A UNIFORMLY CONVERGENT METHOD FOR A SINGULARLY PERTURBED SEMILINEAR REACTION–DIFFUSION PROBLEM WITH MULTIPLE SOLUTIONS

GUANGFU SUN AND MARTIN STYNES

ABSTRACT. This paper considers a simple central difference scheme for a singularly perturbed semilinear reaction–diffusion problem, which may have multiple solutions. Asymptotic properties of solutions to this problem are discussed and analyzed. To compute accurate approximations to these solutions, we consider a piecewise equidistant mesh of Shishkin type, which contains $O(N)$ points. On such a mesh, we prove existence of a solution to the discretization and show that it is accurate of order $N^{-2} \ln^2 N$, in the discrete maximum norm, where the constant factor in this error estimate is independent of the perturbation parameter ε and N. Numerical results are presented that verify this rate of convergence.

1. INTRODUCTION

Singularly perturbed nonlinear boundary value problems occur frequently in engineering applications such as catalytic reactions or absorption processes and fluid dynamics; see Smith [16, Chapter 10].

We consider the semilinear problem

$$F_\varepsilon u(x) \equiv -\varepsilon^2 u''(x) + b(x, u) = 0 \quad \text{for } x \in (0, 1), \tag{1.1a}$$

$$u(0) = u(1) = 0, \tag{1.1b}$$

where ε is a small positive parameter. Set $X = [0, 1]$. We shall assume that $b \in C^\infty(X \times \mathcal{R}^1)$ for convenience.

Asymptotic and numerical solutions of problem (1.1) have been considered by many authors, under various hypotheses on $b(x, u)$. See for example Chang and Howes [1], D'Annunzio [2], Fife [4], Herceg [7], Herceg and Petrović [8] and Lorenz [9].

One of the conditions occurring frequently in the literature is

$$b_u(x, u) > b_0^2 > 0 \quad \text{for all } (x, u) \in X \times \mathcal{R}^1. \tag{1.2}$$

Under this condition, problem (1.1) has a unique solution $u \in C^\infty(X)$; see Lorenz [9].

The reduced problem of (1.1) is defined by

$$b(x, u_0) = 0 \quad \text{for } x \in X. \tag{1.3}$$

Under condition (1.2), this reduced problem has a unique solution $u_0 \in C^\infty(X)$, as can be seen using the implicit function theorem and the compactness of X. Note that in general, u_0 does not satisfy either of the boundary conditions in (1.1b).

The reduced problem (1.3) may have more than one solution if condition (1.2) is not satisfied. Fife [4], D'Annunzio [2] and O'Malley [11] consider problem (1.1) under the assumptions that:

(i) it has a *stable reduced solution*, i.e., there exists a solution $u_0 \in C^\infty(X)$ of (1.3) such that

(1.4a) $$b_u(x, u_0) > b_0^2 > 0 \quad \text{for all } x \in X;$$

(ii) it has *stable boundary layers*, i.e., the stable reduced solution u_0 of (i) satisfies

$$\int_{u_0(0)}^{\tau} b(0, s)ds > 0 \text{ for } \begin{cases} \tau \in (u_0(0), 0], & \text{whenever } 0 > u_0(0), \quad \text{(1.4b)} \\ \tau \in [0, u_0(0)), & \text{whenever } u_0(0) > 0, \quad \text{(1.4c)} \end{cases}$$

and

$$\int_{u_0(1)}^{\tau} b(1, s)ds > 0 \text{ for } \begin{cases} \tau \in (u_0(1), 0], & \text{whenever } 0 > u_0(1), \quad \text{(1.4d)} \\ \tau \in [0, u_0(1)), & \text{whenever } u_0(1) > 0. \quad \text{(1.4e)} \end{cases}$$

The conditions (1.4) are obviously weaker than condition (1.2). Condition (1.4a) implies that any solution of (1.3) is locally unique. Problem (1.1) under the conditions (1.4) may exhibit multiple solutions. D'Annunzio [2] uses degree theory to prove existence and local uniqueness of a solution satisfying (1.1) and (1.4).

In what follows, we shall refer to (1.1) under condition (1.2) as problem (A) and (1.1) under conditions (1.4) as problem (B).

A solution $u(x)$ of (1.1) usually exhibits sharp boundary layers at the endpoints of the interval X when the parameter ε is near zero. When polynomial–based numerical methods are applied to (1.1), one does not obtain accurate results on all of X, even in the linear case. This has led to the development of numerical methods that are uniformly convergent with respect to the perturbation parameter.

Let u be a solution of (1.1). Consider a difference scheme for solving (1.1). Suppose that this scheme has a solution u^N that satisfies

(1.5) $$\|u - u^N\| \le Cg(N),$$

where N, independent of ε, is the number of subintervals in the mesh used, C is a positive constant independent of N and ε, and $g(N)$ is a function of N but is independent of ε. If $g(N) \to 0$ as $N \to \infty$, then we say that the scheme is uniformly convergent to u, with respect to the norm $\| \cdot \|$. Furthermore, we shall say that the scheme is uniformly convergent with order $g(N)$ in the norm $\| \cdot \|$. In this paper, we consider only uniform convergence with respect to the discrete L^∞ norm.

In the linear case, many authors consider both uniformly convergent exponentially fitted schemes on equidistant meshes and uniformly convergent polynomial–based schemes on special meshes; see Doolan et al. [3], Hegarty et al. [6], Niijima [10], O'Riordan and Stynes [12], Roos [14] and Vulanović [18].

Uniformly convergent methods for the semilinear problem (A) have also been examined. Vulanović [17] applies a central difference scheme to the semilinear problem (A). He obtains second-order uniform convergence of the scheme on a special graded mesh of Bakhvalov type. Herceg [7] investigates a scheme for problem (A) under

extra somewhat restrictive conditions on the problem. He achieves fourth-order uniform convergence of the scheme, again on a graded mesh of Bakhvalov type.

D'Annunzio [2] uses a simple central difference scheme on a special locally quasi–equidistant mesh to solve the more general problem (B). This mesh contains $O(h^{-1}\ln 1/\varepsilon)$ mesh points when $\varepsilon \leq h$, where h is the maximum mesh spacing over the interval X. She shows existence of a solution to the discrete problem and $O(h)$ convergence of this solution to a solution of problem (B). The constant factor in the error estimate is independent of ε. The method is not however uniformly convergent in the sense of (1.5), since the number of mesh points depends on the perturbation parameter ε.

In the present paper, we consider a uniformly convergent method for problem (B). This method is D'Annunzio's scheme on a special piecewise equidistant mesh. Such meshes, which were recently introduced by Shishkin [15], are much simpler than the graded meshes of Vulanović [17], Herceg [7] and D'Annunzio [2]. We use degree theory to prove existence of a solution to the scheme. We construct super and sub solutions that are within $O(\varepsilon^2 \ln^2(1/\varepsilon))$ of a solution of problem (B); we also consider their discrete analogues for the scheme. Then we deduce uniform convergence of $O(N^{-2}\ln^2 N)$ for the scheme under the nonrestrictive assumption that $\varepsilon \leq N^{-1}$. This result is a significant improvement over the first-order convergence obtained by D'Annunzio [2] for the same scheme on a different mesh.

A summary of the paper is as follows. Section 2 contains results concerning the exact solutions of problem (B). In §3, we bound truncation errors of the central difference scheme on a piecewise equidistant Shishkin mesh. In §4, we prove existence of solutions and the almost second-order uniform accuracy of the scheme for problem (B). Section 5 presents numerical computations that confirm our results.

Notation. Throughout this paper we let C, sometimes subscripted, denote a generic positive constant that may take different values in different formulas, but is always independent of N and ε.

2. The continuous problem

In this section, we discuss the properties of exact solutions of problem (B). We shall suppose, without loss of generality, that $u_0(0) < 0$ and $u_0(1) < 0$, as other cases can be handled similarly.

The concepts of super and sub solutions are important for the study of problem (B). Suppose that there exist two functions α and $\beta \in C^2(X)$ with the following properties:

$$F_\varepsilon \alpha(x) \leq 0 \leq F_\varepsilon \beta(x) \quad \text{for } x \in X,$$
$$\alpha(0) \leq 0 \leq \beta(0),$$
$$\alpha(1) \leq 0 \leq \beta(1),$$
$$\alpha(x) \leq \beta(x) \quad \text{for } x \in X.$$

Then $\beta(x)$ and $\alpha(x)$ are said to be super and sub solutions respectively of problem (B).

In order to prove higher-order convergence of a central difference scheme for problem (B), we shall introduce super and sub solutions that are more accurate than those in D'Annunzio [2]. Let us first give some notation and definitions.

We use a cutoff function $\sigma(x)$ defined by

$$\sigma(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1/4, \\ 0 & \text{for } 1/2 \leq x \leq 1, \end{cases}$$

where $\sigma \in C^\infty(X)$ and σ is monotonically decreasing.

Let $v \in C^\infty(0, \infty)$. Let J denote a positive integer. Let b_1 be a positive constant. If for each $\delta \in (0, b_1)$ there exists a positive constant C_δ, depending on δ and J, such that

$$\left| v^{(j)}(\eta) \right| \leq C_\delta \exp(-(b_1 - \delta)\eta),$$

for $\eta > 0$ and $j = 0, 1, \ldots, J$, then we say that the function $v(\eta)$ belongs to the class $e(b_1, J)$.

In the rest of this section, we shall use J to denote an arbitrary positive integer. In fact, we shall take $J = 4$ in the analysis of §§3 and 4.

The following two lemmas are modifications of Lemmas 2.1 and 2.2 of Fife [4].

Lemma 2.1. *Let $\lambda > 0$ be a constant. Let $g \in C^\infty[0, \infty)$ satisfy $g(0) = 0$, $g'(0) > 0$ and*

$$\int_0^\tau g(s) > 0 \quad \text{for } \tau \in (0, \lambda].$$

Then for $\eta \geq 0$, there exists a unique strictly decreasing solution $v(\eta)$ of

$$(2.1) \qquad\qquad\qquad v'' - g(v) = 0 \quad \text{for } \eta > 0,$$

$$(2.2) \qquad\qquad\qquad v(0) = \lambda, \quad v(\infty) = 0.$$

Furthermore, v belongs to the class $e(b_1, J)$ with $b_1 = \sqrt{g'(0)}$.

Proof. By Lemma 2.1 of Fife [4], the solution v of (2.1) and (2.2) exists, is strictly decreasing and satisfies

$$C_\delta^{-1} \exp(-(b_1 + \delta)\eta) \leq v^{(j)}(\eta) \leq C_\delta \exp(-(b_1 - \delta)\eta),$$

for $j = 0, 1$ and $\eta > 0$, where $b_1 = \sqrt{g'(0)}$, $\delta \in (0, b_1)$ and $C_\delta > 0$ are constants.

Since $g(0) = 0$ and $g'(s)$ is bounded for $s \in (0, \lambda)$, we have from (2.1)

$$|v''(\eta)| = |g'(v^*)| \, v(\eta), \quad \text{where } v^* \in (0, v) \subseteq (0, \lambda),$$
$$\leq C_\delta \exp(-(b_1 - \delta)\eta) \quad \text{for } \eta > 0,$$

where we recall that C_δ is a generic constant. The result then follows from differentiating (2.1) repeatedly and induction on j, since the derivatives of $g(s)$ up to any prescribed order are bounded for $s \in (0, \lambda)$.   □

Lemma 2.2. *Let λ and $g(s)$ be as in Lemma 2.1. Let $v(\eta)$ be the strictly decreasing solution of (2.1) and (2.2). Let $a(\eta)$ belong to the class $e(b_1, J)$ with $b_1 = \sqrt{g'(0)}$, and let λ_1 be a given constant. Then there exists a unique solution $v_1(\eta)$ of*

$$v_1''(\eta) - g'(v(\eta))v_1(\eta) = a(\eta) \quad \text{for } \eta > 0,$$
$$v_1(0) = \lambda_1, \quad v_1(\infty) = 0.$$

Moreover, $v_1(\eta)$ belongs to the class $e(b_1, J)$.

Proof. The result follows easily from an inspection of the proof of Lemma 2.2 in Fife [4].   □

The next lemma is a modification of Lemma 3.1 of D'Annunzio [2].

Lemma 2.3. *Let λ and $g(s)$ be as in Lemma 2.1. Let p be a constant. Then there is a $p_0 \in \left(0, g'(0)\right)$ such that if $|p| < p_0$, there exists a unique solution $v(\eta, p)$ of*

$$(2.3) \qquad \ddot{v} - g(v) = -pv \quad \text{for } \eta > 0,$$

$$(2.4) \qquad v(0, p) = \lambda, \quad v(\infty, p) = 0.$$

(Here and in what follows a dot denotes partial differentiation with respect to η.) For each fixed $p \in (-p_0, p_0)$, the solution $v(\eta, p)$ is strictly decreasing in η and belongs to the class $e(b_1, J)$, where $b_1 = \sqrt{g'(0) - p}$.

The derivative $\phi(\eta, p) = \frac{\partial v}{\partial p}(\eta, p)$ exists and satisfies

$$(2.5) \qquad \ddot{\phi} - \left(g'(v) - p\right)\phi = -v \quad \text{for } \eta > 0,$$

$$(2.6) \qquad \phi(0, p) = \phi(\infty, p) = 0$$

and

$$(2.7) \qquad \phi(\eta, p) > 0 \quad \text{for } \eta > 0.$$

Furthermore, there exist positive constants C_1 and C_2, independent of p, such that

$$(2.8) \qquad |\phi(\eta, p)| \leq C_1 \eta e^{-C_2 \eta} \quad \text{for } (\eta, p) \in (0, \infty) \times [-p_0, p_0].$$

Proof. The argument is similar to that of Lemma 3.1 of D'Annunzio [2]. We give an outline of the proof for completeness.

Let $g_p(s) = g(s) - ps$. We can easily show that there exists a $p_0 \in \left(0, g'(0)\right)$ such that if $|p| < p_0$, then $g_p(s)$ satisfies the conditions on $g(s)$ in Lemma 2.1. Therefore, the problem (2.3) – (2.4) has a unique solution $v(\eta, p)$. For each fixed $p \in (-p_0, p_0)$, the solution $v(\eta, p)$ is strictly decreasing in η and belongs to the class $e(b_1, J)$ with $b_1 = \sqrt{g'(0) - p}$.

Also, from the proof of Lemma 2.1, we see that

$$(2.9) \qquad C_\delta^{-1} \exp(-(b_1 + \delta)\eta) \leq \frac{\partial^j v}{\partial \eta^j}(\eta, p) \leq C_\delta \exp(-(b_1 - \delta)\eta),$$

for $j = 0, 1$ and $\eta > 0$, where $\delta \in (0, b_1)$ and $C_\delta > 0$ are constants.

Moreover, we can show that $v(\eta, p)$ is differentiable with respect to p and satisfies $\frac{\partial v}{\partial p}(\eta, p)\Big|_{\eta=0} = 0$. Set

$$\phi(\eta, p) = \frac{\partial v}{\partial p}(\eta, p) \quad \text{and} \quad \psi(\eta, p) = -\frac{\partial v}{\partial \eta}(\eta, p).$$

Then $\phi(\eta, p)$ satisfies (2.5) and (2.6), while $\psi(\eta, p)$ satisfies the homogeneous version of (2.5). Using the method of variation of parameters, we obtain

$$(2.10) \qquad \phi(\eta, p) = -\psi(\eta, p) \int_0^\eta \psi^{-2}(\xi, p) \int_\xi^\infty [-v(\zeta, p)]\psi(\zeta, p)\, d\zeta d\xi.$$

Now (2.7) follows from $v(\eta, p) > 0$ and $\psi(\eta, p) > 0$. Using (2.9) and (2.10), we conclude that (2.8) holds. $\square$

We now define the required boundary layer functions. These are more accurate than those of D'Annunzio. They will be used to construct our super and sub

solutions. Let

$$(2.11) \qquad w(x,\varepsilon,p) = \begin{cases} \left(v_0^0\left(x/\varepsilon,p\right) + \varepsilon v_1^0\left(x/\varepsilon\right)\right)\sigma(x) & \text{for } 0 \le x \le 1/2, \\ \left(v_0^1\left((1-x)/\varepsilon,p\right) + \varepsilon v_1^1\left((1-x)/\varepsilon\right)\right)\sigma(1-x) \\ & \text{for } 1/2 < x \le 1, \end{cases}$$

where $v_0^0(\eta,p)$, $v_0^1(\eta,p)$, $v_1^0(\eta)$, and $v_1^1(\eta)$ are respectively defined by

$$(2.12) \qquad \ddot{v}_0^0 - b\left(0, u_0(0) + v_0^0\right) = -p v_0^0 \quad \text{for } \eta > 0,$$

$$(2.13) \qquad v_0^0(0,p) = -u_0(0), \quad v_0^0(\infty,p) = 0,$$

$$(2.14) \qquad \ddot{v}_0^1 - b\left(1, u_0(1) + v_0^1\right) = -p v_0^1 \quad \text{for } \eta > 0,$$

$$(2.15) \qquad v_0^1(0,p) = -u_0(1), \quad v_0^1(\infty,p) = 0,$$

(2.16)
$$\ddot{v}_1^0 - b_u\left(0, u_0(0) + v_0^0(\eta,0)\right) v_1^0$$
$$= \eta\left[b_x\left(0, u_0(0) + v_0^0(\eta,0)\right) + b_u\left(0, u_0(0) + v_0^0(\eta,0)\right) u_0'(0)\right] \quad \text{for } \eta > 0,$$

$$(2.17) \qquad v_1^0(0) = 0, \quad v_1^0(\infty) = 0,$$

and

(2.18)
$$\ddot{v}_1^1 - b_u\left(1, u_0(1) + v_0^1(\eta,0)\right) v_1^1$$
$$= \eta\left[b_x\left(1, u_0(1) + v_0^1(\eta,0)\right) + b_u\left(1, u_0(1) + v_0^1(\eta,0)\right) u_0'(1)\right] \quad \text{for } \eta > 0,$$

$$(2.19) \qquad v_1^1(0) = 0, \quad v_1^1(\infty) = 0.$$

We remark that D'Annunzio uses only the first terms of our expansions, i.e., $v_1^0 \equiv v_1^1 \equiv 0$ in [2].

Lemmas 2.1–2.3 imply that there exists $p_0 > 0$, independent of ε, such that $w(x,\varepsilon,p)$ is well defined for $|p| < p_0$. The function w essentially models boundary layers at $x = 0$ and $x = 1$, as we shall see in the course of proving the next lemma.

Lemma 2.4. *Set $p_\varepsilon = \varepsilon^2 \ln^2(1/\varepsilon)$. Then we can choose positive constants C_1 and C_2, which are independent of ε, such that when ε is sufficiently small, $w(x,\varepsilon,C_1 p_\varepsilon)$ and $w(x,\varepsilon,-C_1 p_\varepsilon)$ are well defined, and*

$$(2.20) \qquad \beta(x,\varepsilon) = u_0(x) + w(x,\varepsilon,C_1 p_\varepsilon) + C_2 p_\varepsilon$$

and

$$(2.21) \qquad \alpha(x,\varepsilon) = u_0(x) + w(x,\varepsilon,-C_1 p_\varepsilon) - C_2 p_\varepsilon$$

are super and sub solutions respectively of problem (B).

Proof. Fix $\varepsilon \in (0,1]$. We shall specify C_1 and C_2 later in the proof. It is easy to see from (2.7) that

$$\alpha(x,\varepsilon) < \beta(x,\varepsilon) \quad \text{for } x \in X.$$

By the construction of $w(x,\varepsilon,p)$, we have

$$\alpha(0,\varepsilon) = -C_2 p_\varepsilon < 0 < C_2 p_\varepsilon = \beta(0,\varepsilon),$$
$$\alpha(1,\varepsilon) = -C_2 p_\varepsilon < 0 < C_2 p_\varepsilon = \beta(1,\varepsilon).$$

To be a super solution, β must satisfy $F_\varepsilon \beta \ge 0$ for $x \in X$. We shall prove this inequality only for $x \in [0,1/2]$, since the argument for $x \in [1/2,1]$ is similar. In the

rest of this proof, the notation $\zeta = O(M)$ stands for $|\zeta| \leq \bar{C}M$, where $\bar{C} > 0$ is any constant independent of C_1, C_2 and ε.

We set

$$\hat{w}(\eta, \varepsilon, p) = \left(v_0^0(\eta, p) + \varepsilon v_1^0(\eta)\right)\sigma(\varepsilon\eta)$$

and

$$\Phi(\eta, \varepsilon, p) = -\frac{\partial^2 \hat{w}(\eta, \varepsilon, p)}{\partial \eta^2} + b\left(\varepsilon\eta, u_0(\varepsilon\eta) + \hat{w}(\eta, \varepsilon, p)\right) - p\hat{w}(\eta, \varepsilon, p);$$

these functions are defined for all sufficiently small $|p|$. Then from (2.12),

(2.22) $$\Phi(\eta, 0, p) = 0.$$

This yields

(2.23) $$\Phi(\eta, 0, 0) = 0$$

and

(2.24) $$\frac{\partial \Phi(\eta, \varepsilon, p)}{\partial p}\Big|_{(\eta,0,0)} = 0.$$

Next, (2.16) implies that

(2.25) $$\frac{\partial \Phi(\eta, \varepsilon, p)}{\partial \varepsilon}\Big|_{(\eta,0,0)} = 0.$$

By Taylor's theorem and (2.23)–(2.25), we have

(2.26) $$\Phi(\eta, \varepsilon, p) = \frac{1}{2}\left(\varepsilon\frac{\partial}{\partial \varepsilon} + p\frac{\partial}{\partial p}\right)^2 \Phi(\eta, \varepsilon, p)\Big|_{(\eta,\theta\varepsilon,\theta p)}$$

for $\eta > 0$, $\ 0 < \varepsilon \leq 1$ and $0 < p \leq p_0$ (p_0 chosen so that $\hat{w}$ is well defined for $|p| < p_0$), and some $\theta \in (0,1)$.

Lemmas 2.1–2.3 imply that

(2.27) $$0 < \frac{\partial \hat{w}(\eta, \varepsilon, p)}{\partial p} \leq C$$

and

(2.28) $$\left|\frac{\partial^j \hat{w}(\eta, \varepsilon, p)}{\partial \eta^j}\right| \leq C \exp\left(-(\bar{b} - \delta)\eta\right),$$

for $\eta > 0$ and $j = 0, 1, \ldots, J$, where $b_0^2 > p$, $\bar{b} = \sqrt{b_0^2 - p}$ (b_0 is given by (1.2)) and δ is any fixed number in $(0, \bar{b})$. Hence there exists a constant C such that

$$\left|\frac{\partial^2 \Phi(\eta, \varepsilon, p)}{\partial \varepsilon^i \partial p^{2-i}}\Big|_{(\eta,\theta\varepsilon,\theta p)}\right| \leq C,$$

for $i = 0, 1, 2$, $\ \eta > 0$, $\ 0 < \varepsilon \leq 1$ and $0 < p \leq p_0$. Thus (2.26) implies that

(2.29) $$|\Phi(\eta, \varepsilon, p)| \leq C(\varepsilon^2 + p^2),$$

for $\eta > 0$, $\ 0 < \varepsilon \leq 1$ and $0 < p \leq p_0$.

We are now ready to show that $\beta(x, \varepsilon)$ is a super solution of problem (B). For, observing that $w(x, \varepsilon, p) = \hat{w}(\eta, \varepsilon, p)$ when $\eta = x/\varepsilon$, we have

$$
\begin{aligned}
F_\varepsilon \beta(x, \varepsilon) &= -\varepsilon^2 \beta''(x, \varepsilon) + b(x, \beta) \\
&= -\varepsilon^2 u_0''(x) + \Phi(\eta, \varepsilon, C_1 p_\varepsilon) + C_1 p_\varepsilon \hat{w}(\eta, \varepsilon, C_1 p_\varepsilon) \\
&\quad + b\big(x, u_0(x) + \hat{w}(\eta, \varepsilon, C_1 p_\varepsilon) + C_2 p_\varepsilon\big) \\
&\quad - b\big(x, u_0(x) + \hat{w}(\eta, \varepsilon, C_1 p_\varepsilon)\big) \\
&= C_1 p_\varepsilon \hat{w}(\eta, \varepsilon, C_1 p_\varepsilon) + C_2 p_\varepsilon b_u\big(x, u_0(x) + \hat{w}(\eta, \varepsilon, C_1 p_\varepsilon) + \hat{\theta} C_2 p_\varepsilon\big) \\
&\quad + O(\varepsilon^2 + p_\varepsilon^2),
\end{aligned}
$$

(2.30)

by (2.29), for some $\hat{\theta} \in (0, 1)$.

Now (1.4a) states that $b_u(x, u_0(x)) \geq b_0^2 > 0$, so by the compactness of X there exists a constant $C_3 > 0$ such that

$$
b_u\big(x, u_0(x) + g(x)\big) \geq b_0^2/2 \quad \text{for } x \in X,
$$

for all functions g satisfying $|g(x)| \leq C_3$ on X.

Choose $C_2 = C_3/2$, so that in (2.30) we get $|\hat{\theta} C_2 p_\varepsilon| \leq C_3/2$. Set

$$
\begin{aligned}
C_4 = \max\{|b_u(x, u_0(x) + \hat{w}(\eta, \varepsilon, r) + s)| : 0 \leq x \leq 1, \quad \eta > 0, \\
0 < \varepsilon \leq 1, \quad 0 \leq r \leq p_0, \quad 0 \leq s \leq C_3/2\};
\end{aligned}
$$

it follows from (2.27) and (2.28) that this maximum is well defined. Now (2.30) and Lemmas 2.1–2.3 imply that

$$
F_\varepsilon \beta(x, \varepsilon) \geq \begin{cases} p_\varepsilon(C_2 b_0^2/2 - \varepsilon C_1 \|v_1^0\|_\infty) + O(\varepsilon^2 + p_\varepsilon^2) & \text{if } \hat{w} \leq C_3/2, \\ (C_1 - C_4) C_3 p_\varepsilon/2 + O(\varepsilon^2 + p_\varepsilon^2) & \text{if } \hat{w} > C_3/2, \end{cases}
$$

where $\|v_1^0\|_\infty = \max_{\eta \geq 0} |v_1^0(\eta)|$. Choose $C_1 = 2C_4$. Then for all sufficiently small ε, we have

$$
F_\varepsilon \beta(x, \varepsilon) > 0 \quad \text{for } x \in [0, 1/2].
$$

Analogously, one may show that $F_\varepsilon \beta(x, \varepsilon) > 0$ for $x \in [1/2, 1]$, and that $F_\varepsilon \alpha(x, \varepsilon) < 0$ for $x \in X$. $\qquad \square$

Theorem 2.1. *Under the same hypotheses as in Lemma 2.4, problem* (B) *has a solution* $u(x)$, *which is the only solution satisfying*

(2.31) $\qquad\qquad \alpha(x, \varepsilon) \leq u(x) \leq \beta(x, \varepsilon) \quad \text{for } x \in X.$

Here, $\beta(x, \varepsilon)$ *and* $\alpha(x, \varepsilon)$ *are the super and sub solutions given by* (2.20) *and* (2.21).

Proof. D'Annunzio [2, Corollary 3.1] proves the following extension of Nagumo's theorem: if problem (B) has a super solution $\beta(x, \varepsilon)$ and a sub solution $\alpha(x, \varepsilon)$, then there exists a solution $u(x)$ of problem (B) such that

$$
\alpha(x, \varepsilon) \leq u(x) \leq \beta(x, \varepsilon) \quad \text{for } x \in X.
$$

Hence the existence of a solution is implied by Lemma 2.4 above. The uniqueness of the solution satisfying (2.31) can be shown by arguments similar to those of [2, Theorem 3.6], using degree theory. $\qquad \square$

Recall that $x(x, \varepsilon, p) = \hat{w}(\eta, \varepsilon, p)$ when $\eta = x/\varepsilon$. From (2.27) and the definitions of our super solution $\beta(x, \varepsilon)$ and sub solution $\alpha(x, \varepsilon)$, we see that

$$
|\beta(x, \varepsilon) - \alpha(x, \varepsilon)| \leq C\varepsilon^2 \ln^2(1/\varepsilon) \quad \text{for } x \in X.
$$

This shows that we have tighter control on the solution $u(x)$ of Theorem 2.1 than in Corollary 3.4 in D'Annunzio [2], where the super and sub solutions yield only an $O(\varepsilon)$ estimate of u.

In principle one could obtain an approximation to u by explicitly computing $\alpha(x)$ or $\beta(x)$. This would entail solving two nonlinear and two linear second–order differential equations (see $(2.12) - (2.19)$) and would be more complicated than using the difference scheme below to solve (1.1) directly.

3. A CENTRAL DIFFERENCE SCHEME ON A SHISHKIN MESH

We analyze the truncation error of a central difference scheme applied to problem (B) on a Shishkin mesh.

For a given positive integer N, we denote by X^N an arbitrary mesh

$$0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1,$$

with $h_i = x_i - x_{i-1}$, for $i = 1, \dots, N$, and $\bar{h}_i = (h_i + h_{i+1})/2$, for $i = 1, \dots, N-1$.

We use $\mathcal{R}^{N+1}$ to denote the real $(N + 1)$-dimensional linear space of all column vectors

$$z = (z_0, z_1, \dots, z_N)^T.$$

In what follows, for any function $y \in C(X)$, we shall abuse the notation by also writing $y \in \mathcal{R}^{N+1}$ with $y_i = y(x_i)$ for $i = 0, 1, \dots, N$.

We equip the space $\mathcal{R}^{N+1}$ with the usual l_∞–norm:

$$\|z\|_\infty = \max_{0 \le i \le N} |z_i|.$$

The induced norm of a linear mapping $\bar{A} = (a_{ij}) : \mathcal{R}^{N+1} \to \mathcal{R}^{N+1}$ is

$$\|\bar{A}\|_\infty = \max_{0 \le i \le N} \sum_{j=0}^N |a_{ij}|.$$

Let

$$A = \begin{pmatrix} -\varepsilon^{-2} & 0 & 0 & & & & \\ r_1^- & r_1^c & r_1^+ & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & r_{N-1}^- & r_{N-1}^c & r_{N-1}^+ \\ & & & & 0 & 0 & -\varepsilon^{-2} \end{pmatrix}$$

be an $(N + 1) \times (N + 1)$ tridiagonal matrix, where

$$r_i^- = \frac{1}{h_i \bar{h}_i}, \quad r_i^c = -\frac{2}{h_i h_{i+1}}, \quad r_i^+ = \frac{1}{h_{i+1} \bar{h}_i}.$$

Let $B : \mathcal{R}^{N+1} \to \mathcal{R}^{N+1}$ be the mapping:

$$(Bz)_i = \begin{cases} 0 & \text{for } i = 0, \\ b(x_i, z_i) & \text{for } i = 1, \dots, N-1, \\ 0 & \text{for } i = N. \end{cases}$$

Set

$$F = -\varepsilon^2 A + B.$$

We shall use $\{F, X^N\}$ to denote the three–point central difference scheme

$$(3.1) \qquad\qquad\qquad\qquad F u^N = 0.$$

Let $y \in C^2(X)$. Define $(F_\varepsilon y)(0) = y(0)$ and $(F_\varepsilon y)(1) = y(1)$. The truncation error of F in approximating F_ε in terms of y is defined to be $\|Fy - F_\varepsilon y\|_\infty$. It is clear that $(Fy)_0 = (F_\varepsilon y)(0)$ and $(Fy)_N = (F_\varepsilon y)(1)$. We shall bound $|(Fy)_i - (F_\varepsilon y)(x_i)|$, for $i = 1, 2, \ldots, N-1$, in the truncation error analysis of this section.

Since $u'(x)$ is in general unbounded in the boundary layers at $x = 0$ and $x = 1$ when $\varepsilon \to 0$, a polynomial-based discretization cannot be consistent, uniformly in ε, unless it is constructed on a special mesh. In the literature, several types of special graded meshes have been introduced for singularly perturbed two–point boundary value problems; see Herceg [7], D'Annunzio [2] and Gartland [5]. In the present paper we shall use a Shishkin mesh [15], which is piecewise equidistant and consequently much simpler than the above meshes.

Given a positive integer N, where N is divisible by 4, we divide the interval $[0,1]$ into the three subintervals

$$[0, \sigma], \quad [\sigma, 1-\sigma], \quad \text{and} \quad [1-\sigma, 1].$$

We use equidistant meshes on each of these subintervals, with $1 + N/4$ points in each of $[0, \sigma]$ and $[1-\sigma, 1]$, and $1 + N/2$ points in $[\sigma, 1-\sigma]$. Set $\tilde{b}_0 = \min\{b_0, 1\}$. We define the parameter σ by

$$(3.2) \qquad\qquad\qquad \sigma = \min\left\{1/4, \quad 4\tilde{b}_0^{-1}\varepsilon \ln N\right\},$$

which depends on ε and N. The basic idea here is to use a fine mesh to resolve part of the boundary layers.

More explicitly, we define

$$X_s^N : 0 = x_0 < x_1 < \cdots < x_{i_0} < \cdots < x_{N-i_0} < \cdots < x_N = 1,$$

with $i_0 = N/4$, $x_{i_0} = \sigma$, $x_{N-i_0} = 1 - \sigma$, and

$$(3.3) \qquad\qquad h_i = 4\sigma N^{-1} \quad \text{for } i = 1, \ldots, i_0, N - i_0 + 1, \ldots, N,$$

$$(3.4) \qquad\qquad h_i = 2(1 - 2\sigma)N^{-1} \quad \text{for } i = i_0 + 1, \ldots, N - i_0.$$

If $\sigma = 1/4$, i.e., $1/4 \le 4\tilde{b}_0^{-1}\varepsilon \ln N$, then N^{-1} is very small relative to ε. This is unlikely in practice (and in this case the method can be analyzed using standard techniques). We therefore assume that

$$(3.5) \qquad\qquad\qquad\qquad \sigma = 4\tilde{b}_0^{-1}\varepsilon \ln N.$$

From (3.3) and (3.4), it is clear that the interval lengths satisfy

$$(3.6) \qquad\qquad\qquad h_i = 16\tilde{b}_0^{-1}\varepsilon N^{-1} \ln N,$$

for $i = 1, \ldots, i_0, N - i_0 + 1, \ldots, N$, and

$$(3.7) \qquad\qquad\qquad\qquad N^{-1} \le h_i \le 2N^{-1},$$

for $i = i_0 + 1, \ldots, N - i_0$.

Lemma 3.1. *Let $y \in C^4(X)$. Suppose that $y = Y + V$, where*

$$(3.8) \qquad\qquad\qquad\qquad \left|Y^{(j)}(x)\right| \le C$$

and

$$(3.9) \qquad \left| V^{(j)}(x) \right| \le C\varepsilon^{-j} \left[\exp\left(-\tilde{b}_0 x / 2\varepsilon \right) + \exp\left(-\tilde{b}_0 (1-x)/2\varepsilon \right) \right],$$

for $x \in X$ and $j = 0, \ldots, 4$. Then on the Shishkin mesh X_s^N, the truncation error of the scheme (3.1) satisfies

$$(3.10) \qquad \|Fy - F_\varepsilon y\|_\infty \le C \left(\varepsilon^2 N^{-1} + N^{-2} \ln^2 N \right).$$

Proof. Suppose first that x_i is inside the fine mesh, i.e., that

$$i \in \{1, \ldots, i_0 - 1\} \cup \{N - i_0 + 1, \ldots, N - 1\}.$$

By a Taylor expansion, there exist $\xi_i \in (x_{i-1}, x_i)$ and $\eta_i \in (x_i, x_{i+1})$ such that

$$
\begin{aligned}
(3.11) \qquad & (Fy)_i - (F_\varepsilon y)(x_i) \\
& = \frac{h_i^2 - h_{i+1}^2}{6\bar{h}_i} \varepsilon^2 y'''(x_i) - \frac{h_i^3}{24\bar{h}_i} \varepsilon^2 y^{(4)}(\xi_i) - \frac{h_{i+1}^3}{24\bar{h}_i} \varepsilon^2 y^{(4)}(\eta_i) \\
& = -\frac{h_i^3}{24\bar{h}_i} \varepsilon^2 y^{(4)}(\xi_i) - \frac{h_{i+1}^3}{24\bar{h}_i} \varepsilon^2 y^{(4)}(\eta_i),
\end{aligned}
$$

since the mesh is equidistant on $[0, \sigma] \cup [1 - \sigma, 1]$. It is then easy to see, from (3.6), (3.8) and (3.9), that

$$(3.12) \qquad |(Fy)_i - (F_\varepsilon y)(x_i)| \le C N^{-2} \ln^2 N.$$

Now suppose that x_i is no longer inside the fine mesh, i.e., suppose that $i \in \{i_0, \ldots, N - i_0\}$. From (3.9) we see that as $\varepsilon \to 0$ with N fixed, $\varepsilon^2 |y'''(x)|$ is unbounded for $x \in J_\varepsilon \equiv [\sigma, \sigma_\varepsilon] \cup (1 - \sigma_\varepsilon, 1 - \sigma]$, where $\sigma_\varepsilon = \min\{1/4, 4\tilde{b}_0^{-1}\varepsilon \ln(1/\varepsilon)\}$. Recall that the Shishkin mesh X_s^N is coarse on J_ε. Hence, (3.11) will not yield a bound for $|(Fy)_i - (F_\varepsilon y)(x_i)|$ that is uniform in ε.

We therefore use a Taylor expansion with integral remainder to control V. The truncation error of the scheme may be split in the form

$$(3.13) \qquad (Fy)_i - (F_\varepsilon y)(x_i) = (I_Y)_i + (I_V)_i.$$

Here (see (3.11)),

$$(I_Y)_i = \frac{h_i^2 - h_{i+1}^2}{6\bar{h}_i} \varepsilon^2 Y'''(x_i) - \frac{h_i^3}{24\bar{h}_i} \varepsilon^2 Y^{(4)}(\xi_i) - \frac{h_{i+1}^3}{24\bar{h}_i} \varepsilon^2 Y^{(4)}(\eta_i),$$

where $\xi_i \in (x_{i-1}, x_i)$ and $\eta_i \in (x_i, x_{i+1})$ depend now on the function Y, and

$$(3.14)$$
$$(I_V)_i = \frac{\varepsilon^2}{2h_i \bar{h}_i} \int_{x_{i-1}}^{x_i} (s - x_{i-1})^2 V'''(s)\, ds - \frac{\varepsilon^2}{2h_{i+1} \bar{h}_i} \int_{x_i}^{x_{i+1}} (x_{i+1} - s)^2 V'''(s)\, ds.$$

Then we easily get

$$(3.15) \qquad |(I_Y)_i| \le C\varepsilon^2 N^{-1},$$

by (3.7) and (3.8). As for $|(I_V)_i|$, (3.14) and (3.9) give

$$|(I_V)_i| \leq C\varepsilon^{-1} \int_{x_{i-1}}^{x_{i+1}} \left(\exp(-\tilde{b}_0 s/2\varepsilon) + \exp(-\tilde{b}_0(1-s)/2\varepsilon) \right) ds$$

$$\leq C\varepsilon^{-1} \int_{x_{i_0-1}}^{x_{N-i_0+1}} \left(\exp(-\tilde{b}_0 s/2\varepsilon) + \exp(-\tilde{b}_0(1-s)/2\varepsilon) \right) ds$$

(3.16)
$$= \frac{2C}{\tilde{b}_0} \left(\exp(-\tilde{b}_0 x_{i_0-1}/2\varepsilon) - \exp(-\tilde{b}_0 x_{N-i_0+1}/2\varepsilon) \right)$$

$$\leq C \exp(-\tilde{b}_0 x_{i_0-1}/2\varepsilon)$$

$$= CN^{-2} \exp(\tilde{b}_0 h_{i_0}/2\varepsilon), \text{ since } x_{i_0} = \sigma = 4\varepsilon \tilde{b}_0^{-1} \ln N,$$

$$\leq CN^{-2},$$

by (3.6).

Thus, from (3.13) – (3.16), we obtain

$$|(Fy)_i - (F_\varepsilon y)(x_i)| \leq C \left(\varepsilon^2 N^{-1} + N^{-2} \right) \quad \text{for } i \in \{i_0, \ldots, N-i_0\}.$$

Combining this with (3.12) completes the proof.   □

Under the reasonable assumption $\varepsilon \leq N^{-1}$, the estimate (3.10) becomes $\|Fy - F_\varepsilon y\|_\infty \leq CN^{-2} \ln^2 N$. This is much better than the $O(h)$ result obtained by D'Annunzio [2] for the same scheme with a more complicated mesh, where h is the maximum mesh spacing.

4. UNIFORM CONVERGENCE

We use degree theory to investigate the existence and uniform convergence of solutions of the central difference scheme on the Shishkin mesh X_s^N for problem (B). We shall prove that the method is uniformly convergent of order $N^{-2} \ln^2 N$ on this piecewise equidistant mesh.

For this purpose, we imbed problem (B) in the following family of problems:

(4.1) $\tilde{F}_\varepsilon(\tilde{u}, t) \equiv -\varepsilon^2 \tilde{u}_{xx}(x, t) + \tilde{b}(x, t, \tilde{u}(x, t)) = 0$ for $x \in (0, 1)$,

(4.2) $\tilde{u}(0, t) = \tilde{u}(1, t) = 0$,

where $t \in [0, 1]$ is a parameter,

(4.3) $\tilde{b}(x, t, \tilde{u}(x, t)) = tb(x, \tilde{u}(x, t)) + (1 - t) \left(\tilde{u}(x, t) - u_0(x) \right),$

for $(x, t, \tilde{u}) \in [0, 1] \times [0, 1] \times \mathcal{R}^1$, and u_0 is the solution of (1.3). Note that for each x and t we have $\tilde{b}(x, t, u_0(x)) = 0$.

Recall $\tilde{b}_0 = \min\{b_0, 1\}$. We have

$$\tilde{b}_u(x, t, u) = tb_u(x, u) + (1 - t)$$

(4.4)
$$= tb_0^2 + (1 - t)$$

$$\geq \tilde{b}_0^2,$$

for all $(x, t, u) \in [0, 1] \times [0, 1] \times \mathcal{R}^1$. Hence, for each t, problem (4.1) – (4.2) is of the same type as problem (B).

Define the mapping $\tilde{F}(\cdot, \cdot) : \mathcal{R}^{N+1} \times [0, 1] \rightarrow \mathcal{R}^{N+1}$ by

$$\tilde{F}(z, t) = -\varepsilon^2 Az + \tilde{B}(z, t),$$

where $\tilde{B}(\cdot, \cdot) : \mathcal{R}^{N+1} \times [0,1] \to \mathcal{R}^{N+1}$ is given by

$$(4.5) \qquad (\tilde{B}(z,t))_i = \begin{cases} 0 & \text{for } i = 0, \\ \tilde{b}(x_i, t, z_i) & \text{for } i = 1, \dots, N-1, \\ 0 & \text{for } i = N. \end{cases}$$

Then the scheme (3.1) for problem (B) is imbedded in the family of schemes

$$(4.6) \qquad \tilde{F}(z,t) = 0.$$

Let us introduce some more notation and definitions.

For z^1 and $z^2 \in \mathcal{R}^{N+1}$, we denote by $z^1 \leq z^2$ (or $z^1 < z^2$) the natural partial ordering on $\mathcal{R}^{N+1}$, i.e., $z_i^1 \leq z_i^2$ (or $z_i^1 < z_i^2$) for $i = 0, 1, \dots, N$.

Let $M : \mathcal{R}^{N+1} \to \mathcal{R}^{N+1}$ be a mapping. Let $\alpha, \beta \in \mathcal{R}^{N+1}$. If

$$\begin{aligned} M\alpha &< 0, \\ M\beta &> 0, \end{aligned}$$

and

$$\alpha < \beta,$$

then β and α are said to be super and sub solutions of $Mz = 0$, respectively.

Let $\alpha, \beta \in \mathcal{R}^{N+1}$ satisfy $\alpha < \beta$. Let G be a mapping: $\mathcal{R}^{N+1} \to \mathcal{R}^{N+1}$. Define $G^m : \mathcal{R}^{N+1} \to \mathcal{R}^{N+1}$ by

$$(4.7) \qquad (G^m z)_i = \begin{cases} (G\beta)_i + (z_i - \beta_i) & \text{if } z_i > \beta_i, \\ (Gz)_i & \text{if } \alpha_i \leq z_i \leq \beta_i, \\ (G\alpha)_i + (\alpha_i - z_i) & \text{if } z_i < \alpha_i, \end{cases}$$

for $i = 0, 1, \dots, N$. Then G^m is called a modification of G.

We give a strengthening of Theorem 5.1 of D'Annunzio [2].

Lemma 4.1. *Let $D = (d_{ij})$ be an $(N+1) \times (N+1)$ matrix satisfying*

$$(4.8) \qquad d_{ij} \leq 0 \quad \text{for } 0 \leq i, j \leq N, \quad i \neq j,$$

and

$$(4.9) \qquad \sum_{j=1}^{N} d_{ij} \geq 0 \quad \text{for } 0 \leq i \leq N.$$

Let $G : \mathcal{R}^{N+1} \to \mathcal{R}^{N+1}$ be a mapping. Let $\alpha, \beta \in \mathcal{R}^{N+1}$ satisfy $\alpha < \beta$. Let G^m be as in (4.7). Define $M : \mathcal{R}^{N+1} \to \mathcal{R}^{N+1}$ by

$$M = D + G^m.$$

If

$$(4.10) \qquad Mz = 0,$$

$$(4.11) \qquad M\alpha < 0$$

and

$$(4.12) \qquad M\beta > 0,$$

then

$$\alpha < z < \beta.$$

Proof. We shall prove only that $z < \beta$, since $z > \alpha$ may be shown analogously.

Set $\nu = z - \beta$. We prove that $\nu < 0$. Suppose that $\nu < 0$ is false. Then for some $i \in \{0, 1, \ldots, N\}$, we have $\nu_i \geq 0$. Let k be an integer such that

$$(4.13) \qquad \nu_k = \max_{0 \leq i \leq N} \{\nu_i\}.$$

Clearly

$$(4.14) \qquad \nu_k \geq 0.$$

By (4.10),

$$\begin{aligned} 0 &= (Dz)_k + (G^m z)_k \\ &= (Dz)_k + (G\beta)_k + (z_k - \beta_k), \end{aligned}$$

from (4.7) and (4.14). Hence, using (4.7) and (4.12), we get

$$\begin{aligned} -\nu_k &= (Dz)_k + (G\beta)_k \\ &= (Dz)_k + (G^m \beta)_k \\ &> (Dz)_k - (D\beta)_k \\ &= (D\nu)_k \\ &= \sum_{j=0}^{N} d_{kj} \nu_j \\ &\geq \left(\sum_{j=0}^{N} d_{kj} \right) \nu_k, \quad \text{by (4.8) and (4.13),} \\ &\geq 0, \end{aligned}$$

by (4.9) and (4.14). That is, $\nu_k < 0$. This contradicts (4.14) and the proof of Lemma 4.1 is completed. $\qquad \square$

D'Annunzio [2] proved the same result under the extra conditions $d_{ii} > 0$ for $i = 0, 1, \ldots, N$, while assuming that strict inequality holds in (4.9) for at least one i.

For each $t \in [0, 1]$, set

$$\tilde{w}(x, t, \varepsilon, p) = \begin{cases} \left(\tilde{v}_0^0 \left(x/\varepsilon, t, p \right) + \varepsilon \tilde{v}_1^0 \left(x/\varepsilon, t \right) \right) \sigma(x) & \text{for } 0 \leq x \leq 1/2, \\ \left(\tilde{v}_0^1 \left((1-x)/\varepsilon, t, p \right) + \varepsilon \tilde{v}_1^1 \left((1-x)/\varepsilon, t \right) \right) \sigma(1-x) \\ \hspace{7cm} \text{for } 1/2 < x \leq 1, \end{cases}$$

where $\tilde{v}_0^0(\eta, t, p)$, $\tilde{v}_0^1(\eta, t, p)$, $\tilde{v}_1^0(\eta, t)$, and $\tilde{v}_1^1(\eta, t)$ are respectively defined by

$$\begin{aligned} \ddot{\tilde{v}}_0^0 - \tilde{b} \left(0, t, u_0(0) + \tilde{v}_0^0 \right) &= -p\tilde{v}_0^0 \quad \text{for } \eta > 0, \\ \tilde{v}_0^0(0, t, p) = -u_0(0), & \quad \tilde{v}_0^0(\infty, t, p) = 0, \end{aligned}$$

$$\begin{aligned} \ddot{\tilde{v}}_0^1 - \tilde{b} \left(1, t, u_0(1) + \tilde{v}_0^1 \right) &= -p\tilde{v}_0^1 \quad \text{for } \eta > 0, \\ \tilde{v}_0^1(0, t, p) = -u_0(1), & \quad \tilde{v}_0^1(\infty, t, p) = 0, \end{aligned}$$

$$\overset{\cdot\cdot}{\tilde{v}}{}_1^0 - \tilde{b}_u\left(0, t, u_0(0) + \tilde{v}_0^0(\eta, 0)\right)\tilde{v}_1^0$$

$$= \eta\left[\tilde{b}_x\left(0, t, u_0(0) + \tilde{v}_0^0(\eta, 0)\right) + \tilde{b}_u\left(0, t, u_0(0) + \tilde{v}_0^0(\eta, 0)\right)u_0'(0)\right]$$

$$\text{for } \eta > 0,$$

$$\tilde{v}_1^0(0, t) = 0, \quad \tilde{v}_1^0(\infty, t) = 0,$$

and

$$\overset{\cdot\cdot}{\tilde{v}}{}_1^1 - \tilde{b}_u\left(1, t, u_0(1) + \tilde{v}_0^1(\eta, 0)\right)\tilde{v}_1^1$$

$$= \eta\left[\tilde{b}_x\left(1, t, u_0(1) + \tilde{v}_0^1(\eta, 0)\right) + \tilde{b}_u\left(1, t, u_0(1) + \tilde{v}_0^1(\eta, 0)\right)u_0'(1)\right]$$

$$\text{for } \eta > 0,$$

$$\tilde{v}_1^1(0, t) = 0, \quad \tilde{v}_1^1(\infty, t) = 0.$$

Recall that $\tilde{b}_0$ is independent of t in (4.4). One may show, by the arguments of §2, that there is a $\tilde{p}_0 > 0$, independent of ε and t, such that $w(x, t, \varepsilon, p)$ is well defined for $|p| \le \tilde{p}_0$. Furthermore, we have

(4.15) $$0 \le \frac{\partial \tilde{w}}{\partial p}(x, t, \varepsilon, p) \le C$$

and

(4.16) $$\left|\frac{\partial^j \tilde{w}}{\partial x^j}(x, t, \varepsilon, p)\right| \le C\varepsilon^{-j}\left(\exp\left(-(\tilde{b} - \delta)x/\varepsilon\right) + \exp\left(-(\tilde{b} - \delta)(1 - x)/\varepsilon\right)\right),$$

for $(x, t) \in [0, 1] \times [0, 1]$ and $j = 0, 1, 2, 3, 4$. Here, $\tilde{b}_0^2 > p$ and $\tilde{b} = \sqrt{\tilde{b}_0^2 - p}$ with $\tilde{b}_0$ given by (4.4) and δ any fixed number in $(0, \tilde{b})$.

Note that problem (4.1) – (4.3) becomes (1.1) when $t = 1$. We have

(4.17) $$\tilde{w}(x, 1, \varepsilon, p) = w(x, \varepsilon, p) \quad \text{for } x \in [0, 1].$$

Assumption 4.1. In what follows, we shall assume that $\varepsilon \le N^{-1}$, which is non-restrictive in practice.

Lemma 4.2. *Set* $\tilde{p}_N = N^{-2}\ln^2 N$. *Let* $t \in [0, 1]$. *Then we can choose positive constants* $\tilde{C}_1 > 0$ *and* $\tilde{C}_2 > 0$, *which are independent of* N, ε *and* t, *and a positive integer* N_0, *which depends on* $\tilde{C}_1$ *and* $\tilde{C}_2$ *but is independent of* ε *and* t, *such that for each fixed* $t \in [0, 1]$, *when* $N \ge N_0$, *the functions* $\tilde{w}(x, t, \varepsilon, \tilde{C}_1\tilde{p}_N)$ *and* $\tilde{w}(x, t, \varepsilon, -\tilde{C}_1\tilde{p}_N)$ *are well defined, and*

(4.18) $$\tilde{\beta}^N(x, t) = u_0(x) + \tilde{w}(x, t, \varepsilon, \tilde{C}_1\tilde{p}_N) + \tilde{C}_2\tilde{p}_N$$

and

(4.19) $$\tilde{\alpha}^N(x, t) = u_0(x) + \tilde{w}(x, t, \varepsilon, -\tilde{C}_1\tilde{p}_N) - \tilde{C}_2\tilde{p}_N$$

are super and sub solutions, respectively, of (4.6) *on the Shishkin mesh* X_4^N.

Proof. Let

$$\tilde{\beta}^N(x, t) = u_0(x) + \tilde{w}(x, t, \varepsilon, \tilde{C}_1\tilde{p}_N) + \tilde{C}_2\tilde{p}_N$$

and

$$\tilde{\alpha}^N(x, t) = u_0(x) + \tilde{w}(x, t, \varepsilon, -\tilde{C}_1\tilde{p}_N) - \tilde{C}_2\tilde{p}_N,$$

where $\tilde{C}_1 > 0$ and $\tilde{C}_2 > 0$ will be chosen later.

For each $t \in [0, 1]$, it is clear that

$$\tilde{\alpha}^N(\cdot, t) < \tilde{\beta}^N(\cdot, t).$$

We now prove that $\tilde{F}(\tilde{\beta}^N, t) > 0$. From the definitions of the terms involved,

$$(4.20) \qquad \left(\tilde{F}(\tilde{\beta}^N, t)\right)_0 = \left(\tilde{F}(\tilde{\beta}^N, t)\right)_N = \tilde{C}_1 \tilde{p}_N > 0.$$

Fix $i \in \{1, 2, \ldots N - 1\}$ and $t \in [0, 1]$. We have

$$\left(\tilde{F}(\tilde{\beta}^N, t)\right)_i = \left(\left(\tilde{F}(\tilde{\beta}^N, t)\right)_i - \left(\tilde{F}_\varepsilon(\tilde{\beta}^N, t)\right)(x_i, t, \varepsilon)\right)$$
$$+ \left(\tilde{F}_\varepsilon(\tilde{\beta}^N, t)\right)(x_i, t, \varepsilon).$$

We separately analyze these two terms.

First, take $N_1 > 0$ such that $\tilde{C}_1 \tilde{p}_N < \tilde{b}_0^2/4$ for $N \geq N_1$. Then for $N \geq N_1$ and $\delta = \tilde{b}_0/4$ in (4.16), we have

$$\left|\frac{\partial^j \tilde{w}}{\partial x^j}(x, t, \varepsilon, \tilde{C}_1 \tilde{p}_N)\right| \leq \tilde{C}\varepsilon^{-j}\left(\exp\left(-\tilde{b}_0 x/2\varepsilon\right) + \exp\left(-\tilde{b}_0(1 - x)/2\varepsilon\right)\right),$$

for $(x, t) \in [0, 1] \times [0, 1]$ and $j = 0, \ldots, 4$. On the Shishkin mesh X_4^N one has, using Lemma 3.1 and $\varepsilon \leq N^{-1}$,

$$\left|\left(\tilde{F}(\tilde{\beta}^N, t)\right)_i - \left(\tilde{F}_\varepsilon(\tilde{\beta}^N, t)\right)(x_i, t, \varepsilon)\right| \leq \tilde{C}\tilde{p}_N,$$

for some positive constant $\tilde{C}$ that is independent of $\tilde{C}_1, \tilde{C}_2, N, \varepsilon$ and t.

Next, we can easily adapt the proof of Lemma 2.4 to show that

$$\tilde{F}_\varepsilon(\tilde{\beta}^N, t)(x_i, t, \varepsilon) \geq 2\tilde{C}\tilde{p}_N,$$

for sufficiently large $\tilde{C}_1$ and $\tilde{C}_2$. Hence,

$$\left(\tilde{F}(\tilde{\beta}^N, t)\right)_i > 0 \quad \text{for } i = 1, \ldots, N - 1.$$

Combining this with (4.20) yields $\tilde{F}(\tilde{\beta}^N, t) > 0$.

We can similarly show that $\tilde{F}(\tilde{\alpha}^N, t) > 0$, to complete the proof. $\qquad \square$

We now introduce a modified problem corresponding to (4.6). Consider

$$\tilde{F}^m(z, t) = 0,$$

where the mapping $\tilde{F}^m(\cdot, \cdot) : \mathcal{R}^{N+1} \times [0, 1] \to \mathcal{R}^{N+1}$ is defined by

$$\tilde{F}^m(z, t) = -\varepsilon^2 A z + \tilde{B}^m(z, t).$$

Here, $\tilde{B}^m(\cdot, t)$ is the modification of $\tilde{B}(\cdot, t)$, with $\tilde{\beta}^N$ and $\tilde{\alpha}^N$ given by (4.18) and (4.19), respectively, for each t; see (4.7).

Define an open and bounded set $D_t \subset \mathcal{R}^{N+1}$ for each $t \in [0, 1]$ by

$$D_t = \left\{z \in \mathcal{R}^{N+1} : \tilde{\alpha}^N(\cdot, t) < z < \tilde{\beta}^N(\cdot, t)\right\}.$$

We shall denote by $\bar{D}_t$ and ∂D_t the closure and the boundary, respectively, of D_t in $\mathcal{R}^{N+1}$.

Define the mapping $T(\cdot, \cdot) : \bar{D}_1 \times [0, 1] \to \mathcal{R}^{N+1}$ by

$$(T(z, t))_i = \left(z_i - \tilde{\alpha}^N(x_i, 1)\right) \frac{\tilde{\beta}^N(x_i, t)}{\tilde{\beta}^N(x_i, 1) - \tilde{\alpha}^N(x_i, 1)}$$

$$+ \left(\tilde{\beta}^N(x_i, 1) - z_i\right) \frac{\tilde{\alpha}^N(x_i, t)}{\tilde{\beta}^N(x_i, 1) - \tilde{\alpha}^N(x_i, 1)},$$

for $i = 0, 1, \dots, N$. It is easy to see that, for each $t \in [0, 1]$, $T(\cdot, t)$ is a linear transformation from $\bar{D}_1$ onto $\bar{D}_t$.

We finally define a mapping $\tilde{H}(\cdot, \cdot) : \bar{D}_1 \times [0, 1] \to \mathcal{R}^{N+1}$ by

$$\tilde{H}(z, t) = \tilde{F}^m(T(z, t), t) \quad \text{for } (z, t) \in \bar{D}_1 \times [0, 1].$$

This is a continuously differentiable mapping and satisfies

(4.21) $$\tilde{H}(z, 1) = \tilde{F}^m(z, 1) = \tilde{F}(z, 1) = Fz,$$

for $z \in \bar{D}_1$. We shall prove that

$$\left| \text{Deg}(\tilde{H}(\cdot, 1), D_1, 0) \right| = 1,$$

where Deg denotes topological degree (see, e.g., Ortega and Rheinboldt [13]), by using the Homotopy Invariance Theorem [13, Theorem 6.2.2].

We first show the following:

Lemma 4.3. *There holds*

$$\tilde{H}(z, t) \neq 0 \quad \text{for all } (z, t) \in \partial D_1 \times [0, 1].$$

Proof. Suppose that $\tilde{H}(z^*, t^*) = 0$ for some $(z^*, t^*) \in \bar{D}_1 \times [0, 1]$. Set $T^* = T(z^*, t^*)$. Then $T^* \in \bar{D}_{t^*}$ satisfies

(4.22) $$\tilde{F}^m(T^*, t^*) = 0.$$

From the definition of $\tilde{F}^m(\cdot, \cdot)$ and Lemma 4.2, we have

(4.23) $$\tilde{F}^m(\tilde{\alpha}^N(\cdot, t^*), t^*) = \tilde{F}(\tilde{\alpha}^N(\cdot, t^*), t^*) < 0,$$

(4.24) $$\tilde{F}^m(\tilde{\beta}^N(\cdot, t^*), t^*) = \tilde{F}(\tilde{\beta}^N(\cdot, t^*), t^*) > 0.$$

If we set $D = -\varepsilon^2 A$, the conditions (4.8) and (4.9) are satisfied. Combining (4.22) – (4.24) with Lemma 4.1 yields

$$\tilde{\alpha}^N(\cdot, t^*) < T^* < \tilde{\beta}^N(\cdot, t^*).$$

From the definition of $T(\cdot, \cdot)$, we obtain

$$z_i^* = \left(T_i^* - \tilde{\alpha}^N(x_i, t^*)\right) \frac{\tilde{\beta}^N(x_i, 1)}{\tilde{\beta}^N(x_i, t^*) - \tilde{\alpha}^N(x_i, t^*)}$$

$$+ \left(\tilde{\beta}^N(x_i, t^*) - T_i^*\right) \frac{\tilde{\alpha}^N(x_i, 1)}{\tilde{\beta}^N(x_i, t^*) - \tilde{\alpha}^N(x_i, t^*)},$$

for $i = 0, 1, \dots, N$. Hence,

$$\tilde{\alpha}^N(\cdot, 1) < z^* < \tilde{\beta}^N(\cdot, 1),$$

i.e., $z^* \notin \partial D_1$, which is the desired result. $\square$

Now we have

Lemma 4.4. *If $\tilde{C}_1$ and $\tilde{C}_2$ in (4.18) and (4.19) are chosen sufficiently large, then*

$$\left| \mathrm{Deg}(\tilde{H}(\cdot, 0), D_1, 0) \right| = 1.$$

Proof. We start with the problem

$$(4.25) \qquad \tilde{F}(z, 0) = 0 \quad \text{for } z \in \mathcal{R}^{N+1}.$$

Set

$$S = \begin{pmatrix} 0 & 0 & 0 & & & & \\ 0 & 1 & 0 & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & 0 & 1 & 0 \\ & & & & 0 & 0 & 0 \end{pmatrix}.$$

Then (4.25) can be written in the form

$$(-\varepsilon^2 A + S)z - S u_0 = 0,$$

from (4.3) and (4.5). The matrix $-\varepsilon^2 A + S$ is an M-matrix and thus has a positive inverse. Consequently, (4.25) has a unique solution

$$z^* = (-\varepsilon^2 A + S)^{-1} S u_0 \in \mathcal{R}^{N+1}.$$

We wish to prove that $z^* \in D_0$. We have

$$\begin{aligned} (-\varepsilon^2 A + S)\left(\tilde{\beta}^N(\cdot, 0) - z^*\right) &= \tilde{F}(\tilde{\beta}^N(\cdot, 0), 0) - \tilde{F}(z^*, 0) \\ &= \tilde{F}(\tilde{\beta}^N(\cdot, 0), 0) \\ &> 0, \end{aligned}$$

by (4.25) and (4.24). Hence,

$$\tilde{\beta}^N(\cdot, 0) > z^*.$$

Similarly,

$$\tilde{\alpha}^N(\cdot, 0) < z^*.$$

That is, $z^* \in D_0$.

We now consider the problem

$$(4.26) \qquad \tilde{H}(z, 0) = 0 \quad \text{for } z \in \bar{D}_1.$$

As $T(z, 0) \in \bar{D}_0$, the problem (4.26) is equivalent to $\tilde{F}(T(z, 0), 0) = 0$. But from above, (4.25) has a unique solution $z^* \in D_0$. Consequently, we need look only for solutions $z \in \bar{D}_1$ of

$$(4.27) \qquad T(z, 0) = z^*.$$

Recalling that $T(\cdot, 0)$ is a linear mapping from $\bar{D}_1$ onto $\bar{D}_0$, so $\partial D_0 = T(\partial D_1, 0)$, we conclude that (4.27) has a unique solution $\tilde{z} \in D_1$. That is, (4.26) has a unique solution, which lies in D_1.

Furthermore, we have for $z \in D_1$,

$$\frac{\partial \tilde{H}}{\partial z}(z,0) = \frac{\partial \tilde{F}^m}{\partial T}(T(z,0),0)\frac{\partial T}{\partial z}(z,0)$$

$$= \frac{\partial \tilde{F}}{\partial T}(T(z,0),0)\frac{\partial T}{\partial z}(z,0)$$

$$= (-\varepsilon^2 A + S)\frac{\partial T}{\partial z}(z,0).$$

From above, we know that

$$\det(-\varepsilon^2 A + S) \neq 0.$$

Since $\tilde{\alpha}^N(\cdot,0) < \tilde{\beta}^N(\cdot,0)$, we have

$$\det\left(\frac{\partial T}{\partial z}(z,0)\right) \neq 0 \quad \text{for all } z \in \mathcal{R}^{N+1}.$$

Therefore,

$$\det\left(\frac{\partial \tilde{H}}{\partial z}(z,0)\right) \neq 0 \quad \text{for all } z \in D_1.$$

We have shown that (4.26) has a unique solution z^*, which lies in D_1, with $\det\left(\left.\frac{\partial \tilde{H}}{\partial z}\right|_{z=z^*}\right) \neq 0$. This completes the proof. $\qquad \square$

Theorem 4.1. *Let $u(x)$ be the solution of problem* (B) *guaranteed by Theorem* 2.1. *Assume that $\varepsilon \leq N^{-1}$. For N sufficiently large, independently of ε, the scheme $\{F, X_s^N\}$ has a solution u^N such that*

$$\|u - u^N\|_\infty \leq CN^{-2}\ln^2 N.$$

Proof. Let α and β be given by (2.19) and (2.20). Then

(4.28) $$\alpha \leq u \leq \beta,$$

by Theorem 2.1.

On the other hand, from Lemma 4.3,

$$\text{Deg}(\tilde{H}(\cdot,t), D_1, 0) \text{ is constant for } t \in [0,1],$$

by the Homotopy Invariance Theorem [13, Theorem 6.2.2]. Hence,

$$\left|\text{Deg}(\tilde{H}(\cdot,1), D_1, 0)\right| = \left|\text{Deg}(\tilde{H}(\cdot,0), D_1, 0)\right| = 1,$$

by Lemma 4.4. This implies that the equation

$$\tilde{H}(z,1) = 0$$

has at least one solution $u^N \in D_1$. Recall (4.21). We see that $\{F, X_s^N\}$ has a solution that satisfies

(4.29) $$\tilde{\alpha}^N(\cdot,1) < u^N < \tilde{\beta}^N(\cdot,1).$$

Choose $\tilde{C}_1$ and $\tilde{C}_2$ in Lemma 4.2 sufficiently large such that $\tilde{C}_1 \geq C_1$ and $\tilde{C}_2 \geq C_2$, where C_1 and C_2 are given in Lemma 2.4. Then $C_1 p_\varepsilon \leq \tilde{C}_1 \tilde{p}_N$ and

$C_2 p_\varepsilon \leq \tilde{C}_2 \tilde{p}_N$, since $\varepsilon \leq N^{-1}$. Hence, for $i = 0, 1, \dots, N$,

$$
\begin{aligned}
\beta(x_i, \varepsilon) &= u_0(x_i) + w(x_i, \varepsilon, C_1 p_\varepsilon) + C_2 p_\varepsilon \\
&= u_0(x_i) + \tilde{w}(x_i, 1, \varepsilon, C_1 p_\varepsilon) + C_2 p_\varepsilon \\
&\leq u_0(x_i) + \tilde{w}(x_i, 1, \varepsilon, \tilde{C}_1 \tilde{p}_N) + \tilde{C}_2 \tilde{p}_N \\
&= \tilde{\beta}^N(x_i, 1),
\end{aligned}
$$

by (4.17) and (4.15). Similarly, for $i = 0, 1, \dots, N$,

$$
\tilde{\alpha}^N(x_i, 1) \leq \alpha(x_i, \varepsilon).
$$

That is,

$$
\tag{4.30} \tilde{\alpha}^N(\cdot, 1) \leq \alpha < \beta \leq \tilde{\beta}^N(\cdot, 1).
$$

We have

$$
\tag{4.31}
\begin{aligned}
\tilde{\beta}^N(x_0, 1) - \tilde{\alpha}^N(x_0, 1) &= \tilde{\beta}^N(x_N, 1) - \tilde{\alpha}^N(x_N, 1) \\
&= 2\tilde{C}_2 N^{-2} \ln^2 N
\end{aligned}
$$

and, for $i = 1, \dots, N - 1$,

$$
\tag{4.32}
\begin{aligned}
& \left| \tilde{\beta}^N(x_i, 1) - \tilde{\alpha}^N(x_i, 1) \right| \\
& \leq 2\tilde{C}_1 N^{-2} \ln^2 N \left| \frac{\partial w}{\partial p}(x_i, \varepsilon, p^*) \right| + 2\tilde{C}_2 N^{-2} \ln^2 N, \\
& \qquad\qquad\qquad\qquad \text{where } p^* \in (-\tilde{C}_1 \tilde{p}_N, \tilde{C}_1 \tilde{p}_N), \\
& \leq C N^{-2} \ln^2 N,
\end{aligned}
$$

by (2.27) and the relation $w(x, \varepsilon, p) = \hat{w}(x/\varepsilon, \varepsilon, p)$.

Therefore, from (4.28) and (4.29) – (4.32),

$$
\|u - u^N\|_\infty \leq \|\tilde{\beta}^N(\cdot, 1) - \tilde{\alpha}^N(\cdot, 1)\|_\infty \leq C N^{-2} \ln^2 N,
$$

which is the desired result. $\square$

Theorem 4.1 achieves uniform accuracy that is almost one order higher than that of D'Annunzio [2], who uses a more complicated locally quasi–equidistant mesh.

5. NUMERICAL RESULTS

In this section we present numerical results to confirm the uniform accuracy of the scheme $\{F, X_s^N\}$.

The nonlinear system of equations is solved using Newton's method with the initial guess $u^{N,0} = (0, u_0(x_1), \dots, u_0(x_{N-1}), 0)^T$. Here, u_0 is a stable reduced solution with stable boundary layers. We iteratively compute $u^{N,k}$, for $k = 1, 2, \dots$, as successive approximations to u^N. The stopping criterion used is

$$
\max \left\{ \|F u^{N,k}\|_\infty, \|u^{N,k} - u^{N,k-1}\|_\infty \right\} < 0.1 N^{-2}.
$$

For each N and ε in the tables, it takes only about five iterations to satisfy this criterion.

The exact solutions of our test problems are unknown. We use a double-mesh method [3] to compute the experimental rates of convergence. In order to do this, we shall in addition to computing u^N also compute another approximate solution $\tilde{u}^N$ that we now describe.

Let $\tilde{u}^N \in \mathcal{R}^{N+1}$ be a solution of $\{F, \tilde{X}_s^N\}$, where $\tilde{X}_s^N$ is a Shishkin mesh with the mesh parameter σ of (3.2) altered slightly to

$$\tilde{\sigma} = \min\{1/4, 4\tilde{b}_0^{-1}\varepsilon \ln(N/2)\}.$$

Then for $i = 0, 1, \dots, N$, the ith point of the mesh X_s^N coincides with the $(2i)$th point of the mesh $\tilde{X}_s^{2N}$.

By inspecting the arguments of §§3 and 4, one may see that when $\varepsilon \leq N^{-1}$,

$$\|u - \tilde{u}^N\|_\infty \leq C(N^{-1}\ln N)^2,$$

where C is independent of N and ε. Hence, for $i = 0, 1, \dots, N$,

$$|(u^N)_i - (\tilde{u}^{2N})_{2i}| \leq C(N^{-1}\ln N)^2.$$

For each N and ε, we shall report

$$\tilde{E}_\varepsilon^N = \max_{0 \leq i \leq N} |(u^N)_i - (\tilde{u}^{2N})_{2i}|$$

in the error tables below.

Assuming convergence of order $(N^{-1}\ln N)^r$ for some r, we estimate the classical convergence rate r from

$$
\begin{aligned}
R_\varepsilon^N &= \frac{\ln \tilde{E}_\varepsilon^N - \ln \tilde{E}_\varepsilon^{2N}}{\ln\left(\frac{2\ln N}{\ln 2N}\right)} \\
&= \frac{\ln \tilde{E}_\varepsilon^N - \ln \tilde{E}_\varepsilon^{2N}}{\ln\left(\frac{2k}{k+1}\right)}, \quad \text{for } N = 2^k \text{ and } k = 5, 6, \dots, 11.
\end{aligned}
$$

The last row of each rate table is the *uniform* convergence rate,

$$R^N = \frac{\ln \tilde{E}^N - \ln \tilde{E}^{2N}}{\ln\left(\frac{2k}{k+1}\right)},$$

where $\tilde{E}^N = \max_\varepsilon \tilde{E}_\varepsilon^N$.

Example. Consider the following problem of Herceg [7]:

(5.1a) $-\varepsilon^2 u'' + (u^2 + u - 0.75)(u^2 + u - 3.75) = 0$ for $x \in (0, 1)$,

(5.1b) $u(0) = u(1) = 0.$

We have

$$b_u(x, u) = (2u + 1)(2u^2 + 2u - 4.5).$$

The reduced problem

$$b(x, u) = 0$$

has four solutions $u_1 = -2.5$, $u_2 = -1.5$, $u_3 = 0.5$ and $u_4 = 1.5$. It is easy to see that

$$b_u(x, u_1) = -12, \quad b_u(x, u_2) = 6, \quad b_u(x, u_3) = -6 \text{ and } b_u(x, u_4) = 12.$$

Hence, u_1 and u_3 are not stable reduced solutions of (5.1). A calculation shows that u_2 and u_4 satisfy the conditions (1.4). Thus (5.1) is a problem of type (B) with two stable reduced solutions u_2 and u_4. Each of u_2 and u_4 is "close" (in the sense of Theorem 2.1) to a solution of (5.1) when ε is sufficiently small. We apply the scheme $\{F, X_s^N\}$ to compute these solutions of (5.1).

TABLE 5.1. Errors for solution near u_2

ε	N=64	128	256	512	1024
2.500000e-01	3.4316e-04	8.6686e-05	2.1679e-05	5.4227e-06	1.3557e-06
6.250000e-02	3.5179e-03	1.1715e-03	3.4384e-04	8.6876e-05	2.1726e-05
1.562500e-02	3.5180e-03	1.1715e-03	3.7359e-04	1.1596e-04	3.5084e-05
3.906250e-03	3.5180e-03	1.1715e-03	3.7359e-04	1.1596e-04	3.5084e-05
9.765625e-04	3.5179e-03	1.1715e-03	3.7359e-04	1.1596e-04	3.5084e-05
2.441406e-04	3.5179e-03	1.1715e-03	3.7359e-04	1.1596e-04	3.5084e-05
6.103516e-05	3.5179e-03	1.1715e-03	3.7359e-04	1.1596e-04	3.5084e-05
1.525879e-05	3.5179e-03	1.1715e-03	3.7359e-04	1.1596e-04	3.5084e-05
3.814697e-06	3.5179e-03	1.1715e-03	3.7359e-04	1.1596e-04	3.5084e-05
9.536743e-07	3.5179e-03	1.1715e-03	3.7359e-04	1.1596e-04	3.5084e-05

TABLE 5.2. Convergence rates for solution near u_2

ε	N=64	128	256	512
2.500000e-01	2.55	2.48	2.41	2.36
6.250000e-02	2.04	2.19	2.39	2.36
1.562500e-02	2.04	2.04	2.03	2.03
3.906250e-03	2.04	2.04	2.03	2.03
9.765625e-04	2.04	2.04	2.03	2.03
2.441406e-04	2.04	2.04	2.03	2.03
6.103516e-05	2.04	2.04	2.03	2.03
1.525879e-05	2.04	2.04	2.03	2.03
3.814697e-06	2.04	2.04	2.03	2.03
9.536743e-07	2.04	2.04	2.03	2.03
R^N	2.04	2.04	2.03	2.03

TABLE 5.3. Errors for solution near u_4

ε	N=64	128	256	512	1024
2.500000e-01	1.1820e-03	2.9456e-04	7.3582e-05	1.8397e-05	4.5991e-06
6.250000e-02	5.5968e-03	1.8000e-03	5.6725e-04	1.7490e-04	5.2884e-05
1.562500e-02	5.6164e-03	1.8021e-03	5.6737e-04	1.7490e-04	5.2884e-05
3.906250e-03	5.6057e-03	1.8022e-03	5.6760e-04	1.7493e-04	5.2885e-05
9.765625e-04	5.5976e-03	1.8008e-03	5.6743e-04	1.7493e-04	5.2888e-05
2.441406e-04	5.5951e-03	1.8002e-03	5.6731e-04	1.7491e-04	5.2886e-05
6.103516e-05	5.5944e-03	1.8000e-03	5.6727e-04	1.7490e-04	5.2884e-05
1.525879e-05	5.5943e-03	1.8000e-03	5.6726e-04	1.7490e-04	5.2884e-05
3.814697e-06	5.5942e-03	1.8000e-03	5.6726e-04	1.7490e-04	5.2884e-05
9.536743e-07	5.5942e-03	1.8000e-03	5.6726e-04	1.7490e-04	5.2884e-05

The numerical results for the example show that the scheme is capable of computing those solutions of problem (B) that lie close to particular reduced solutions. Furthermore, the scheme achieves second-order accuracy for this difficult problem, confirming our theoretical results.

TABLE 5.4. Convergence rates for solution near u_4

ε	N=64	128	256	512
2.500000e-01	2.58	2.48	2.41	2.36
6.250000e-02	2.10	2.06	2.04	2.03
1.562500e-02	2.11	2.07	2.05	2.04
3.906250e-03	2.11	2.06	2.05	2.04
9.765625e-04	2.10	2.06	2.05	2.04
2.441406e-04	2.10	2.06	2.04	2.03
6.103516e-05	2.10	2.06	2.04	2.03
1.525879e-05	2.10	2.06	2.04	2.03
3.814697e-06	2.10	2.06	2.04	2.03
9.536743e-07	2.10	2.06	2.04	2.03
R^N	2.10	2.06	2.04	2.03

$\triangledown$ the computed solution u^{12}

— Piecewise linear interpolant of the computer solution u^{256}
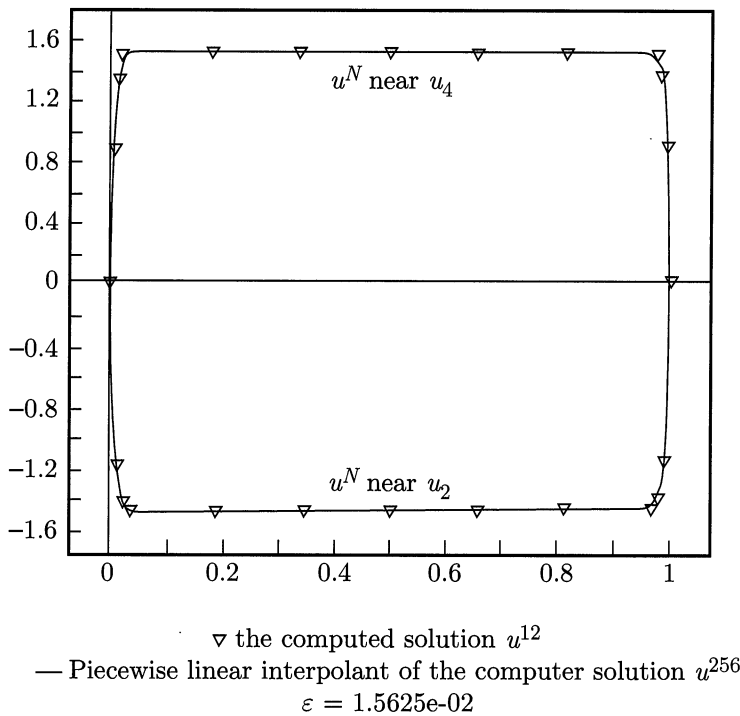
$\varepsilon = 1.5625$e-02

FIGURE 1

In Figure 1 we display the computed solutions of (5.1) that lie near u_2 and u_4 when $\varepsilon=1.5625$e–02, with $N=12$ (discrete points marked by triangles) and $N=256$ (continuous piecewise linear interpolant to the computed solution). The proximity of the solutions for $N=12$ and $N=256$ demonstrates the accuracy of the method when only a small number of points is used.

Figure 2 shows the behavior of the same problem inside the layer region $0 \le x \le 0.04$. It compares the solution for $N = 24$ (discrete points marked by triangles) with the continuous piecewise linear interpolant to the computed solution for $N = 256$.
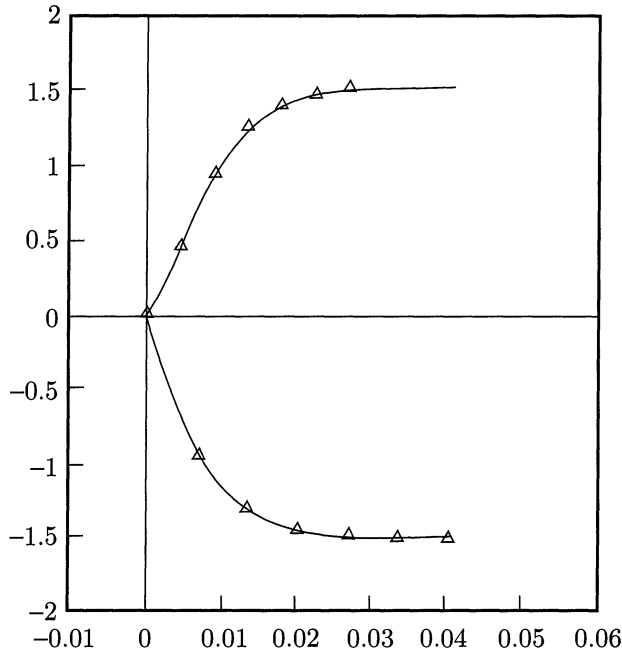
Enlargement of computer solutions near $x = 0$;
$\triangle$ is u^{24} and the continuous curve is
the piecewise linear interpolant of u^{256}

FIGURE 2

Clearly, the method tracks the layer accurately even when using relatively few points.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE, CORK, IRELAND
E-mail address: stynes@ucc.ie