

INTEGRATION OF POLYHARMONIC FUNCTIONS

DIMITAR K. DIMITROV

ABSTRACT. The results in this paper are motivated by two analogies. First, m -harmonic functions in \mathbf{R}^n are extensions of the univariate algebraic polynomials of odd degree $2m-1$. Second, Gauss' and Pizzetti's mean value formulae are natural multivariate analogues of the rectangular and Taylor's quadrature formulae, respectively. This point of view suggests that some theorems concerning quadrature rules could be generalized to results about integration of polyharmonic functions. This is done for the Tchakaloff-Obrechhoff quadrature formula and for the Gaussian quadrature with two nodes.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let \mathbf{R}^n be the real n -dimensional Euclidean space. The points of \mathbf{R}^n are denoted by $x = (x_1, x_2, \dots, x_n)$ and $|x|$ denotes the nonnegative value of $(\sum_{i=1}^n x_i^2)^{1/2}$. For any positive r the open ball $B(r)$ and the hypersphere $S(r)$ with center 0 and radius r in \mathbf{R}^n are defined by

$$B(r) = \{x : |x| < r\}$$

and

$$S(r) = \{x : |x| = r\},$$

respectively. The closed ball $\bar{B}(r)$ is $\bar{B}(r) = B(r) \cup S(r)$. If $r = 1$, then sometimes the argument in the notations for the unit open and closed balls and hypersphere will be omitted. The inner normal derivative on S is denoted by $\frac{\partial}{\partial \nu}$. We denote by dx Lebesgue measure in \mathbf{R}^n and by $d\sigma$ the $(n-1)$ -dimensional surface measure on $S(r)$. Note that the area of the unit sphere S in \mathbf{R}^n is $\sigma_n = 2\pi^{n/2}/\Gamma(n/2)$, where Γ is the Gamma function.

The iterates Δ^m of the Laplace operator in \mathbf{R}^n are defined recursively by

$$\begin{aligned}\Delta &= \Delta^1 = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}, \\ \Delta^m &= \Delta \Delta^{m-1}.\end{aligned}$$

Received by the editor August 15, 1994 and, in revised form, May 22, 1995.

1991 *Mathematics Subject Classification*. Primary 31B30; Secondary 65D32.

Key words and phrases. Polyharmonic function, extended cubature formula, polyharmonic order of precision, polyharmonic monospline.

Research supported by the Bulgarian Ministry of Science under Grant MM-414 and The Royal Society London.

The identity operator is Δ^0 . The function u is called polyharmonic of order m , or m -harmonic, in B if u belongs to the space

$$H^m(B) := \{u \in C^{2m-1}(\bar{B}) \cap C^{2m}(B) : \Delta^m u = 0 \text{ on } B\}.$$

In particular, if $m = 1$ or $m = 2$, u is said to be harmonic or biharmonic, respectively. For any $p \geq 1$ we denote by $L_p(B)$ the L_p space on B equipped with the norm

$$\begin{aligned} \|f\|_p &:= \left(\int_B |f(x)|^p dx \right)^{1/p}, \quad 1 \leq p < \infty, \\ \|f\|_\infty &:= \sup_{x \in B} |f(x)|. \end{aligned}$$

The Sobolev space is defined by

$$H_p^m(B) := \{u \in C^{2m-1}(\bar{B}) : \Delta^m u \text{ exists a.e. in } B \text{ and } \Delta^m u \in L_p(B)\}.$$

As null spaces of the even-order differential operator Δ^m the polyharmonic functions of order m not surprisingly inherit most of the remarkable properties of the univariate algebraic polynomials of odd degree $2m - 1$. Thus, polyharmonic functions are appropriate for the approximation of multivariate functions. We refer to the volume [5] for some recent results. The main purpose of this paper is to extend two well-known quadrature formulae, which are precise for all algebraic polynomials of degree $2m - 1$, to the corresponding multivariate analogues, which are exact for the m -harmonic functions. First we introduce a notion similar to the algebraic degree of precision. Every linear functional $Q(f)$ approximating the integral $I(f) = \int_B f(x) dx$ in terms of values of $\Delta^i f$, $i = 0, \dots$, at certain points and/or surface integrals of them and their normal derivatives is called an *extended cubature formula* or *extended cubature rule*. The relation between the functionals I and Q is described as $I(F) \approx Q(f)$. An extended formula is said to have *polyharmonic order of precision m* , $PHOP(Q) = m$ if $I(F) = Q(f)$ for all $f \in H^m(B)$ and there exists a function f such that $\Delta^m f \neq 0$ in B and $I(F) \neq Q(f)$.

These definitions are justified by the analogy between some well-known theorems concerning integration of polyharmonic functions and results about quadrature formulae. For example the Gaussian mean value property

$$\int_B u(x) dx = [\pi^{n/2} / \Gamma(n/2 + 1)] u(0)$$

of harmonic functions is a multivariate analogue of the rectangular quadrature formula

$$\int_{-1}^1 f(x) dx \approx 2 f(0),$$

which is precise for linear functions. More generally, Pizzetti's mean value formula

$$\int_B u(x) dx = \sum_{j=0}^{m-1} \frac{\pi^{n/2}}{2^{2j} j! \Gamma(n/2 + j + 1)} \Delta^j u(0),$$

which holds for all m -harmonic functions u , extends the quadrature formula

$$\int_{-1}^1 f(x) dx \approx \sum_{j=0}^{m-1} \frac{2}{(2j + 1)!} f^{(2j)}(0),$$

which is precise for all the algebraic polynomials of degree $2m - 1$. An immediate consequence of the first Green formula is the equality

$$(1) \quad \int_B u(x) \, dx = 1/n \int_S u(x) \, d\sigma$$

for harmonic functions. This is an analogue of the trapezoid quadrature rule

$$\int_{-1}^1 f(x) \, dx \approx f(-1) + f(1),$$

which is precise for polynomials of first degree. This analogy suggests that we consider each hypersphere in \mathbf{R}^n as an extension of a pair of nodes in \mathbf{R} .

The natural question arises if it is possible to extend (1) in order to increase the polyharmonic order of precision. The first possibility is to try to express the integral over B in terms of linear combination of integrals of u and differential operators of u over S . The second way of extending (1) is to approximate $I(u)$ by a multiple of the integral of u over a sphere which is concentric to S . The main results of this paper are explicit representations and sharp error bounds for such extensions. Note that if an extended cubature formula Q has $PHOP(Q) = m$, then it is precise for every n -variate algebraic polynomial of total degree $2m - 1$. Thus, our results allow the problem of constructing ordinary cubature formulae (linear combinations of values of the integrand and differential operators applied to it at certain points) of highest possible total algebraic degree of precision to be reduced to constructing ordinary cubature formulae for integrals on spheres. Furthermore, these integrals are given explicitly.

The first quadrature formula we will extend is the ‘‘Tchakaloff-Obrechhoff quadrature formula’’

$$\int_{-1}^1 f(x) \, dx \approx \sum_{j=0}^{m-1} c_j \left(f^{(j)}(-1) + (-1)^j f^{(j)}(1) \right),$$

where

$$c_j = c_j(m) = 2^{j+1} \frac{m!(2m - j - 1)!}{(2m)!(j + 1)!(m - j - 1)!}.$$

It is precise for every algebraic polynomial of degree $2m - 1$, and the error term is

$$\frac{(-1)^m}{(2m)!} \int_{-1}^1 (1 - x^2)^m f^{(2m)}(x) \, dx.$$

Pochhammer’s symbol $(a)_k$ is defined by $(a)_k := a(a + 1) \cdots (a + k - 1)$, $k > 0$, $(a)_0 := 1$. We shall prove

Theorem 1. *For every $m \in \mathbf{N}$ the extended cubature formula*

$$(2) \quad \begin{aligned} \int_B u(x) \, dx &\approx \sum_{j=0}^{[(m-1)/2]} C_j^{(e)} \int_S \Delta^j u \, d\sigma \\ &+ \sum_{j=0}^{[m/2]-1} C_j^{(o)} \int_S \frac{\partial}{\partial \nu} \Delta^j u \, d\sigma \\ &=: Q_m(u), \end{aligned}$$

where

$$(3) \quad C_j^{(e)} = C_j^{(e)}(m, n) = \frac{(-1)^j \Gamma(n/2)\Gamma(n/2 + m - j)}{2^{2j+1} \Gamma(n/2 + m)\Gamma(n/2 + j + 1)} \frac{(j + 1 - m)_j}{j!}$$

and

$$(4) \quad C_j^{(o)} = C_j^{(o)}(m, n) = \frac{(-1)^{j+1} \Gamma(n/2)\Gamma(n/2 + m - j - 1)}{2^{2j+2} \Gamma(n/2 + m)\Gamma(n/2 + j + 1)} \frac{(j + 1 - m)_{j+1}}{(j + 1)!},$$

has polyharmonic order of precision m . Moreover, if $u \in C^{2m}(B)$, then

$$R_m(u) := I(u) - Q_m(u) = \int_B \varphi_m(x) \Delta^m u(x) dx,$$

where φ_m is a radial function defined by

$$\varphi_m(x) = \varphi_m(r) = \frac{(-1)^m}{2^{2m}} (m!(n/2)_m)^{-1} (1 - r^2)^m.$$

For any $u \in H_p^m(B)$, $p \geq 1$, we have

$$|R_m(u)| \leq \|\varphi_m\|_q \|\Delta^m u\|_p,$$

where $1/p + 1/q = 1$.

Note that if $m = 1$, then the second sum in (2) disappears and Q_1 coincides with (1). It is worth mentioning also that, applying (3) and (4) formally for $n = 1$, we obtain the weights $c_j(m)$, namely,

$$C_j^{(e)}(m, 1) = c_{2j}(m), \quad C_j^{(o)}(m, 1) = c_{2j+1}(m).$$

This holds because $C_j^{(e)}$ and $C_j^{(o)}$ are the weights associated with surface integrals of even-order $2j$ and odd-order $2j + 1$ differential operators, respectively. Similarly, setting $n = 1$ in the formula for φ_m , we obtain the univariate error function.

It is well known that there exists a unique quadrature rule of the form $\int_{-1}^1 f(x) dx \approx c_1 f(x_1) + c_2 f(x_2)$, which is precise for all polynomials of degree three. This rule is $\int_{-1}^1 f(x) dx \approx f(-1/\sqrt{3}) + f(1/\sqrt{3})$, the Gaussian quadrature formula with two nodes. We prove

Theorem 2. *There is no extended cubature formula of the form*

$$(5) \quad \int_B u(x) dx = c \int_{S(R)} u d\sigma$$

with $PHOP > 2$. There exist a unique radius $R > 0$ and a weight $c \in \mathbf{R}$ such that the extended cubature formula (5) has polyharmonic order of precision two. Moreover,

$$R = R(n) = \left(\frac{n}{n + 2} \right)^{1/2}$$

and

$$c = c(n) = \frac{1}{n + 2} \left(\frac{n + 2}{n} \right)^{(n+1)/2}.$$

In what follows, this extended cubature rule will be referred to as Q_G ,

$$Q_G(u) := c(n) \int_{S(R(n))} u \, d\sigma.$$

Let us note that $c(1) = 1$ and $R(1) = 1/\sqrt{3}$. In view of the proposed analogy between hyperspheres and pairs of nodes, this yields the univariate Gaussian quadrature formula with two nodes.

Corollary 1. *For any positive integer n the radial polynomial*

$$P_2(x) = |x|^2 - R^2(n)$$

is orthogonal on B to every harmonic function.

This polynomial is the extension of the Legendre polynomial of second degree. Let BH_p^2 be the unit ball in H_p^2 ,

$$BH_p^2 := \{u \in H_p^2 : \|\Delta^2 u\|_p \leq 1\}.$$

Denote by $R_G(u)$ the error of $Q_G(u)$,

$$R_G(u) := I(u) - Q_G(u).$$

Then

$$R_{p,G} := \sup \{|R_G(u)| : u \in BH_p^2\}$$

is the maximal error of Q_G in BH_p^2 . The next theorem concerns $R_{p,G}$. In order to formulate it, we define, for each $n \geq 2$, a radial function $M(x) = M(|x|) = M(r)$. For $n = 2$ we set

$$M(r) = \begin{cases} \frac{1}{64} \{r^4 - 4(1 - \ln 2)r^2 - (1 - 2 \ln 2)\}, & 0 \leq r \leq 1/\sqrt{2}, \\ \frac{1}{64} \{r^4 + 4r^2 - 5 - 4 \ln r(1 + 2r^2)\}, & 1/\sqrt{2} \leq r \leq 1. \end{cases}$$

For $n = 4$,

$$M(r) = \begin{cases} \frac{1}{192}r^4 - \frac{1}{128}r^2 + \frac{1}{64} - \frac{1}{32} \ln(2/3), & 0 \leq r \leq \sqrt{2/3}, \\ \frac{1}{192} \{r^4 - 6r^2 + 3 + 2r^{-2} + 12 \ln r\}, & \sqrt{2/3} \leq r \leq 1. \end{cases}$$

For all $n > 2, n \neq 4$, we define M in the following way:

$$M(r) = \begin{cases} \frac{r^4}{8n(n+2)} + \frac{r^2}{2n(2-n)} \left(\frac{1}{2} - \frac{1}{n+2} \left(\frac{n+2}{n} \right)^{n/2} \right) + \frac{1}{8(2-n)(4-n)} \\ \quad + \frac{n(n-3)}{4(n+2)^2(4-n)} \left(\frac{n+2}{n} \right)^{n/2}, & 0 \leq r \leq R(n), \\ \frac{r^4}{8n(n+2)} + \frac{r^2}{4n(2-n)} + \frac{1}{8(2-n)(4-n)} - \frac{1}{2n(2-n)(4-n)} r^{4-n} \\ \quad - \frac{1}{2n(2-n)(2+n)} r^{2-n}, & R(n) \leq r \leq 1. \end{cases}$$

Theorem 3. *Let $n \geq 2, 1 < p \leq \infty$, and $R_{p,G}$ and M be defined as above. Then*

$$(6) \quad R_{p,G}(u) = \|M(x)\|_q,$$

where $1/p + 1/q = 1$.

2. POLYHARMONIC MONOSPINES

The principal tool in the proofs is an extension of the notion of univariate monosplines introduced by Schoenberg [4]. Monosplines are Peano kernels associated with the functional of the error of quadrature formulae. Every monospline M of degree N is a piecewise polynomial of degree N with leading coefficient $(-1)^N/N!$ so that $M^{(N)} \equiv (-1)^N$ on every interval of the partition $a = x_0 < x_1 < \dots < x_k < x_{k+1} = b$, $M^{(N)} \equiv 0$ for $x \notin [a, b]$. The knots x_i coincide with the nodes of the associated quadrature formula

$$(7) \quad \int_a^b f(x)dx \approx \sum_{i=0}^{k+1} \sum_{j=0}^{N-1} a_{ij} f^{(j)}(x_i).$$

Moreover, the one-to-one correspondence between monosplines M of degree N and quadrature formulae (7) which are precise for all algebraic polynomials of degree $N - 1$, is described by the following relation for the weights a_{ij} and the jumps of the derivatives of M at x_i :

$$a_{ij} = (-1)^{N-j} \left(M^{(N-j-1)}(x_{i-}) - M^{(N-j-1)}(x_{i+}) \right).$$

Here as usual we denote $g(x-) := \lim \{g(t) : t \rightarrow x, t < x\}$ and $g(x+) := \lim \{g(t) : t \rightarrow x, t > x\}$. The formula for a_{ij} can be obtained directly by using Peano’s theorem, or alternatively, by N -fold integration by parts of the product $M(x)f^{(N)}(x)$ on the intervals $[x_i, x_{i+1}]$, $i = 0, \dots, k$, and summing the results. We refer to [2, Chapter 7] for a detailed proof using the second approach.

Let Ω be a bounded domain in \mathbf{R}^n , and $\delta = \{\Omega_1, \dots, \Omega_k\}$ be a partition of Ω which consists of subdomains Ω_i , $i = 1, \dots, k$, such that

$$\bar{\Omega} = \bigcup_{i=1}^k \bar{\Omega}_i, \quad \Omega_i \cap \Omega_s = \emptyset \text{ for } i \neq s.$$

We consider the union of the boundaries $\partial\delta = \bigcup_{i=1}^k \partial\Omega_i$ as an analogue of the nodes x_i of (7). Let us denote by $\partial\delta_{is}$ the common boundary of Ω_i and Ω_s if it exists; otherwise $\partial\delta_{is}$ is empty. Then obviously

$$\partial\delta = \left(\bigcup_{i,s=1}^k \partial\delta_{is} \right) \cup \partial\Omega.$$

By a *Lipschitzian graph manifold* we mean a topological submanifold in \mathbf{R}^n such that for each of its points y there exists a neighborhood U of y on the manifold and an $(n - 1)$ -dimensional plane γ in \mathbf{R}^n such that the orthogonal projection Π_γ of U on γ is a bi-Lipschitzian homeomorphism, that is, there exists a constant $K, 0 < K \leq 1$, such that for any two points $y_1, y_2 \in U$ the inequalities

$$K \leq \frac{|\Pi_\gamma y_1 - \Pi_\gamma y_2|}{|y_1 - y_2|} \leq 1$$

hold. On each Lipschitzian submanifold of \mathbf{R}^n there exists an intrinsic measure σ , which is defined by

$$\sigma(E) = \int_{\Pi_\gamma} \frac{D(\Pi_\gamma^{-1}(y))}{D(y)} d_\gamma y$$

for any $E \subset U$. Here, U, γ, Π_γ are defined as above, $D(\Pi_\gamma^{-1}(y))/D(y)$ is the Jacobian of the Lipschitzian mapping Π_γ^{-1} at y , and $d_\gamma y$ is the Lebesgue measure on the hyperplane γ .

Let Ω and δ be such that all the boundaries $\partial\Omega_i$ are Lipschitzian graph manifolds. We fix the orientation on $\partial\Omega_i$ consistent with the orientation of \mathbf{R}^n . Then at almost every point x of $\partial\Omega_i$ there exists a uniquely determined tangent hyperplane γ_x and an inner normal vector $\nu_i = \nu_i(x)$ which is orthogonal to γ_x . We refer to [1, pp. 8-9] for more details about Lipschitzian graph manifolds. In what follows, Ω and δ are supposed to be such that all the boundaries which make up $\partial\delta$ are bounded orientable piecewise smooth Lipschitzian graph manifolds of dimension $n - 1$. Such partitions will be called *regular*. For any nonempty $\partial\delta_{is}$ the inner normal derivatives on $\partial\delta_{is}$ with respect to ν_i and ν_s are denoted by $\frac{\partial}{\partial\nu_i}$ and $\frac{\partial}{\partial\nu_s}$, respectively. Similarly, the limits of $\Delta^j u(x)$ when x approaches $\partial\delta_{is}$ from Ω_i and Ω_s are denoted by $\Delta_i^j u$ and $\Delta_s^j u$, respectively. The function $M(x)$ is said to be an m -harmonic monospline on δ if $M \in C^{2m}(\Omega_i)$, $i = 1, \dots, k$, and

$$\Delta^m M(x) \equiv 1 \quad \text{on every } \Omega_i, \quad i = 1, \dots, k.$$

Lemma 1 (The first Green formula for Δ^m [1, p.10]). *Let Ω be a region in \mathbf{R}^n whose boundary $\partial\Omega$ is an orientable piecewise smooth Lipschitzian graph manifold of dimension $n - 1$. Then the equality*

$$(8) \quad \sum_{j=0}^{m-1} \int_{\partial\Omega} \left(\Delta^j u \frac{\partial}{\partial\nu} \Delta^{m-j-1} v - \Delta^{m-j-1} v \frac{\partial}{\partial\nu} \Delta^j u \right) d\sigma + \int_{\Omega} (u \Delta^m v - v \Delta^m u) dx = 0$$

holds for any $u, v \in C^{2m-1}(\bar{\Omega})$ for which the integral over Ω exists.

Theorem 4. *Let δ be a regular partition of the bounded domain Ω . If M is an m -harmonic monospline on δ such that*

$$(9) \quad \begin{aligned} \Delta_i^{m-j-1} M(x) - \Delta_s^{m-j-1} M(x) &= a_{is}^j(x), \quad x \in \partial\delta_{is}, \\ -\frac{\partial}{\partial\nu_i} \Delta^{m-j-1} M(x) - \frac{\partial}{\partial\nu_s} \Delta^{m-j-1} M(x) &= \bar{a}_{is}^j(x), \quad x \in \partial\delta_{is}, \end{aligned}$$

and

$$(10) \quad \begin{aligned} \Delta^{m-j-1} M(x) &= a_0^j(x), \quad x \in \partial D, \\ -\frac{\partial}{\partial\nu} \Delta^{m-j-1} M(x) &= \bar{a}_0^j(x), \quad x \in \partial D, \end{aligned}$$

then the formula

$$(11) \quad \begin{aligned} \int_{\Omega} u(x) dx &\approx \sum_{i,s=1}^k \left\{ \sum_{j=0}^{m-1} \int_{\partial\delta_{is}} a_{is}^j(x) \frac{\partial}{\partial\nu} \Delta^j u(x) d\sigma + \sum_{j=0}^{m-1} \int_{\partial\delta_{is}} \bar{a}_{is}^j(x) \Delta^j u(x) d\sigma \right\} \\ &+ \sum_{j=0}^{m-1} \int_{\partial\Omega} a_0^j(x) \frac{\partial}{\partial\nu} \Delta^j u(x) d\sigma + \sum_{j=0}^{m-1} \int_{\partial\Omega} \bar{a}_0^j(x) \Delta^j u(x) d\sigma \\ &=: Q(u) \end{aligned}$$

is precise for every m -harmonic function on Ω . Moreover,

$$(12) \quad \int_{\Omega} u(x) dx = Q(u) + \int_{\Omega} M(x) \Delta^m u(x) dx.$$

Conversely, if the formula (11) is precise for every m -harmonic function on Ω , then it defines a unique m -harmonic monospline on Ω which satisfies (9) and (10).

Since the proof is similar to that for the duality of univariate monosplines and quadrature formulae, we omit it. The most interesting case is when the domain Ω admits a partition δ for which there exists an m -harmonic monospline M whose jumps $a_{i_s}^j, \bar{a}_{i_s}^j, a_0^j$ and \bar{a}_0^j are constants. Such partitions will be called *super-regular*.

Corollary 2. *Let δ be a super-regular partition of Ω . Then the extended cubature formula*

$$\int_{\Omega} u(x) dx \approx \sum_{i,s=1}^k \left\{ \sum_{j=0}^{m-1} a_{i_s}^j \int_{\partial\delta_{i_s}} \frac{\partial}{\partial\nu} \Delta^j u d\sigma + \sum_{j=0}^{m-1} \bar{a}_{i_s}^j \int_{\partial\delta_{i_s}} \Delta^j u d\sigma \right\} + \sum_{j=0}^{m-1} a_0^j \int_{\partial\Omega} \frac{\partial}{\partial\nu} \Delta^j u d\sigma + \sum_{j=0}^{m-1} \bar{a}_0^j \int_{\partial\Omega} \Delta^j u d\sigma$$

is precise for every m -harmonic function on Ω if and only if the associated monospline M satisfies (9) and (10) with $a_{i_s}^j(x) = a_{i_s}^j, \bar{a}_{i_s}^j(x) = \bar{a}_{i_s}^j, a_0^j(x) = a_0^j$ and $\bar{a}_0^j(x) = \bar{a}_0^j$.

Obviously, every ball in \mathbf{R}^n admits a super-regular partition which is induced by concentric hyperspheres, and the corresponding monosplines are radial functions.

3. PROOFS OF THE THEOREMS

A multi-index is an n -tuple $q = (q_1, \dots, q_n)$ of nonnegative integers. The following standard notations shall be used:

$$\begin{aligned} |q| &= \sum_{i=1}^n q_i, \\ q! &= q_1! \cdots q_n!, \\ D^q &= \frac{\partial^{|q|}}{\partial x_1^{q_1} \cdots \partial x_n^{q_n}}. \end{aligned}$$

We shall need the following formulae [1, Chapter 1]:

$$(13) \quad \Delta^k r^s = s(s-2) \cdots (s-2k+2)(s-2+n)(s-4+n) \cdots (s-2k+n)r^{s-2k},$$

$$(14) \quad \Delta(r^s \log r) = s(s+n-2)r^{s-2} \log r + (2s+n-2)r^{s-2}$$

and

$$(15) \quad \begin{aligned} \Delta^2(uv) &= u\Delta^2v + v\Delta^2u + 2\Delta u\Delta v \\ &+ 4 \sum_{|q|=1} \{(D^q\Delta u)(D^q v) + (D^q\Delta v)(D^q u)\} \\ &+ 8 \sum_{|q|=2} \frac{1}{q!} (D^q u)(D^q v). \end{aligned}$$

Proof of Theorem 1. Apply (8) for $v = \varphi_m =: \varphi$. The result is

$$\begin{aligned} \int_B \Delta^m \varphi(x) u(x) \, dx &= - \sum_{j=0}^{m-1} \int_S \Delta^j u \frac{\partial}{\partial \nu} \Delta^{m-j-1} \varphi \, d\sigma \\ &\quad + \sum_{j=0}^{m-1} \int_S \frac{\partial}{\partial \nu} \Delta^j u \Delta^{m-j-1} \varphi \, d\sigma \\ &\quad + \int_B \varphi(x) \Delta^m u(x) \, dx. \end{aligned}$$

If $u \in H^m(B)$, then the last integral on the right-hand side vanishes. Taking into account the facts that φ is a radial function and $\frac{\partial}{\partial \nu}$ is the inner normal derivative, we conclude that for any $u \in H^m(B)$ the equality

$$\begin{aligned} \int_B \Delta^m \varphi(x) u(x) \, dx &= \sum_{j=0}^{m-1} \frac{d}{dr} \Delta^{m-j-1} \varphi(r) \Big|_{r=1} \int_S \Delta^j u \, d\sigma \\ &\quad + \sum_{j=0}^{m-1} \Delta^{m-j-1} \varphi(r) \Big|_{r=1} \int_S \frac{\partial}{\partial \nu} \Delta^j u \, d\sigma \end{aligned}$$

holds. The first statement in the theorem will be established once we have proved that

$$(16) \quad \Delta^m (1 - r^2)^m \equiv (-1)^m 2^{2m} m! (n/2)_m \text{ on } B,$$

$$(17) \quad \Delta^k (1 - r^2)^m \Big|_{r=1} = (-1)^k 2^{2k} \frac{m!}{(m-k)!} (n/2)_k \frac{(-k)_{m-k}}{(n/2)_{m-k}},$$

$$(18) \quad \frac{d}{dr} \Delta^k (1 - r^2)^m \Big|_{r=1} = (-1)^{k+1} 2^{2k+1} \frac{m!}{(m-k-1)!} (n/2)_{k+1} \frac{(-k)_{m-k-1}}{(n/2)_{m-k}}.$$

Indeed, it follows from (16) that $\Delta^m \varphi(x) \equiv 1$ on B . Applying (17) and (18) for $k = m - j - 1$, we obtain the equality

$$\begin{aligned} \int_B u(x) \, dx &= \sum_{j=0}^{m-1} \frac{(-1)^j}{2^{2j+1}} \frac{\Gamma(n/2) \Gamma(n/2 + m - j)}{\Gamma(n/2 + m) \Gamma(n/2 + j + 1)} \frac{(j + 1 - m)_j}{j!} \int_S \Delta^j u \, d\sigma \\ &\quad + \sum_{j=0}^{m-1} \frac{(-1)^{j+1}}{2^{2j+2}} \frac{\Gamma(n/2) \Gamma(n/2 + m - j - 1)}{\Gamma(n/2 + m) \Gamma(n/2 + j + 1)} \frac{(j + 1 - m)_{j+1}}{(j + 1)!} \\ &\quad \cdot \int_S \frac{\partial}{\partial \nu} \Delta^j u \, d\sigma, \end{aligned}$$

which holds for every $u \in H^m(B)$. It remains to observe that $(j + 1 - m)_j \neq 0$ if and only if $2j \leq m - 1$, which is equivalent to $j \leq [(m - 1)/2]$, and similarly, $(j + 1 - m)_{j+1} \neq 0$ if and only if $j \leq [m/2] - 1$.

Now we shall prove (16), (17) and (18). Since

$$(1 - r^2)^m = \sum_{i=0}^m (-1)^i \binom{m}{i} r^{2i},$$

we obtain by the linearity of Δ^k and (13)

$$\begin{aligned} \Delta^k(1-r^2)^m &= \sum_{i=0}^m (-1)^i \binom{m}{i} 2i \cdots (2i-2k+2)(2i-2+n) \\ &\quad \cdots (2i-2k+n) r^{2i-2k} \\ &= 2^{2k} \sum_{i=k}^m (-1)^i \frac{m!}{(m-i)!(i-k)!} (i-1+n/2)(i-2+n/2) \\ &\quad \cdots (i-k+n/2) r^{2i-2k} \\ &= (-1)^k 2^{2k} \frac{m!}{(m-k)!} \sum_{i=k}^m (-1)^{i-k} \frac{(m-k)!}{(m-i)!(i-k)!} (n/2+i-k) \\ &\quad \cdots (n/2+i-1) r^{2i-2k} \\ &= (-1)^k 2^{2k} \frac{m!}{(m-k)!} \sum_{\nu=0}^{m-k} (-1)^\nu \binom{m-k}{\nu} (n/2+\nu)_k r^{2\nu}. \end{aligned}$$

In the last equality we used the change of variables $i = k + \nu$ in the sum and the definition of Pochhammer's symbol. On applying the identity

$$(a + \nu)_k = \frac{(a)_k (a + k)_\nu}{(a)_\nu},$$

we obtain

(19)

$$\begin{aligned} \Delta^k(1-r^2)^m &= (-1)^k 2^{2k} \frac{m!}{(m-k)!} (n/2)_k \sum_{\nu=0}^{m-k} (-1)^\nu \frac{(m-k)!}{(m-k-\nu)!(\nu)!} \frac{(n/2+k)_\nu}{(n/2)_\nu} r^{2\nu} \\ &= (-1)^k 2^{2k} \frac{m!}{(m-k)!} (n/2)_k \sum_{\nu=0}^{m-k} \frac{(k-m)_\nu (n/2+k)_\nu}{(n/2)_\nu} \frac{r^{2\nu}}{\nu!} \\ &= (-1)^k 2^{2k} \frac{m!}{(m-k)!} (n/2)_k {}_2F_1(k-m, n/2+k; n/2; r^2), \end{aligned}$$

where ${}_2F_1(a, b; c; x)$ is the hypergeometric function.

Now the identity (16) follows immediately from (19). Equality (17) is a consequence of the well-known Gaussian identity [3, p.104]

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

which holds for $c > a + b$. In order to prove (18), we note that ${}_2F_1(a, b; c; x)$ is a solution of the differential equation

$$x(1-x)y'' + (c - (a+b+1)x)y' - aby = 0.$$

Hence,

$$\begin{aligned} \frac{d}{dr} {}_2F_1(k-m, n/2+k; n/2; r^2)|_{r=1} \\ = 2 \frac{(k-m)(n/2+k)}{m-2k-1} {}_2F_1(k-m, n/2+k; n/2; 1). \end{aligned}$$

Thus, (18) follows from (19). The first statement in the theorem is proved. The second is a consequence of (12) and of Hölder's inequality. \square

Proof of Theorem 2. Let $U(x) = |x|^2 - R^2$. Then for any R we have $\int_B U^2(x)dx > 0$ and $\int_{S(R)} U^2 d\sigma = 0$, which means that the highest possible PHOP of the cubature formulae of the form (5) is two. The uniqueness and the explicit form of the radius $R(n)$ can be obtained immediately by comparing the integrals of U on B and on S . Then the comparison of the integrals of $V(x) \equiv 1$ on B and on S gives the weight $c(n)$. We omit the calculations. We mention, however, that this yields $0 < R < 1$.

We shall establish the existence and uniqueness of Q_G simultaneously. In view of Corollary 2 this can be done if one proves that there exists a unique radial biharmonic monospline $M(x) = M(r)$ on $\delta\{\Omega_1, \Omega_2\}$, $\Omega_1 = B(R)$, $\Omega_2 = B(1) - \bar{B}(R)$ such that $M|_{\Omega_1} = M_1$, $M|_{\Omega_2} = M_2$ and

$$(20) \quad M_2(1) = \frac{dM_2}{dr}(1) = \Delta M_2(1) = \frac{d}{dr} \Delta M_2(1) = 0,$$

$$(21) \quad M_1(R) = M_2(R), \quad \frac{dM_1}{dr}(R) = \frac{dM_2}{dr}(R), \quad \Delta M_1(R) = \Delta M_2(R).$$

We need explicit representations of M_1 and M_2 . It follows from (13) that

$$(22) \quad M_i(r) = \frac{r^4}{8n(n+2)} + \sum_{k=1}^4 d_{ik} y_k(r),$$

where $y_k, k = 1, 2, 3, 4$, are the linearly independent solutions of the homogeneous differential equation $\Delta_r^2 y = 0$. Here, Δ_r is the radial part of Δ . It is well-known that

$$\Delta_r y = y'' + \frac{n-1}{r} y'.$$

Therefore,

$$\frac{d}{dr} \Delta_r y = y''' + \frac{n-1}{r} y'' - \frac{n-1}{r^2} y'$$

and

$$\Delta_r^2 y = y^{(4)} + \frac{n-1}{r} y''' + \frac{(n-1)(n-3)}{r^2} y'' - \frac{(n-1)(n-3)}{r^3} y'.$$

Obviously, for any $n \in \mathbb{N}$ we have that $y_1 = 1$ and $y_2 = r^2$ are solutions of $\Delta_r^2 y = 0$. The remaining two solutions are: for $n = 2$, $y_3 = \log r$ and $y_4 = r^2 \log r$, for $n = 4$, $y_3 = \log r$ and $y_4 = r^{-2}$, for $n \neq 2, 4$, $y_3 = r^{2-n}$ and $y_4 = r^{4-n}$. Recall that we require M to be $C^4(\Omega_i), i = 1, 2$. Since y_3 and y_4 are not C^4 at $r = 0$, we set $d_{13} = d_{14} = 0$ in (22).

On using (22), (13) and (14), the coefficients d_{2k} in the representation of M_2 can be uniquely determined from (20). Then we find d_{11}, d_{12} and R from (21). Finally, (9) yields

$$c(n) = \frac{d}{dr} \Delta_r M_1(R) - \frac{d}{dr} \Delta_r M_2(R).$$

First we perform this procedure for $n \in \mathbb{N}, n \neq 2, 4$. In this case, (20) reduces to

$$\begin{aligned} d_{21} + d_{23} + d_{22} + d_{24} &= -[8n(n+2)]^{-1}, \\ (2-n)d_{23} + 2d_{22} + (4-n)d_{24} &= -[2n(n+2)]^{-1}, \\ 2nd_{22} + 2(4-n)d_{24} &= -[2n]^{-1}, \\ 2(2-n)(4-n)d_{24} &= -1/n. \end{aligned}$$

Solving this linear system, we obtain the explicit representation of M_2 . Taking into account that $M_1(r) = r^4/[8n(2+n)] + d_{12}r^2 + d_{11}$, we conclude that (21) is equivalent to the system

$$\begin{aligned} d_{12}R^2 + d_{11} &= \frac{R^2}{4n(2-n)} - \frac{R^{4-n}}{2n(2-n)(4-n)} - \frac{R^{2-n}}{2n(2-n)(2+n)} \\ &\quad + \frac{1}{8(2-n)(4-n)}, \\ 2Rd_{12} &= \frac{R}{2n(2-n)} - \frac{R^{3-n}}{2n(2-n)} - \frac{R^{1-n}}{2n(2+n)}, \\ 2nd_{12} &= \frac{1}{2(2-n)} - \frac{R^{1-n}}{n(2-n)}. \end{aligned}$$

Subtracting the third equation from the second multiplied by n/R , we obtain $R^2 = n/(n+2)$. Then the coefficients d_{12} and d_{11} are subsequently derived from the third and first equations. The equalities $\frac{d}{dr}\Delta_r M_2(R) = R/n - R^{1-n}/n$ and $\frac{d}{dr}\Delta_r M_2(R) = R/n$ together with $R = (n/(n+2))^{1/2}$ yield

$$c(n) = R^{1-n}/n = \frac{R^2}{n} R^{-n-1} = \frac{1}{n+2} \left(\frac{n+2}{n}\right)^{(n+1)/2}.$$

The cases $n = 2$ and $n = 4$ are treated in the same way. We omit the details and write only the systems to which (20) and (21) are reduced. For $n = 2$ they are

$$\begin{aligned} d_{21} + d_{22} &= -1/64, \\ d_{23} + 2d_{22} + d_{24} &= -1/16, \\ 4d_{22} + 4d_{24} &= -1/4, \\ 4d_{24} &= -1/2 \end{aligned}$$

and

$$\begin{aligned} d_{12}R^2 + d_{11} &= \frac{1}{64} \{4R^2 - 5 - 4 \log R(1 + 2R^2)\}, \\ 2Rd_{12} &= -\frac{1}{16} \{1/R + 4R \log R\}, \\ 4d_{12} &= -\frac{1}{4} \{1 + 2 \log R\}. \end{aligned}$$

For $n = 4$ the corresponding systems are

$$\begin{aligned} d_{21} + d_{24} + d_{22} &= -1/192, \\ -2d_{24} + 2d_{22} + d_{23} &= -1/48, \\ 8d_{22} + 8d_{23} &= -1/8, \\ -4d_{23} &= -1/4 \end{aligned}$$

and

$$\begin{aligned} d_{12}R^2 + d_{11} &= \frac{1}{192} \{-6R^2 + 3 + 12 \log R + 2R^{-2}\}, \\ 2Rd_{12} &= -\frac{1}{48} \{3R - 31/R + 1/R^3\}, \\ 8d_{12} &= -\frac{1}{8} \{2 - R^{-2}\}. \end{aligned}$$

The theorem is proved. \square

Proof of Corollary 1. It must be proved that $\int_B u(x)P_2(x)dx = 0$ for every function u which is harmonic on B . Note that (13) yields $\Delta P_2 \equiv 2n$ and then $D^q \Delta P_2 \equiv 0$ for $|q| = 1$. It is easily seen that $D^2 P_2 \equiv 2$ if $D^2 = \partial^2/\partial x_i^2$ for some $i, 1 \leq i \leq n$, and for the other values of $q, |q| = 2, D^q P_2 \equiv 0$. Hence, applying (15) for u and $v = P_2$, we obtain

$$\Delta^2(uP_2) = 8 \sum_{|q|=2} \frac{1}{q!} (D^q u)(D^q P_2) = 8 \sum_{i=1}^n \partial^2 u / \partial x_i^2 \equiv 0.$$

Hence, the product uP_2 is biharmonic in B . Therefore, Q_G is precise for it and

$$\int_B u(x)P_2(x)dx = c(n) \int_{S(R(n))} uP_2 d\sigma = Q_G(uP_2).$$

Since P_2 vanishes for $|x| = R(n)$, we have $Q_G(uP_2) = 0$. Thus $\int_B uP_2 dx = 0$. □

Proof of Theorem 3. It follows from (12) that for every $u \in H_2^2$ we have

$$R_G(u) = \int_B M(x)\Delta^2 u(x) dx,$$

where M is the radial biharmonic monospline defined in the first section and obtained in the proof of Theorem 2. On applying Hölder's inequality, we obtain

$$|R_G(u)| \leq \|M\|_q \|\Delta^2 u\|_p.$$

If $u \in BH_p^2$, then $|R_G(u)| \leq \|M\|_q$. Hence, $|R_{p,G}(u)| \leq \|M\|_q$. Let $1 < p \leq \infty$. Equality (6) will be proved if we find a function $u \in BH_p^2$ for which $|R_G(u)| = \|M\|_q$. Every function $u \in H_p^2$ such that

$$\Delta^2 u(x) = \left(\int_B |M(x)|^q dx \right)^{-1/p} |M(x)|^{q-1} \text{sign} M(x)$$

belongs to BH_p^2 and obviously $|R_G(u)| = \|M\|_q \|\Delta^2 u\|_p$. □

ACKNOWLEDGEMENTS

The author is grateful to the referees for their helpful remarks.

REFERENCES

1. N.Aronszajn, T.M.Greese and L.J.Lipkin, *Polyharmonic Functions*, Clarendon Press, Oxford, 1983. MR **86g**:31001
2. B.D.Bojanov, H.Hakopian and A.Sahakian, *Spline Functions and Multivariate Interpolations*, Kluwer Academic Publishers, Dordrecht,1993. MR **94k**:41001
3. A.Erdélyi et al., *Higher Transcendental Functions I*, McGraw-Hill, New York, 1953. MR **15**:419i
4. I.J.Schoenberg, *Spline functions, convex curves and mechanical quadratures*, Bull. Amer. Math. Soc. **64**(1958), 352-357. MR **20**:7174
5. *Approximation by Solutions of Partial Differential Equations*, B.Fuglede et al., eds., Kluwer Academic Publishers, Dordrecht, 1992. MR **93a**:31001

DEPARTAMENTO DE CIÊNCIAS DE COMPUTAÇÃO E ESTATÍSTICA, IBILCE, UNIVERSIDADE ESTADUAL PAULISTA, 15054-000 SÃO JOSÉ DO RIO PRETO, SP, BRAZIL
E-mail address: dimitrov@nimitz.ibilce.unesp.br