertices. Then $(\underset{\sim}{v}, \underset{\sim}{w})_h$ is defined by

$$(\underset{\sim}{v}, \underset{\sim}{w})_h := h^2 \sum a_i \alpha_i + h^4 \sum b_j \beta_j. \tag{5.5}$$

The inner product $(\cdot,\cdot)_H$ is defined similarly.

The theory developed in §2 cannot be directly applied to the problem here because of the divergence-free constraint. To be more specific, the v_j's defined in the proof of Lemma 2.3 do not satisfy the divergence-free constraint.

We will modify the theory in the following manner. We establish assumptions (A.1a) and (A.2), and then Lemmas 2.1 and 2.2 remain valid. We will then prove Lemma 2.3 directly by exploiting the connection between the divergence-free P1 nonconforming finite element and the Morley finite element. Since Lemma 2.4 remains unchanged, Theorem 2.1 then holds.

Assumption (A.2) is trivially true. In order to define I_H^h and verify assumptions (A.1a) and (A.1b), we need to consider the connection between V_h (resp., V_j, V_H) and the Morley finite element spaces defined in §4, which we denote by M_h (resp., M_j, M_H) here.

There is an isomorphism between M_h and V_h (cf. [12]) given by the operator $\underset{\sim}{\mathrm{curl}}_h$, where

$$(5.6)\qquad (\underset{\sim}{\mathrm{curl}}_h\psi)|_T = \underset{\sim}{\mathrm{curl}}\,(\psi|_T) \quad \forall T \in \mathcal{T}_h.$$

The isomorphism $\underset{\sim}{\mathrm{curl}}_H : M_H \longrightarrow V_H$ is defined similarly. Note that $\underset{\sim}{\mathrm{curl}}_h|_{M_j}$ is an isomorphism from M_j onto V_j. The inverse of $\underset{\sim}{\mathrm{curl}}_h$ (resp., $\underset{\sim}{\mathrm{curl}}_H$) will be denoted by $\underset{\sim}{\mathrm{curl}}_h^{-1}$ (resp., $\underset{\sim}{\mathrm{curl}}_H^{-1}$).

In terms of the basis B_h of V_h, we have a simple description of $\underset{\sim}{\mathrm{curl}}_h$. Let $\psi \in M_h$. Then we have

$$(5.7)\qquad \underset{\sim}{\mathrm{curl}}_h\psi = \sum a_i \underset{\sim}{\mu}_{e_i} + \sum b_j \underset{\sim}{\nu}_{p_j},$$

where $a_i = \frac{\partial\psi}{\partial n_{e_i}}(m_i)$, m_i is the midpoint of edge e_i, $\underset{\sim}{t}_{e_i}$ is obtained by rotating $\underset{\sim}{n}_{e_i}$ clockwise through a right angle, and $b_j = \psi(p_j)$.

We are now ready to define the operator I_H^h. Let $\tilde{I}_H^h$ be the intergrid transfer operator between the Morley spaces M_H and M_h defined in §4. The operator $I_H^h : V_H \longrightarrow V_h$ is defined by

$$(5.8)\qquad I_H^h := \underset{\sim}{\mathrm{curl}}_h \circ \tilde{I}_H^h \circ \underset{\sim}{\mathrm{curl}}_H^{-1}.$$

The relations between these operators are illustrated by the commutative diagram in Fig. 6.

$$\begin{array}{ccc} V_h & \xleftarrow{\ \underset{\sim}{\mathrm{curl}}_h\ } & M_h \\ {\scriptstyle I_H^h}\big\uparrow & & \big\uparrow{\scriptstyle \tilde{I}_H^h} \\ V_H & \xrightarrow[\ \underset{\sim}{\mathrm{curl}}_H^{-1}\ ]{} & M_H \end{array}$$

FIGURE 6

In view of (5.7), I_H^h is represented by a sparse banded matrix with respect to the bases B_h and B_H.

Note that we have the trivial identities

$$(5.9a)\qquad |\underset{\sim}{\mathrm{curl}}_h\psi|_{H^1(\mathcal{T}_h)} = |\psi|_{H^2(\mathcal{T}_h)}, \quad \|\underset{\sim}{\mathrm{curl}}_h\psi\|_{L^2(\Omega)} = |\psi|_{H^1(\mathcal{T}_h)}$$

and

$$(5.9b)\qquad |\underset{\sim}{\mathrm{curl}}_H\phi|_{H^1(\mathcal{T}_H)} = |\phi|_{H^2(\mathcal{T}_H)}, \quad \|\underset{\sim}{\mathrm{curl}}_H\phi\|_{L^2(\Omega)} = |\phi|_{H^1(\mathcal{T}_H)},$$

where ψ (resp., ϕ) is piecewise C^1 with respect to $\mathcal{T}_h$ (resp., $\mathcal{T}_H$).

Lemma 5.1. *Assumption* (A.1a) (*for* $k = 1$) *holds for* I_H^h *defined by* (5.8).

Proof. It was established in Proposition 4.1 that $\tilde{I}_H^h$ has the following property

$$|\tilde{I}_H^h \psi|_{H^2(\mathcal{T}_h)} \le C\,|\psi|_{H^2(\mathcal{T}_H)} \quad \forall\, \psi \in M_H. \tag{5.10}$$

Assumption (A.1a) follows immediately from (5.8), (5.9) and (5.10). □

Finally, we establish Lemma 2.3 in the present context.

Lemma 5.2. *Given any* $\underset{\sim}{v} \in V_h$*, there exists* $\underset{\sim}{v}_0 \in V_H$*,* $\underset{\sim}{v}_j \in V_j$ $(1 \le j \le J)$ *such that*

$$\underset{\sim}{v} = I_H^h \underset{\sim}{v}_0 + \sum_{j=1}^{J} \underset{\sim}{v}_j \tag{5.11}$$

and

$$a_H(\underset{\sim}{v}_0, \underset{\sim}{v}_0) + \sum_{j=1}^{J} a_h(\underset{\sim}{v}_j, \underset{\sim}{v}_j) \le C\, N_c \left(1 + \left(\frac{H}{\delta}\right)^4\right) a_h(\underset{\sim}{v}, \underset{\sim}{v}). \tag{5.12}$$

Proof. Let $\psi = \underset{\sim}{\mathrm{curl}}_h^{-1} \underset{\sim}{v} \in M_h$. It follows from Proposition 4.1 that Lemma 2.3 holds for the Morley finite element spaces. Therefore, there exists $\psi_0 \in M_H$, $\psi_j \in M_j$ $(1 \le j \le J)$ such that

$$\psi = \tilde{I}_H^h \psi_0 + \sum_{j=1}^{J} \psi_j \tag{5.13}$$

and

$$|\psi_0|^2_{H^2(\mathcal{T}_H)} + \sum_{j=1}^{J} |\psi_j|^2_{H^2(\mathcal{T}_h)} \le C\, N_c \left(1 + \left(\frac{H}{\delta}\right)^4\right) |\psi|^2_{H^2(\mathcal{T}_h)}. \tag{5.14}$$

Let $\underset{\sim}{v}_0 = \underset{\sim}{\mathrm{curl}}_H \psi_0$ and $\underset{\sim}{v}_j = \underset{\sim}{\mathrm{curl}}_h \psi_j$ for $1 \le j \le J$. Then, using (5.8) and (5.13), we obtain

$$\begin{aligned}
\underset{\sim}{v} &= \underset{\sim}{\mathrm{curl}}_h \psi \\
&= \underset{\sim}{\mathrm{curl}}_h \left(\tilde{I}_H^h \psi_0 + \sum_{j=1}^{J} \psi_j \right) \\
&= \underset{\sim}{\mathrm{curl}}_h \left(\tilde{I}_H^h \underset{\sim}{\mathrm{curl}}_H^{-1} \underset{\sim}{v}_0 + \sum_{j=1}^{J} \underset{\sim}{\mathrm{curl}}_h^{-1} \underset{\sim}{v}_j \right) \\
&= I_H^h \underset{\sim}{v}_0 + \sum_{j=1}^{J} \underset{\sim}{v}_j.
\end{aligned}$$

The estimate (5.14) can be rewritten, using (5.9), as

$$|\underset{\sim}{v}_0|^2_{H^1(\mathcal{T}_H)} + \sum_{j=1}^{J} |\underset{\sim}{v}_j|^2_{H^1(\mathcal{T}_h)} \le C\, N_c \left(1 + \left(\frac{H}{\delta}\right)^4\right) |\underset{\sim}{v}|^2_{H^1(\mathcal{T}_h)},$$

which is equivalent to (5.14) by (A.2). □

As was pointed out earlier, Lemmas 5.1 and 5.2 yield the following theorem.

Theorem 5.1. *The two-level additive Schwarz preconditioner B for the divergence-free P1 nonconforming finite element method defined by* (2.16) *satisfies*

$$\frac{\lambda_{\max}(BA_h)}{\lambda_{\min}(BA_h)} \le C\,\frac{\omega_1}{\omega_0} N_c^2 \left(1 + \left(\frac{H}{\delta}\right)^4\right),$$

where A_h is the operator respresenting the discretized stationary Stokes equations.

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Department of Mathematics and Computer Science, Clarkson University, Potsdam, New York 13699-5815

Current address: Department of Mathematics, University of South Carolina, Columbia, South Carolina 29208

E-mail address: brenner@math.sc.edu