ON QUADRATURE CONVERGENCE OF EXTENDED LAGRANGE INTERPOLATION

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ABSTRACT. Quadrature convergence of the extended Lagrange interpolant $L_{2n+1}f$ for any continuous function f is studied, where the interpolation nodes are the n zeros τ_i of an orthogonal polynomial of degree n and the $n+1$ zeros $\hat{\tau}_j$ of the corresponding "induced" orthogonal polynomial of degree $n+1$. It is found that, unlike convergence in the mean, quadrature convergence *does* hold for all four Chebyshev weight functions. This is shown by establishing the positivity of the underlying quadrature rule, whose weights are obtained explicitly. Necessary and sufficient conditions for positivity are also obtained in cases where the nodes τ_i and $\hat{\tau}_j$ interlace, and the conditions are checked numerically for the Jacobi weight function with parameters α and β. It is conjectured, in this case, that quadrature convergence holds for $|\alpha| \leq \frac{1}{2}$, $|\beta| \leq \frac{1}{2}$.

1. INTRODUCTION

If $\pi_n(\cdot\,;w)$, $n \geq 1$, denotes the nth-degree orthogonal polynomial on $[-1,1]$ with respect to a positive weight function w, and $(L_nf)(\cdot)$ the Lagrange interpolation polynomial of degree $< n$ interpolating f at the zeros $\{\tau_i\}$ of π_n, it is a well-known result of Erdös and Turán [2] that L_nf converges in the mean to f for any continuous function f. That is,

$$(1.1) \qquad \lim_{n\to\infty} \| f - L_nf \|_w = 0, \text{ all } f \in C[-1,1],$$

where $\| g \|_w = \left(\int_{-1}^1 g^2(t)w(t)dt\right)^{1/2}$. Attempts have been made in the past to obtain an analogous result for the extended Lagrange interpolant $(\hat{L}_{2n+1}f)(\cdot)$ interpolating f at $2n + 1$ points — the n points $\{\tau_i\}$ and $n + 1$ additional points $\{\hat{\tau}_j\}$ suitably chosen. A particularly interesting choice of the $\hat{\tau}_j$, first suggested by Bellen [1], is given by the zeros of $\hat{\pi}_{n+1}$, the polynomial $\hat{\pi}_{n+1}(\cdot) = \pi_{n+1}(\cdot\,;\pi_n^2 w)$ of degree $n + 1$ "induced by π_n", i.e., orthogonal relative to the weight function $\pi_n^2 w$ (cf. [5]). Concrete results have only been obtained in the case of Chebyshev weight functions. The one of the second kind, $w(t) = (1 - t^2)^{1/2}$, is particularly easy, since in this case $\{\tau_i\} \cup \{\hat{\tau}_j\}$ are precisely the zeros of $U_nT_{n+1} = U_{2n+1}$ (cf. [1]), and one is led back to the Erdös-Turán result. For all other three Chebyshev weight functions, however, one of us [3] has shown that mean convergence cannot hold for all continuous functions.

It may be interesting to ask the same question for what Erdös and Turán called *quadrature convergence*. In their scenario, that would mean

$$(1.2) \qquad \lim_{n \to \infty} \int_{-1}^{1} [f(t) - (L_n f)(t)] w(t) dt = 0, \text{ all } f \in C[-1, 1],$$

which is obviously true, since the integral over $L_n f$ is just the n-point Gauss quadrature sum relative to the weight function w. Is it true that the same holds for extended interpolation,

$$(1.3) \qquad \lim_{n \to \infty} \int_{-1}^{1} [f(t) - (\hat{L}_{2n+1} f)(t)] w(t) dt = 0, \text{ all } f \in C[-1, 1]?$$

The answer is yes, if the underlying quadrature rule has all weights positive, as follows from a classical result of Pólya [6]. We will show in §2 of this note that this is indeed the case, for all four Chebyshev weight functions, and in the process also determine explicitly the weights of the quadrature rules involved. Moreover, it will be shown in §3 that positivity also holds if the nodes τ_i and $\hat{\tau}_j$ interlace, provided the Gauss weights for the weight function w satisfy certain inequalities. The latter are checked numerically for the Jacobi weight function $w^{(\alpha,\beta)}(t) = (1-t)^\alpha (1+t)^\beta$, and evidence is produced suggesting that the quadrature weights in question are indeed positive if $|\alpha| \le \frac{1}{2}$, $|\beta| \le \frac{1}{2}$.

One could be tempted to take the zeros of π_{n+1} as the additional nodes $\hat{\tau}_j$ since interlacing is then guaranteed. However, the quadrature rule implied by (1.3) is then simply the $(n+1)$-point Gaussian rule for w (all nodes τ_i receive weight zero), and we are back again to the Erdös-Turán result!

2. CHEBYSHEV WEIGHT FUNCTIONS

The weights of the interpolatory quadrature rule implied by (1.3) are given by

$$(2.1) \qquad \lambda_i = \int_{-1}^{1} \frac{\pi_n(t) \hat{\pi}_{n+1}(t)}{(t - \tau_i) \pi_n'(\tau_i) \hat{\pi}_{n+1}(\tau_i)} w(t) dt, \qquad i = 1, 2, \ldots, n;$$

$$(2.2) \qquad \mu_j = \int_{-1}^{1} \frac{\pi_n(t) \hat{\pi}_{n+1}(t)}{(t - \hat{\tau}_j) \pi_n(\hat{\tau}_j) \hat{\pi}_{n+1}'(\hat{\tau}_j)} w(t) dt, \qquad j = 1, 2, \ldots, n+1,$$

where $\pi_n(\cdot) = \pi_n(\cdot; w)$ and $\hat{\pi}_{n+1}(\cdot) = \pi_{n+1}(\cdot; \pi_n^2 w)$. The rule has degree of exactness equal to $2n$. For reasons indicated in the Introduction, it suffices to look at Chebyshev weights of the first, third, and fourth kind.

2.1. Chebyshev weight of the first kind.
Here the weight function is $w_1(t) = (1 - t^2)^{-1/2}$, and π_n is the Chebyshev polynomial of the first kind,

$$(2.3) \qquad \pi_n(t) = T_n(t), \qquad T_n(\cos \theta) = \cos n\theta,$$

whereas $\hat{\pi}_{n+1}$ is given by [3]

$$(2.4) \qquad \hat{\pi}_{n+1}(t) = T_{n+1}(t) - \tfrac{1}{2} T_{n-1}(t), \quad n \ge 1.$$

Theorem 2.1. *For $w_1(t) = (1-t^2)^{-1/2}$, the quadrature weights λ_i and μ_j in (2.1), (2.2) are given by*

$$(2.5) \qquad \lambda_i = \frac{\pi}{3n}, \qquad i = 1, 2, \ldots, n;$$

(2.6)
$$\mu_j = \frac{2\pi}{3} \frac{1}{n + \dfrac{3}{9 - 8\hat{\tau}_j^2}}, \qquad j = 1, 2, \dots, n+1,$$

where $\hat{\tau}_j$ are the zeros of $\hat{\pi}_{n+1}$. All weights are positive.

Proof. It follows easily from (2.3) and (2.4) that $\pi_n'(\tau_i) = n(-1)^{i-1}/\sin\theta_i$ and $\hat{\pi}_{n+1}(\tau_i) = \frac{3}{2}(-1)^i \sin\theta_i$, where $\theta_i = (2i-1)\pi/2n$, so that

(2.7)
$$\pi_n'(\tau_i)\hat{\pi}_{n+1}(\tau_i) = -\tfrac{3}{2}\,n.$$

It remains, for λ_i, to evaluate the integral

$$\int_{-1}^{1} \frac{T_n(t)[T_{n+1}(t) - \frac{1}{2} T_{n-1}(t)]}{t - \tau_i}\, w_1(t)dt.$$

Since τ_i is a zero of T_n, the integral, by orthogonality of the T_m, reduces to

$$-\frac{1}{2} \int_{-1}^{1} \frac{T_n(t)}{t - \tau_i}\, T_{n-1}(t)w_1(t)dt,$$

which in turn is equal to $-\frac{\pi}{2}$. This follows by observing, if $n > 1$, that

$$\frac{T_n(t)}{t - \tau_i} = 2T_{n-1}(t) + \text{ lower-degree terms},$$

by orthogonality, and by using

$$\int_{-1}^{1} T_m^2(t)w_1(t)dt = \frac{\pi}{2}, \qquad m \geq 1.$$

For $n = 1$, the reasoning is the same except for the factor and divisor 2 in the last two formulae, which must be replaced by 1. The result (2.5) now follows immediately.

To evaluate the constant in the denominator of (2.2), we let

$$\hat{\tau}_j = \cos\hat{\theta}_j$$

and obtain from

$$\hat{\pi}_{n+1}(\cos\theta) = \cos(n+1)\theta - \tfrac{1}{2}\cos(n-1)\theta$$

by differentiation and the addition formula for the sine

(2.8)
$$\hat{\pi}_{n+1}'(\hat{\tau}_j) = \frac{1}{2\sin\hat{\theta}_j}\{(n+3)\sin n\hat{\theta}_j \cos\hat{\theta}_j + (3n+1)\cos n\hat{\theta}_j \sin\hat{\theta}_j\}.$$

Since

$$\cos(n+1)\hat{\theta}_j - \tfrac{1}{2}\cos(n-1)\hat{\theta}_j = 0,$$

and using here the addition formula for the cosine, we find

$$\sin n\hat{\theta}_j = \frac{1}{3}\frac{\cos n\hat{\theta}_j \cos\hat{\theta}_j}{\sin\hat{\theta}_j}.$$

Together with (2.8), this yields after a simple computation

$$\pi_n(\hat{\tau}_j)\hat{\pi}_{n+1}'(\hat{\tau}_j) = \frac{1}{2} T_n^2(\hat{\tau}_j)\left\{\frac{n+3}{3}\frac{\hat{\tau}_j^2}{1 - \hat{\tau}_j^2} + 3n + 1\right\}.$$

It is known from [3, Eq. (2.7)] that $T_n^2(t) = 9(1-t^2)/(9-8t^2)$ for $t = \hat{\tau}_j$. Therefore,

$$(2.9) \qquad \pi_n(\hat{\tau}_j)\hat{\pi}'_{n+1}(\hat{\tau}_j) = \frac{3(9n+3-8n\hat{\tau}_j^2)}{2(9-8\hat{\tau}_j^2)} = \frac{3}{2}\left(n + \frac{3}{9-8\hat{\tau}_j^2}\right).$$

For the integral in (2.2), we proceed as follows:

$$(2.10) \qquad \int_{-1}^{1} \frac{T_n(t)[T_{n+1}(t) - \frac{1}{2}T_{n-1}(t)]}{t - \hat{\tau}_j} w_1(t)dt = 2\int_{-1}^{1} T_n^2(t)w_1(t)dt = \pi.$$

The first equality is a result of the orthogonality of the T_m and the fact that

$$\frac{T_{n+1}(t) - \frac{1}{2}T_{n-1}(t)}{t - \hat{\tau}_j} = 2T_n(t) + \text{ lower-degree terms}.$$

Combining (2.10) and (2.9) yields (2.6).

The positivity of the quadrature weights is an immediate consequence of $-1 < \hat{\tau}_j < 1$ for the μ_j, and trivial for the λ_i. $\qquad\square$

Since $\sum_{i=1}^{n} \lambda_i + \sum_{j=1}^{n+1} \mu_j = \pi$, it follows from Theorem 2.1 that the nodes $\hat{\tau}_j$ must satisfy

$$(2.11) \qquad \sum_{j=1}^{n+1} \frac{1}{n + \dfrac{3}{9-8\hat{\tau}_j^2}} = 1.$$

2.2. Chebyshev weights of the third and fourth kind.

Because of the remark at the beginning of §3.2 below, it suffices to examine the Chebyshev weight function of the third kind, $w_3(t) = (1-t)^{-1/2}(1+t)^{1/2}$, for which the relevant polynomials are

$$(2.12) \qquad \pi_n(t) = V_n(t), \qquad V_n(\cos\theta) = \frac{\cos\left(n+\frac{1}{2}\right)\theta}{\cos\frac{1}{2}\theta}$$

and [3, Eq. (2.17)]

$$(2.13) \qquad \hat{\pi}_{n+1}(t) = T_{n+1}(t) - \frac{1}{2}T_n(t), \qquad n \geq 1.$$

Theorem 2.2. *For $w_3(t) = (1-t)^{-1/2}(1+t)^{1/2}$, the quadrature weights λ_i and μ_j in (2.1), (2.2) are given by*

$$(2.14) \qquad \lambda_i = \frac{2\pi}{3}\frac{1+\tau_i}{2n+1}, \qquad i = 1,2,\ldots,n;$$

$$(2.15) \qquad \mu_j = \frac{2\pi}{3}\frac{1+\hat{\tau}_j}{n + \dfrac{4-2\hat{\tau}_j}{5-4\hat{\tau}_j}}, \qquad j = 1,2,\ldots,n+1,$$

where τ_i and $\hat{\tau}_j$ are the zeros of π_n and $\hat{\pi}_{n+1}$, respectively. All weights are positive.

Proof. From (2.12) and (2.13), one obtains by an elementary computation that $\pi'_n(\tau_i) = \left(n+\frac{1}{2}\right)(-1)^{i-1}/\left(\cos\frac{1}{2}\theta_i\sin\theta_i\right)$ and $\hat{\pi}_{n+1}(\tau_i) = \frac{3}{2}(-1)^i\sin\frac{1}{2}\theta_i$, where $\theta_i = (2i-1)\pi/(2n+1)$, so that the constant in the integral of (2.1) is

$$(2.16) \qquad \pi'_n(\tau_i)\hat{\pi}_{n+1}(\tau_i) = -\frac{3}{4}\frac{2n+1}{1+\tau_i}.$$

The integral itself is

$$I_n = \int_{-1}^{1} \frac{V_n(t)}{t - \tau_i} [T_{n+1}(t) - \frac{1}{2} T_n(t)] w_3(t) dt$$

$$= \int_{-1}^{1} \frac{V_n(t)}{t - \tau_i} (1 + t)[T_{n+1}(t) - \frac{1}{2} T_n(t)] w_1(t) dt.$$

Since, by the recurrence relation for the T_m, we have

$$(1 + t)[T_{n+1}(t) - \tfrac{1}{2} T_n(t)] = \tfrac{1}{2} T_{n+2}(t) + \tfrac{3}{4} T_{n+1}(t) - \tfrac{1}{4} T_{n-1}(t),$$

we can use the orthogonality of the T_m with respect to w_1 to simplify:

$$I_n = -\frac{1}{4} \int_{-1}^{1} \frac{V_n(t)}{t - \tau_i} T_{n-1}(t) w_1(t) dt.$$

Now $V_n(t)$ has leading coefficient 2^n, if $n \geq 2$, so that

$$I_n = -\int_{-1}^{1} T_{n-1}^2(t) w_1(t) dt = -\frac{\pi}{2}, \qquad n \geq 2.$$

The same result holds also for $n = 1$. Combining it with (2.16) yields (2.14).

Letting as before $\hat{\tau}_j = \cos \hat{\theta}_j$, putting $t = \cos \theta$ in (2.13), and differentiating with respect to θ gives

$$\hat{\pi}'_{n+1}(\hat{\tau}_j) = \frac{n + 2}{4} \frac{\sin \left(n + \frac{1}{2} \right) \hat{\theta}_j}{\sin \frac{1}{2} \hat{\theta}_j} + \frac{3n + 2}{4} \frac{\cos \left(n + \frac{1}{2} \right) \hat{\theta}_j}{\cos \frac{1}{2} \hat{\theta}_j}.$$

Since

$$\cos(n + 1)\hat{\theta}_j - \tfrac{1}{2} \cos n\hat{\theta}_j = 0,$$

this simplifies to

$$\hat{\pi}'_{n+1}(\hat{\tau}_j) = \frac{V_n(\hat{\tau}_j)}{4} \left(\frac{n + 2}{3} \frac{1 + \hat{\tau}_j}{1 - \hat{\tau}_j} + 3n + 2 \right).$$

From [3, Eq. (2.22)] it is known that

$$V_n^2(t) = \frac{9(1 - t)}{(1 + t)(5 - 4t)} \text{ for } t = \hat{\tau}_j,$$

so that the constant in the integral of (2.2) becomes

(2.17) $$\pi_n(\hat{\tau}_j)\hat{\tau}'_{n+1}(\hat{\tau}_j) = \frac{3}{2} \frac{n + \dfrac{4 - 2\hat{\tau}_j}{5 - 4\hat{\tau}_j}}{1 + \hat{\tau}_j}.$$

The integral, on the other hand, is

$$\int_{-1}^{1} \frac{T_{n+1}(t) - \frac{1}{2} T_n(t)}{t - \hat{\tau}_j} V_n(t) w_3(t) dt,$$

which, since

$$\frac{T_{n+1}(t) - \frac{1}{2} T_n(t)}{t - \hat{\tau}_j} = V_n(t) + \text{ lower-degree terms,}$$

reduces to

$$\int_{-1}^{1} V_n^2(t) w_3(t) dt = \pi, \qquad n \geq 1.$$

Together with (2.17), this yields (2.15).

The positivity of the weights is evident from (2.14), (2.15). $\qquad\qquad\square$

Analogously to (2.11) one finds, after an elementary calculation, that

$$(2.18) \qquad \sum_{j=1}^{n+1} \frac{1+\hat\tau_j}{n+\dfrac{4-2\hat\tau_j}{5-4\hat\tau_j}} = 1.$$

3. JACOBI WEIGHT FUNCTIONS

For more general weight functions, in particular the Jacobi weight function $w(t) = w^{(\alpha,\beta)}(t)$, where $w^{(\alpha,\beta)}(t) = (1-t)^\alpha(1+t)^\beta$, we have only conjectural results based on numerical experimentation. We are especially interested in cases where the nodes $\{\tau_i\}$ and $\{\hat\tau_j\}$ interlace,

$$(3.1) \qquad \hat\tau_{n+1} < \tau_n < \hat\tau_n < \tau_{n-1} < \cdots < \hat\tau_2 < \tau_1 < \hat\tau_1.$$

We shall assume in this section (in slight contrast to §2) that the polynomials π_n and $\hat\pi_{n+1}$ are monic.

3.1. Quadrature weights for interlacing nodes. We assume, as in §2, that n is given and fixed. Our computations are based on the following theorem.

Theorem 3.1. *Let w be any (positive) weight function for which the nodes $\{\tau_i\}$, $\{\hat\tau_j\}$ (defined in §2) interlace. Then the quadrature weights λ_i and μ_j in (2.1), (2.2) are all positive if and only if*

$$(3.2) \qquad \lambda_i^G > \frac{\|\pi_n\|_w^2}{|\pi_n'(\tau_i)\hat\pi_{n+1}(\tau_i)|}, \qquad i = 1, 2, \dots, n,$$

where λ_i^G are the Christoffel numbers of the n-point Gaussian quadrature rule for the weight function w, and $\|\pi_n\|_w^2 = \int_{-1}^1 \pi_n^2(t)w(t)dt$.

Proof. We first show that the interlacing property implies $\mu_j > 0$. It is clear from (3.1) that

$$(3.3) \qquad \pi_n(\hat\tau_j)\hat\pi_{n+1}'(\hat\tau_j) > 0, \qquad j = 1, 2, \dots, n+1.$$

Thus the constant in the denominator of (2.2) is positive. In the integral that remains, the integrand is a monic polynomial of degree $2n$. Its $(2n)$th derivative divided by $(2n)!$ is therefore constant equal to 1, and the n-point Gauss formula with remainder term yields

$$\int_{-1}^1 \frac{\pi_n(t)\hat\pi_{n+1}(t)}{t-\hat\tau_j}\, w(t)dt = \sum_{k=1}^n \lambda_k^G \frac{\pi_n(\tau_k)\hat\pi_{n+1}(\tau_k)}{\tau_k-\hat\tau_j}$$
$$+ \int_{-1}^1 \pi_n^2(t)w(t)dt = \|\pi_n\|_w^2,$$

since $\pi_n(\tau_k) = 0$ for all k. Therefore,

$$(3.4) \qquad \mu_j = \frac{\|\pi_n\|_w^2}{\pi_n(\hat\tau_j)\hat\pi_{n+1}'(\hat\tau_j)}, \qquad j = 1, 2, \dots, n+1,$$

and the positivity of the μ_j follows from (3.3).

Similarly, for the λ_i we have

$$\int_{-1}^{1} \frac{\pi_n(t)}{t - \tau_i} \hat{\pi}_{n+1}(t) w(t) dt = \lambda_i^G \pi_n'(\tau_i) \hat{\pi}_{n+1}(\tau_i) + \| \pi_n \|_w^2,$$

so that from (2.1)

$$(3.5) \qquad \lambda_i = \lambda_i^G + \frac{\| \pi_n \|_w^2}{\pi_n'(\tau_i) \hat{\pi}_{n+1}(\tau_i)}, \qquad i = 1, 2, \ldots, n.$$

Now, however, interlacing implies

$$\pi_n'(\tau_i) \hat{\pi}_{n+1}(\tau_i) < 0, \qquad i = 1, 2, \ldots, n,$$

so that $\lambda_i > 0$ for all i if and only if (3.2) holds.   □

3.2. Numerical results for the Jacobi weight function.

For $w(t) = w^{(\alpha, \beta)}(t)$ it suffices to consider $\beta \geq \alpha > -1$, since an interchange of α and β only changes the sign of the argument t in $\pi_n(t)$ and $\hat{\pi}_{n+1}(t)$, hence the signs of the zeros τ_i and $\hat{\tau}_j$, and the weights λ_i and μ_j in (2.1), (2.2) remain the same, as is easily seen.

In order to check the positivity of the weights λ_i and μ_j numerically, we used Theorem 3.1 and examined, first of all, whether interlacing of the zeros holds, and if so, whether or not the inequalities (3.2) are valid for all n up to some large limit (below we take $n \leq 160$). For computational purposes we found it convenient to write these inequalities in the form

$$(3.2') \qquad \lambda_i^G > \frac{\beta_0 \beta_1 \cdots \beta_n}{\prod_{\substack{k=1 \\ k \neq i}}^{n} |\tau_i - \tau_k| \prod_{j=1}^{n+1} |\tau_i - \hat{\tau}_j|}, \qquad i = 1, 2, \ldots, n,$$

where the β's are the coefficients in the recurrence relation

$$(3.6) \qquad \begin{aligned} \pi_{\nu+1}(t) &= (t - \alpha_\nu) \pi_\nu(t) - \beta_\nu \pi_{\nu-1}(t), \qquad \nu = 0, 1, 2, \ldots, \\ \pi_0(t) &= 1, \ \pi_{-1}(t) = 0 \end{aligned}$$

for the polynomials $\pi_\nu(\cdot) = \pi_\nu(\cdot; w^{(\alpha, \beta)})$. To generate these coefficients, and with them the polynomial π_n and its zeros τ_i, we used the routines recur and gauss in [4]. Similarly for the polynomial $\hat{\pi}_{n+1}$, where we used the routines indp and gauss. All calculations were done in double precision on a Sun SPARCstation IPX, using Fortran Version 2.0, for $n = 1(1)160$. We found that interlacing and/or positivity fails for $\alpha < -\frac{1}{2}$ and $\alpha > 1$, and also for $-\frac{1}{2} \leq \alpha \leq 1$ and $\beta > 1$. On the other hand, there is strong evidence for both interlacing and positivity to hold if $|\alpha| \leq \frac{1}{2}$, $|\beta| \leq \frac{1}{2}$. Both may even hold for somewhat larger values of α and β, as suggested in Fig. 3.1, where they seem to hold in the triangular-like region, and its reflection with respect to the diagonal $\alpha = \beta$, bounded on the left by the line $\alpha = -\frac{1}{2}$, below by $\alpha = \beta$, and on top by the dashdotted line (for $1 \leq n \leq 40$), the dashed line (for $1 \leq n \leq 80$), and the solid line (for $1 \leq n \leq 160$). We say "seem to hold" since interlacing and the inequality (3.2') were verified numerically only for discrete points in the (α, β)-plane spaced apart by .1 in most of the region, and by .001 (in the β-values) near the top of the region. We also verified the failure (for some n) of either interlacing or (3.2') for $\alpha = -\frac{1}{2} - .01$ and $\beta = -\frac{1}{2}(.1)1$, as well as for $\alpha = \beta = 1(.1)2$. It seems safe, therefore, to state the following conjecture.
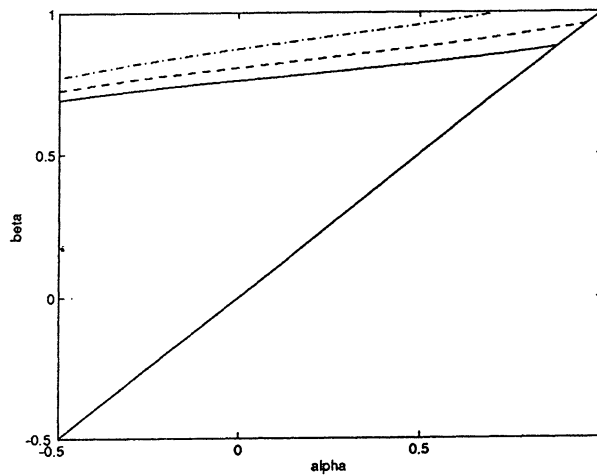
FIGURE 3.1. Positivity of quadrature weights for the Jacobi weight function $w^{(\alpha,\beta)}$

Conjecture 3.1. *For the Jacobi weight function $w(t) = w^{(\alpha,\beta)}(t)$ the quadrature weights λ_i and μ_j in (2.1), (2.2) are all positive if (α,β) is in the square $|\alpha| \leq \frac{1}{2}$, $|\beta| \leq \frac{1}{2}$.*

The positivity expressed in Conjecture 3.1 has been proved in §2 at the four corner points of the square.

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