MATHEMATICS OF COMPUTATION
Volume 65, Number 216
October 1996, Pages 1439–1461

A MUSCL METHOD SATISFYING ALL THE NUMERICAL ENTROPY INEQUALITIES

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ABSTRACT. We consider here second-order finite volume methods for one-dimensional scalar conservation laws. We give a method to determine a slope reconstruction satisfying *all* the *exact* numerical entropy inequalities. It avoids inhomogeneous slope limitations and, at least, gives a convergence rate of $\Delta x^{1/2}$. It is obtained by a theory of second-order entropic projections involving values at the nodes of the grid and a variant of error estimates, which also gives new results for the first-order Engquist-Osher scheme.

1. INTRODUCTION

Second-order upwind schemes for scalar conservation laws, based on ideas of B.Van Leer [28], rely on two steps. First, the application of an upwind solver to a piecewise linear function, then a reconstruction step in order to build this piecewise linear function. The "slope reconstruction" is crucial, and is performed using a minmod limitation, so as to satisfy the total variation diminishing (TVD) property (see A. Harten [11], P.K. Sweby [25]). This procedure is usually called MUSCL method. Unfortunately, this property cannot hold in several dimensions on a non-Cartesian grid, and appears only in a weaker form (S. Champier, T. Gallouët and R. Herbin [5], F. Coquel and P. LeFloch [8], A. Szepessy [26], J.P. Vila [30]).

An entropy inequality is also necessary in order to compute the physical shocks, and is not easily checked when dealing with second-order schemes. It was obtained in various situations by S. Osher [19, 20], S. Osher and S.R. Chakravarthy [21], S. Osher and E.Tadmor [22], J.P. Vila [29]. The numerical entropy inequality is usually obtained for the entropy $S(u) = u^2/2$, with a first-order approximation, under a supplementary nonhomogeneous limitation on the slopes depending on the grid size. Many works are devoted to avoiding this inhomogeneous limitation. H. Yang [31] proposes an approach in that direction. Also, using Hamilton-Jacobi equations, P.L. Lions and P. Souganidis [17] could avoid this kind of supplementary limitations in the case of a convex flux for the implicit scheme. For finite element methods, these problems are also relevant. J. Jaffre, C. Johnson and A. Szepessy [12] have developed a high-order multidimensional discontinuous Galerkin method, which satisfies all the entropy conditions, but again with a nonhomogeneous artificial viscosity term. In a simpler context, G. Jiang and C.-W. Shu [13] have presented a simple approach to get this inequality without unnatural limitation or viscosity.

Received by the editor August 4, 1994 and, in revised form, August 24, 1995.
1991 *Mathematics Subject Classification.* Primary 65M15, 35Q53, 35L65.
Key words and phrases. Scalar conservation laws, MUSCL method, discrete entropy inequality, kinetic schemes, entropic slope reconstruction.

In this paper, we present a second-order MUSCL-type scheme which satisfies the entropy conditions for general one-dimensional scalar conservation laws. It does not use any grid size dependent limiters. The key of the construction is to evolve not only the cell averages but also the solution values at half nodes. Hence the result does not contradict the (almost) impossibility of such a second-order scheme within the class of schemes evolving only cell averages proved by S. Osher and E. Tadmor [22]. The abstract form of our scheme is very simple. Starting from a piecewise linear function, one first evolves it exactly or approximately (as done in practice). One then projects the solution at the next time level back to a piecewise linear function. Our major contribution is to give such an abstract projection which diminishes all entropies (Lemma 3.2). In order to make the scheme effective, some technical modifications are needed. They lead to easy-to-code schemes in several situations which we present first. Our most general approach is presented in Theorem 3.6.

We consider a one-dimensional scalar conservation law

$$\begin{cases} \partial_t v + \partial_x A(v) = 0, \quad t \geq 0, x \in \mathbb{R}, \\ v(0,x) = v^0(x). \end{cases} \tag{1.1}$$

Second-order finite volume approximations of $v(x)$ are developed as follows:

$$\Delta x_i (u_i^{n+1} - u_i^n) + \Delta t (A_{i+1/2}^n - A_{i-1/2}^n) = 0. \tag{1.2}$$

We will construct numerical approximations $A_{i+1/2}^n$ of the exact flux

$$A(t_n, x_{i+1/2}) = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} A(v(s, x_{i+1/2}))ds \tag{1.3}$$

such that the scheme satisfies *exactly all* the numerical entropy inequalities

$$\Delta x_i (S_i^{n+1} - S_i^n) + \Delta t (\eta_{i+1/2}^n - \eta_{i-1/2}^n) \leq 0, \tag{1.4}$$

hence recovering in the limit the exact entropy solution, i.e.,

$$\partial_t S(v) + \partial_x \eta(v) \leq 0, \tag{1.5}$$

for all convex functions S, with $\eta' = S'A'$. We are concerned with second-order schemes, which means that, for smooth solutions, the numerical fluxes are second-order approximations of the exact fluxes (1.3).

As usual for finite volume methods, in (1.2), u_i^n is an approximation of the average of the solution v at time $t_n = n\Delta t$ on the cell $C_i = (x_{i-1/2}, x_{i+1/2})$ of length $\Delta x_i = x_{i+1/2} - x_{i-1/2}$ and center $x_i = (x_{i+1/2} + x_{i-1/2})/2$. These cells are supposed to cover $\mathbb{R}$, but their sizes are not supposed to be uniform nor to vary smoothly from i to $i+1$. We set

$$h = \sup_{i \in \mathbb{Z}} \Delta x_i. \tag{1.6}$$

In MUSCL-type methods, one constructs a piecewise linear approximation of $v(t_n, x)$,

$$u^n(x) = u_i^n + s_i^n (x - x_i), \quad x \in C_i; \tag{1.7}$$

we will denote by V^1 the vector space of piecewise linear functions. These functions have a possible jump at the point $x_{i+1/2}$:

$$\Sigma_{i+1/2}^n = u^n(x_{i+1/2}^+) - u^n(x_{i+1/2}^-).$$

We will often need a subset of V^1 defined by the *no sawtooth condition*

$$\Sigma^n_{i+1/2} s^n_i \geq 0 \quad \text{or} \quad \Sigma^n_{i+1/2} s^n_{i+1} \geq 0. \tag{1.8}$$

We prove that it is possible to determine the numerical fluxes $A^n_{i+1/2}$ and the slopes s^n_i so that the entropy inequalities (1.4) hold for all convex S with

$$S^n_i = \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} S(u^n_i + s^n_i(x - x_i))dx. \tag{1.9}$$

Notice that all the authors quoted above use the discrete entropy $S(u^n_i)$, whereas our results only hold for (1.9), which seems fairly new. Another difference is that we use the "characteristic" variant of finite volume methods, where not only the average u^n_i of the solution is computed at each time step, but also point values $u^n_{i\pm1/2}$ (see for gas dynamics P. Colella [7], R. Sanders and A. Weiser [24]). This leads to a more precise reconstruction when using a rough grid.

More precisely, to obtain the values $u^{n+1}_{i\pm1/2}$, we use the kinetic interpretation introduced by Y. Brenier [3, 2], Y. Giga and T. Miyakawa [10], which is closely related to the kinetic formulation of (1.1) (see P.L. Lions, B. Perthame and E. Tadmor [16], B. Perthame and E. Tadmor [23]). This means that our method is nothing but a second-order version of the Engquist-Osher scheme [9]. But our reconstruction of the slopes s^n_i *does not involve any nonhomogeneous* limitation, and this is also new.

Another motivation to obtain all entropy inequalities is that apart from the duality method of E. Tadmor [27] it is the main tool, via S.N. Kružkov [14] entropies, to obtain error estimates by the method of N.N. Kuznetsov [15]. As an application, we recover the first-order convergence rate of $h^{1/2}$. For second-order schemes, such a rate is only known for the max-mod scheme of Y. Brenier and S. Osher [4]; this recent result is due to H. Nessyahu, E. Tadmor and T. Tassa [18]. The results of this paper were announced in [1].

The details of the construction of the schemes are given in §2; with our precise results, we first treat, for simplicity, the particular case of a linear equation or of Burgers' equation. Extensions are possible, but they require more technicalities, and we give the general result as well as a general slope reconstruction theory in §3. The other sections are devoted to proofs. In §4, we show that the explicit schemes of §2 are indeed particular cases, or easy variants, of the general result. In §5, we introduce some general tools, which can be useful elsewhere, to prove the convergence rate. These results are used in the Appendix in order to give new convergence rates for Engquist-Osher type schemes: we do not impose any condition on the time step.

2. Notations and second-order entropic schemes

This section is devoted to particular cases of our main result (Theorem 3.6), which are completely explicit. For linear or Burgers' equations, we give the expression of the numerical fluxes $A^n_{i+1/2}$ and of the entropy flux $\eta^n_{i+1/2}$ and state precisely the properties of the resulting method. The derivation of the scheme, and the proofs of the theorems, are given in the next sections. The linear case is very simple, and we give our results in that case after we have introduced some notations. Next, we treat the Burgers equation without sonic point in §2.3. Finally, the case of Burgers' equation with general initial data is treated in §2.4. We do not claim that these results are of practical interest. They only indicate that it is possible to go further in the theory of second-order schemes, thus recovering at least all the entropy conditions and known convergence rates.

2.1. **Notations.** We begin with some notations and assumptions that will be used throughout the paper. For general fluxes $A(v)$ in (1.1) and initial approximation $u_h^\circ \in V^1$ of v° (but we will denote for simplicity $u^\circ = u_h^\circ$), we define

$$a(\cdot) = A'(\cdot), \tag{2.1}$$

$$a_\infty = \sup_{\min\, u^\circ \le \xi \le \max\, u^\circ} |a(\xi)|. \tag{2.2}$$

Also, we will often need the following conditions, which bound the time step in (1.2): the Courant-Friedrichs-Levy condition (CFL in short)

$$a_\infty \Delta t < \min_{i\in\mathbb{Z}} \Delta x_i, \tag{2.3}$$

and the *piecewise nonovertaking* condition

$$\forall i,\ \forall \xi \in\ [\min\, u^\circ, \max\, u^\circ], \quad \Delta t\, s_i^n\, a'(\xi)\ >\ -1. \tag{2.4}$$

Throughout this paper, $TV(u)$ denotes the total variation of u,

$$TV(u) = \int_{\mathbb{R}} |\partial_x u(x)|\, dx.$$

Finally, we recall the definitions of the classical minmod limiter and its extension:

$$\text{minmod}(a,b) = \begin{cases} 0 & \text{if } ab \le 0, \\ \text{sign}\, a\ \min(|a|,|b|) & \text{if } ab > 0, \end{cases}$$

$$\text{minmod}(E) = \begin{cases} \inf(E) & \text{if } E \subset \mathbb{R}_+, \\ \sup(E) & \text{if } E \subset \mathbb{R}_-, \\ 0 & \text{otherwise.} \end{cases}$$

2.2. **The linear case.** In the linear case $A(u) = au, \quad a > 0$ for instance, we define the exact node values and numerical fluxes, and the slopes, by the induction formulae

$$u_{i+1/2}^{n+1} = u_i^n + s_i^n(\Delta x_i/2 - a\Delta t), \tag{2.5}$$

$$A_{i+1/2}^n = au_i^n + a\frac{s_i^n}{2}(\Delta x_i - a\Delta t), \tag{2.6}$$

$$s_i^{n+1} = \frac{2}{\Delta x_i}\, \text{minmod}(u_{i+1/2}^{n+1} - u_i^{n+1},\ u_i^{n+1} - u_{i-1/2}^{n+1}). \tag{2.7}$$

For this numerical flux, the finite volume method (1.2) satisfies

Theorem 2.1 (Linear equation). *Under the CFL condition* (2.3) *and for initial data* $u^\circ(x) \in V^1 \cap BV(\mathbb{R})$ *satisfying the* ***no sawtooth condition*** (1.8), *the scheme* (1.2), (2.5)–(2.7) *is second-order accurate and satisfies:*

(i) *the entropy conditions* (1.4),(1.9) *for all convex functions* S,

(ii) $\min\, u^\circ\ \le\ u^n(x)\ \le\ \max\, u^\circ$,

(iii) *the* ***no sawtooth condition*** (1.8), *for all* $n \ge 0$,

(iv) *the TVD property,* $TV(u^{n+1})\ \le\ TV(u^n)$,

(v) $|v(t_n,\cdot) - u^n(\cdot)|_{L^1(\mathbb{R})} \le CTV(u_h^\circ)h\sqrt{\frac{t_n}{\Delta t}} + |v^\circ - u_h^\circ|_{L^1(\mathbb{R})}$.
The entropy flux associated with S is

$$\eta_{i+1/2}^n = \frac{1}{\Delta t}\int_0^{\Delta t} aS\left(u_i^n + s_i^n(\Delta x_i/2 - at)\right) dt.$$

We remark that the *no sawtooth condition* is not necessary; we just need to replace the minmod in (2.7) by a slightly different formula, see Lemma 4.1.

From the numerical point of view, we have tested this scheme and other variants motivated in §3. We have obtained results whose precision lies between the Van Leer and second-order ENO schemes.

2.3. **Burgers' equation without sonic point.** The problem of computing an exact node value $u_{i+1/2}^n$ and the exact flux $A_{i+1/2}^n$ is more difficult for Burgers' equation,

$$A(v) = v^2/2. \tag{2.8}$$

In this subsection, we only consider the nonsonic case, i.e., $v(x), u^n(x) \ge 0$. Then, we introduce the following scheme obtained by solving exactly the kinetic equation which follows from the kinetic interpretation of the Engquist-Osher scheme [3],

$$u_{i+1/2}^{n+1} = \frac{u_i^n + s_i^n \Delta x_i/2}{1 + \Delta t s_i^n}, \tag{2.9}$$

$$A_{i+1/2}^n = \frac{(u_i^n + s_i^n \Delta x_i/2)^2}{2(1 + \Delta t s_i^n)}, \tag{2.10}$$

$$s_i^{n+1} = \frac{2}{\Delta x_i} \text{ minmod } (u_{i+1/2}^{n+1} - u_i^{n+1}, u_i^{n+1} - u_{i-1/2}^{n+1}). \tag{2.11}$$

For this scheme we obtain the same results as in the linear case:

Theorem 2.2 (Nonsonic Burgers' equation). *We assume the CFL and nonovertaking conditions* (2.3)–(2.4), *and that the initial data $u^\circ(x) \in V^1 \cap BV(\mathbb{R})$ satisfy $u^\circ(x) \ge 0$ and the* ***no sawtooth condition*** (1.8). *Then, the scheme* (1.2), (2.9)–(2.11) *is second-order accurate and satisfies $u^n(x) \ge 0$ for all $n \ge 0$ and the conclusions* (i)–(v) *of Theorem* 2.1. *The entropy flux is given by*

$$\eta_{i+1/2}^n = \frac{1}{\Delta t}\int_0^{\Delta t}\int_{\xi=0}^{\xi_i(t)} \xi S'(\xi) dt d\xi,$$

where

$$\xi_i(t) = \frac{u_i^n + s_i^n \Delta x_i/2}{1 + t s_i^n}.$$

Remark 2.3. The above expression of the entropy flux can be written in a 'characteristic spirit' rather than a 'kinetic spirit', for instance, following [24],

$$\eta_{i+1/2}^n = \eta_+(u_{i+1/2}^{n+1}) - u_{i+1/2}^{n+1} S(u_{i+1/2}^{n+1}) + \frac{1}{\Delta t}\int_{x_{i+1/2} - \Delta t u_{i+1/2}^{n+1}}^{x_{i+1/2}} S\left(u_i^n + s_i^n(x - x_i)\right) dx,$$

$$\eta_+'(\xi) = S'(\xi)\xi_+.$$

2.4. **Sonic Burgers' equation.** To treat the general Burgers equation, we need more complete formulae. They are obtained by refining the mesh by a factor of two in order to avoid mixing some waves. They produce an algorithm which is more complicated but still effective,

$$u_{i+1/2}^{n+1} = \frac{(u_i^n + s_i^n \Delta x_i/2)_+}{1+\Delta t s_i^n} - \frac{(u_{i+1}^n - s_{i+1}^n \Delta x_{i+1}/2)_-}{1+\Delta t s_{i+1}^n}, \tag{2.12}$$

$$A_{i+1/2}^{n} = \frac{(u_i^n + s_i^n \Delta x_i/2)_+^2}{2(1+\Delta t s_i^n)} + \frac{(u_{i+1}^n - s_{i+1}^n \Delta x_{i+1}/2)_-^2}{2(1+\Delta t s_{i+1}^n)}. \tag{2.13}$$

The slopes are computed by means of

$$u_{i,c}^{n+1} = \frac{u_i^n}{1+\Delta t s_i^n}, \quad A_i^n = \frac{(u_i^n)^2}{2(1+\Delta t s_i^n)}, \tag{2.14}$$

$$u_{i\pm 1/4}^{n+1} = u_i^n \pm s_i^n \frac{\Delta x_i}{4} \pm 2\frac{\Delta t}{\Delta x_i}(A_i^n - A_{i\pm 1/2}^n), \tag{2.15}$$

$$Du_{i\pm 1/4}^{n+1} = \pm \frac{4}{\Delta x_i} \operatorname{minmod}(u_{i\pm 1/4}^{n+1} - u_{i,c}^{n+1}, u_{i\pm 1/2}^{n+1} - u_{i\pm 1/4}^{n+1}), \tag{2.16}$$

$$\begin{aligned} s_i^{n+1} = \frac{2}{\Delta x_i} \operatorname{minmod}(u_i^{n+1} - u_{i-1/2}^{n+1}, u_{i+1/2}^{n+1} - u_i^{n+1}, \\ u_i^{n+1} - u_{i-1/4}^{n+1} + Du_{i-1/4}^{n+1}\frac{\Delta x_i}{4}, \\ u_{i+1/4}^{n+1} - u_i^{n+1} + Du_{i+1/4}^{n+1}\frac{\Delta x_i}{4}). \end{aligned} \tag{2.17}$$

Notice that in the nonsonic case, $u^n \geq 0$, this scheme reduces to the nonsonic scheme (2.9)–(2.11). Again, we obtain the entropy and convergence rate properties:

Theorem 2.4 (General Burgers equation). *We assume the nonovertaking condition* (2.4)*, the half CFL condition*

$$a_\infty \Delta t < \frac{1}{2} \min_{i\in\mathbb{Z}} \Delta x_i, \tag{2.18}$$

and that the initial data $u^\circ(x) \in V^1 \cap BV(\mathbb{R})$ satisfy the ***no sawtooth condition*** (1.8)*. Then, the scheme* (1.2), (2.12)–(2.17) *is second-order accurate and satisfies the conclusions* (i)–(v) *of Theorem* 2.1*. The entropy flux is given by*

$$\eta_{i+1/2}^n = \frac{1}{\Delta t}\int_0^{\Delta t} \left(\int_{\xi\in[0,\xi_i^+(t)]} \xi^+ S'(\xi)d\xi - \int_{\xi\in[0,\xi_{i+1}^-(t)]} \xi^- S'(\xi)d\xi \right) dt,$$

where

$$\xi_i^\pm(t) = \frac{u_i^n \pm s_i^n \Delta x_i/2}{1+t s_i^n}.$$

3. Entropic projections and kinetic formalism

The schemes presented in §2 are particular cases of a general theorem that we present in this section. It relies mainly on a new tool that we introduce here: the notion of entropic projections. This means finding a second-order approximation of a function $u^-(x)$ by a piecewise linear function $u(x)$, while decreasing all the convex entropies

$$\int_{C_i} S(u(x))dx \leq \int_{C_i} S(u^-(x))dx. \tag{3.1}$$

Before doing so, we explain how it is possible to reduce the numerical resolution of scalar conservation laws (1.1) to two steps: exact transport and projection, using the kinetic approach. This is a preliminary step to the proof of the theorems presented in §2.

3.1. **Kinetic interpretation of the Engquist-Osher scheme.** The interpretation of the Engquist-Osher scheme, due to Brenier [3, 2], is based on the following approximation (see also [23, 16]). Introduce a real parameter ξ and define

$$\chi(u,\xi) = \begin{cases} +\ 1 & \text{for} \quad 0 < \xi < u, \\ -\ 1 & \text{for} \quad u < \xi < 0, \\ \ \ \ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Given $u^n(x)$, we solve the free transport equation

$$\begin{cases} \partial_t f + a(\xi)\, \partial_x f = 0, \quad t \in [t_n, t_{n+1}),\ x, \xi \in \mathbb{R}, \\ f(t_n, x, \xi) = \chi(u^n(x), \xi). \end{cases} \tag{3.3}$$

We obtain an infinitely accurate in time approximation of the solution to (1.1), just setting

$$u^{n+1,-}(x) = \int_{\mathbb{R}} f(t_{n+1}^-, x, \xi) d\xi. \tag{3.4}$$

Indeed, if u^n is smooth and Δt is small enough, $u^{n+1,-}$ is the solution to the scalar conservation law (1.1) at time t_{n+1} corresponding to the initial data u^n at time t_n. This is the basis of the *Transport Collapse* (TC in short) method of [2]. We thus define

$$T(t)u\ (x) = \int_{\mathbb{R}} \chi\left(u(x - ta(\xi)), \xi\right) d\xi. \tag{3.5}$$

Notice that in the linear case $a(\xi) = a$, we have that $T(t)u\ (x) = u(x - at)$ is the exact solution to the equation (1.1). The TC operator satisfies the following properties:

Lemma 3.1 ([2]). *We have*
(i) $\int_{|x-x_0|<R} |T(t)u - T(t)v| \leq \int_{|x-x_0|<R+|a|_\infty t} |u - v|$,
(ii) *if* $u \leq v$, *then* $T(t)u \leq T(t)v$ *and* $\inf_{\mathbb{R}} u \leq T(t)u \leq \sup_{\mathbb{R}} u$,
(iii) $|T(t)u - T(t)v|_{L^1(\mathbb{R})} \leq |u - v|_{L^1(\mathbb{R})}$,
(iv) $TV(T(t)u) \leq TV(u)$,
(v) $|T(t_1)u - T(t_2)u|_{L^1(\mathbb{R})} \leq |a|_\infty TV(u)|t_1 - t_2|$,
(vi) *for any convex function* $S(\cdot)$,

$$S(T(t)u) - S(u) + \partial_x \int_0^t \int_{\mathbb{R}} S'(\xi)a(\xi)\chi\left(u(x - sa(\xi)), \xi\right) d\xi ds \leq 0. \tag{3.6}$$

These properties are straightforward consequences of the fact that

$$\int_{\mathbb{R}} |\chi(u,\xi) - \chi(v,\xi)| d\xi = |u - v|, \tag{3.7}$$

and that $\chi(u(x - a(\xi)t), \xi)$ solves the linear transport equation (3.3). Also, for convex functions S (see [3, 2, 23, 16]) we have

$$S(T(t)u) \le S(0) + \int_{\mathbb{R}} S'(\xi)\chi(u(x - a(\xi)t), \xi) d\xi, \tag{3.8}$$

because, for any function $f(\xi)$ satisfying $0 \le \operatorname{sign}(\xi)\, f(\xi) \le 1$ we have that, for any convex function S,

$$S\Big(\int_{\mathbb{R}} f(\xi) d\xi\Big) \le S(0) + \int_{\mathbb{R}} S'(\xi) f(\xi)\, d\xi. \tag{3.9}$$

3.2. **Entropic projections.** Up to this point we have developed a good approximation of the solution of the scalar conservation law (1.1) after a time step. But, if u^n is piecewise linear, the approximation $u^{n+1,-} = T(\Delta t)u^n$ is not. Therefore, it remains to explain how to construct a projection u^{n+1} of $u^{n+1,-}$ in V^1, the vector space of piecewise linear functions, which realizes the entropy dissipation (3.1). A general method is as follows.

Lemma 3.2. *Let* $u \in L^1(a,b)$, $c = \dfrac{a+b}{2}$ *and*

$$\underline{u} = \frac{1}{b-a}\int_a^b u(x)dx. \tag{3.10}$$

Define the function $\zeta \in C(a,b)$ *and the approximate derivative* Du *of* u *by*

$$\zeta(y) = \frac{2}{b-a}\left(\frac{1}{b-y}\int_y^b u(x)dx - \frac{1}{y-a}\int_a^y u(x)dx\right), \quad a < y < b, \tag{3.11}$$

$$Du = \operatorname{minmod}_{a<y<b} \zeta(y). \tag{3.12}$$

Then, (i) *for all convex functions* S *and* $\theta \in [0,1]$,

$$\int_a^b S(\underline{u} + \theta Du(x - c))dx \le \int_a^b S(u(x))dx, \tag{3.13}$$

(ii) *if* u *is continuous at the points* a *and* b, *then* $\zeta \in C([a,b])$ *and*

$$\zeta(a) = \frac{2}{b-a}\left(\underline{u} - u(a)\right), \quad \zeta(b) = \frac{2}{b-a}\left(u(b) - \underline{u}\right).$$

Proof. The continuity statements and (ii) are obvious, and we just prove (i). Denote

$$v(x) = \underline{u} + Dv(x - c), \quad Dv = \theta Du.$$

Since (3.13) holds as an equality when S is a linear function, it is enough to prove it for the entropies $S(u) := S_k(u) = (u - k)_+$. We have, by convexity,

$$\int_a^b (S_k(u) - S_k(v)) \ge \int_a^b (u - v)\, I_{v>k} := J, \tag{3.14}$$

and we are going to prove that $J \geq 0$. The result is clear if $Dv = 0$, or more generally, if $v - k$ has a constant sign on (a, b). Thus, we can assume that $Dv > 0$, for instance, and

$$J = \int_a^b (u - v)\, I_{x>y}, \quad a < y = c + \frac{k - \underline{u}}{Dv} < b.$$

Then,

$$\begin{aligned} J &= \int_y^b (u - v) \\ &= \int_y^b u \; - (b - y)\underline{u} - Dv(y - a)(b - y)/2 \\ &= \frac{y - a}{b - a} \int_y^b u - \frac{b - y}{b - a} \int_a^y u - Dv(y - a)(b - y)/2 \\ &= (y - a)(b - y)(\zeta(y) - Dv)/2. \end{aligned}$$

Since $Du > 0$, and from the definition of Du and Dv, we have $\zeta(y) \geq Du \geq Dv$. Hence $J \geq 0$. The case $Dv < 0$ is similar. □

Remark 3.3. (1) Our definition of Du is consistent with the derivative for C^1 functions. Indeed, if u is linear, then Du is just the slope of u. Then, by a convexity argument, one can check that, for all $y \in (a, b)$, there is a point $\eta_y \in [a, b]$ such that

$$\zeta(y) = \frac{2}{b - a} \int_a^b (\frac{x - a}{y - a} I_{x \leq y} + \frac{b - x}{b - y} I_{y < x}) u'(x) dx = u'(\eta_y). \tag{3.15}$$

Therefore, there is also an $\eta \in [a, b]$ such that $Du = u'(\eta)$.

(2) Another way to see the consistency of Du is as follows. If u is convex in $[a, b]$ (resp. concave), then ζ is nondecreasing (resp. nonincreasing) and thus

$$Du = \frac{2}{b - a} \operatorname{minmod}(\underline{u} - u(a), u(b) - \underline{u}).$$

This is a consequence of the following formula, which gives the derivative of ζ:

(3.16)
$$\zeta'(y) = \frac{2}{(b - y)(y - a)} \left(\frac{y - a}{(b - a)(b - y)} \int_y^b u + \frac{b - y}{(b - a)(y - a)} \int_a^y u - u(y) \right),$$

and of the following type of inequalities, in the convex case for instance,

$$u(\frac{y + b}{2}) = u(\frac{1}{b - y} \int_y^b x dx) \leq \frac{1}{b - y} \int_y^b u(x) dx.$$

(3) Still another case where consistency appears clearly is $u' \in L^1(a, b)$; then $\underline{Du} = \operatorname{minmod}\{u'(y), a < y < b\}$ satisfies indeed

$$\underline{Du} = \theta Du \quad \text{for some} \quad 0 \leq \theta \leq 1. \tag{3.17}$$

Therefore, this evaluation $\underline{Du}$ of the derivative, although it is entropic, is not as good as Du. Especially when u has discontinuities, it cannot be used because it is too far from Du and accuracy is lost. □

We can now go back to the numerical schemes for (1.1). Given a piecewise linear function $u^n(x)$, we have developed a second-order approximation of the scalar conservation law $u^{n+1,-} = T(\Delta t)u^n$, using the transport collapse method. We can define another operator and another piecewise linear function (we use the notations of the introduction and §2.1),

$$Q^1(\Delta t) = P^1 \cdot T(\Delta t), \quad u^{n+1} = P^1 u^{n+1,-} = Q^1(\Delta t)u^n; \tag{3.18}$$

the projection P^1 is just defined as above on each cell:

$$P^1 u(x) = u_i + Du_i(x - x_i) \quad \text{for } x \in C_i,$$

$$u_i = \frac{1}{\Delta x_i}\int_{C_i} u(x)dx, \quad Du_i = \text{minmod}\{\zeta_i(x),\ x \in C_i\}, \tag{3.19}$$

and ζ_i is just defined as ζ in (3.11) in each cell C_i by means of Lemma 3.2. We can give some properties of the operator P^1:

Proposition 3.4. *The projection P^1 enjoys the following properties:*

(i) $\forall u \in V^1,\ P^1 u = u$,

(ii) $\inf(u) \le P^1 u \le \sup(u)$,

(iii) *for any convex function S, we have, for all $i \in \mathbb{Z}$,*

$$\int_{C_i} S(P^1 u) \le \int_{C_i} S(u),$$

(iv) $|P^1 u|_{L^p(\mathbb{R})} \le |u|_{L^p(\mathbb{R})}$ *for all* $1 \le p \le \infty$,

(v) *if u is monotone nonincreasing (resp. nondecreasing), so is $P^1 u$,*

(vi) $TV(P^1 u) \le TV(u)$, $|P^1 u - u|_{L^1(\mathbb{R})} \le \frac{3h}{4} TV(u)$,

(vii) *if $u \in BV(\mathbb{R})$ is continuous at the points $x_{i+1/2}$, then $P^1 u$ satisfies the "no-sawtooth condition"* (1.8).

Remark 3.5. (1) The approximation rate given in (vi) is just first-order. This is because we only use the BV regularity of u, the only one available in practice. If $u \in C^2$, one can prove that $|P^1 u - u|_\infty \le |u''|_\infty h^2/2$.

(2) It is easy to check that, except for property (i), Proposition 3.4 holds if we replace Du_i by $\theta_i Du_i$, $0 \le \theta_i \le 1$.

Proof of Proposition 3.4. We use the notation $v = P^1 u$ throughout this proof. (i) is clear because, when u is linear on C_i, then Du_i is just its slope. Next, we let $y > x_{i-1/2}$ tend to $x_{i-1/2}$, and $y < x_{i+1/2}$ tend to $x_{i+1/2}$. We find in the definition of ζ_i that

$$Du_i = \alpha_i \frac{2}{\Delta x_i}(u_i - \lambda_i), \quad Du_i = \beta_i \frac{2}{\Delta x_i}(\mu_i - u_i),$$

for some $0 \le \alpha_i, \beta_i \le 1$, $\inf(u) \le \lambda_i, \mu_i \le \sup(u)$ (if right and left limits exist, in the BV case for instance, then $\lambda_i = u(x^+_{i-1/2}), \mu_i = u(x^-_{i+1/2})$). Hence,

$$v(x^+_{i-1/2}) \in [u_i, \lambda_i], \quad v(x^-_{i+1/2}) \in [u_i, \mu_i], \tag{3.20}$$

and (ii) is proved. (iii) is just the inequality (3.13). (iv) is obtained from (iii) by choosing $S(u) = |u|^p$. Next, if u is nondecreasing for instance, then we obtain

$Du_i \geq 0$ and (3.20) shows that the jumps of v are nondecreasing. This proves (v). The proof of (vi) is more delicate. We have

$$TV(v) = \sum_i |v(x_{i+1/2}^-) - v(x_{i-1/2}^+)| + \sum_i |v(x_{i+1/2}^+) - v(x_{i+1/2}^-)|,$$

and

$$|v(x_{i+1/2}^+) - v(x_{i+1/2}^-)| \leq |u(x_{i+1/2}^+) - u(x_{i+1/2}^-)| + |v(x_{i+1/2}^+) - u(x_{i+1/2}^+)| + |v(x_{i+1/2}^-) - u(x_{i+1/2}^-)|.$$

Hence,

$$\begin{aligned} TV(v) \leq & \sum_i |u(x_{i+1/2}^+) - u(x_{i+1/2}^-)| \\ & + \sum_i \Big(|v(x_{i+1/2}^-) - v(x_{i-1/2}^+)| + |v(x_{i-1/2}^+) - u(x_{i-1/2}^+)| \\ & \qquad + |v(x_{i+1/2}^-) - u(x_{i+1/2}^-)| \Big). \end{aligned}$$

But, by (3.20),

$$\begin{aligned} & |v(x_{i+1/2}^-) - v(x_{i-1/2}^+)| + |v(x_{i-1/2}^+) - u(x_{i-1/2}^+)| + |v(x_{i+1/2}^-) - u(x_{i+1/2}^-)| \\ & \quad = |v(x_{i+1/2}^-) - u_i| + |u_i - v(x_{i-1/2}^+)| + |v(x_{i-1/2}^+) - u(x_{i-1/2}^+)| \\ & \qquad + |v(x_{i+1/2}^-) - u(x_{i+1/2}^-)| \\ & \quad = |u(x_{i+1/2}^-) - u_i| + |u_i - u(x_{i-1/2}^+)|, \end{aligned}$$

which yields

$$\begin{aligned} TV(v) & \leq \sum_i \Big(|u(x_{i+1/2}^-) - u_i| + |u_i - u(x_{i-1/2}^+)| + |u(x_{i+1/2}^+) - u(x_{i+1/2}^-)| \Big) \\ & \leq TV(u). \end{aligned}$$

This is the first inequality of (vi). To prove the second, we use the first-order projector P^0 (on piecewise constant functions):

$$|u - v|_1 \leq |u - P^0 u|_{L^1(\mathbb{R})} + |P^0 u - v|_{L^1(\mathbb{R})} \leq \frac{3}{4} h TV(u),$$

because $|P^0 u - u|_{L^1(\mathbb{R})} \leq hTV(u)/2$, and

$$\begin{aligned} |P^0 u - v|_{L^1(\mathbb{R})} & = \sum_i \int_{C_i} |Du_i||x - x_i| dx = \sum_i \frac{\Delta x_i^{\;2}}{4} |Du_i| \\ & \leq \sum_i \frac{\Delta x_i}{4} |u(x_{i+1/2}^-) - u(x_{i-1/2}^+)| \\ & \leq TV(u)\, h/4, \end{aligned}$$

which gives the second inequality of (vi). Finally, (vii) is also a straightforward consequence of (3.20). □

From these properties of P^1 follow the properties of the operator Q^1, i.e., of the scheme $u^{n+1} = Q^1(\Delta t)u^n$. Although it might look very abstract, this is our main result, because we will show in the next section that Q^1 can be made explicit in particular cases.

Theorem 3.6. *The numerical scheme Q^1 satisfies the following properties*

(i) $\inf(u) \le Q^1(\Delta t)u \le \sup(u)$,

(ii) $|Q^1(\Delta t)u|_{L^p(\mathbb{R})} \le |u|_{L^p(\mathbb{R})}$ *for all* $1 \le p \le \infty$,

(iii) *if u is monotone nonincreasing (resp. nondecreasing), so is* $Q^1(\Delta t)u$,

(iv) $TV(Q^1(\Delta t)u) \le TV(u)$ *and* $|Q^1(\Delta t)u - u|_{L^1(\mathbb{R})} \le TV(u)(a_\infty \Delta t + 3h/4)$,

(v) *for any convex function S, we have, for all i,*

$$\frac{1}{\Delta x_i}\int_{C_i} S\left(Q^1(\Delta t)u\right) - \frac{1}{\Delta x_i}\int_{C_i} S(u) + \frac{\Delta t}{\Delta x_i}(\eta_{i+1/2} - \eta_{i-1/2}) \le 0, \tag{3.21}$$

$$\eta_{i+1/2} = \frac{1}{\Delta t}\int_0^{\Delta t}\int_{\mathbb{R}} S'(\xi)a(\xi)\chi\left(u(x_{i+1/2} - sa(\xi)), \xi\right) d\xi ds, \tag{3.22}$$

(vi) *let $u^\circ \in V^1 \cap BV(\mathbb{R})$, denote $T = n\Delta t$, and let v be the exact entropic solution to* (1.1) *with initial data u°. Then, we have for some absolute constants C,*

$$|Q^1(\Delta t)^n u^\circ - v(T,\cdot)|_{L^1(\mathbb{R})} \le C\,TV(u^\circ)\left(a_\infty\sqrt{T\Delta t} + h\sqrt{T/\Delta t}\right), \tag{3.23}$$

and in the linear case $a(\xi) = a$,

$$|Q^1(\Delta t)^n u^\circ - v(T,\cdot)|_1 \le C\,TV(u^\circ)\, h\sqrt{T/\Delta t}. \tag{3.24}$$

Remark 3.7. (1) These results hold without any CFL condition, and for any flux $A(v)$ in the equation (1.1). They can be seen as an abstract second-order extension of the Transport Collapse method. Under the CFL condition and for $\Delta t \ge \alpha h$ for some $\alpha > 0$, we obtain the classical rate of convergence $h^{1/2}$.

(2) Since P^1 is a conservative operator, we also have a discretized equation on the cell averages,

$$\frac{1}{\Delta x_i}\int_{C_i} Q^1(\Delta t)u - \frac{1}{\Delta x_i}\int_{C_i} u + \frac{\Delta t}{\Delta x_i}(A_{i+1/2} - A_{i-1/2}) = 0,$$

$$A_{i+1/2} = \frac{1}{\Delta t}\int_0^{\Delta t}\int_{\mathbb{R}} a(\xi)\chi(u(x_{i+1/2} - sa(\xi)), \xi) ds d\xi. \tag{3.25}$$

Proof of Theorem 3.6. All these results are straightforward combinations of the corresponding results of Lemma 3.1 and Proposition 3.4. Only the global rate of convergence (vi) is new, and its proof will be given in §5.

4. Proof of the main results

Under some conditions the operator P^1 can be completely identified. Then, our results on the operator Q^1 give the convergence and the entropy inequalities for numerical schemes. This is the case of the three results announced in §2. We detail the explicit computations for the different cases below.

Since all these results are special cases of Theorem 3.6, the fluxes $A^n_{i+1/2}$ and the entropy fluxes $\eta^n_{i+1/2}$, in the theorems of §2, are those given in (3.25) and (3.22), which are explicit for u a piecewise linear function, for a CFL less than one and under the piecewise nonovertaking condition (2.4). They are just those of the (TC) operator. Also second-order accuracy is always maintained because Q^1 is second-order in space and time. It remains to explain how to compute P^1.

4.1. **The linear case.** This case relies on a preliminary lemma.

Lemma 4.1. *With the notations of Lemma 3.2, let $d \in (a,b)$ and assume that u is linear in each subinterval (a,d) and (d,b), with a jump Σ at the point d. Then*

$$Du = \frac{2}{b-a} \operatorname{minmod}(\underline{u} - u(a), u(b) - \underline{u}, \frac{b-a}{2}\zeta(d)). \tag{4.1}$$

If u satisfies the no sawtooth condition

$$u'_l \, \Sigma \geq 0 \quad or \quad u'_r \, \Sigma \geq 0, \tag{4.2}$$

where u'_l, u'_r are the left and right derivatives of u, then

$$Du = \frac{2}{b-a} \operatorname{minmod}(\underline{u} - u(a), u(b) - \underline{u}). \tag{4.3}$$

Proof. Following Lemma 3.2 (ii), we have $\zeta \in C([a,b])$, and one easily computes

$$\zeta(y) = \begin{cases} \frac{2}{b-y}(\underline{u} - u(\frac{a+y}{2})) & \text{if } a \leq y \leq d, \\ \frac{2}{y-a}(u(\frac{b+y}{2}) - \underline{u}) & \text{if } d \leq y \leq b. \end{cases} \tag{4.4}$$

But ζ is monotone on both subintervals $(a,d), (d,b)$, and thus (4.1) follows. The no sawtooth case will be proved in the next subsection (see Lemma 4.2). □

Now, we can complete the proof of the linear case because, under the CFL condition, the projection P^1 can be completely identified. Indeed, after a time step, a piecewise linear no sawtooth function is translated into a new function which satisfies the assumptions of Lemma 4.1. Thus, (4.3) holds, just giving in each cell the slope $s_i^{n+1} = Du_i^{n+1,-}$ used in the scheme of Theorem 2.1. Hence, Theorem 2.1 is nothing but Theorem 3.6 in this particular case. Notice that the no sawtooth condition propagates thanks to Proposition 3.4 (vii), which holds true here.

4.2. **Nonsonic Burgers' equation.** Again, we will prove that the formula given in §2.3 is an explicit expression of the operator Q^1, in the nonsonic Burgers case and under the CFL and nonovertaking conditions (2.3), (2.4). Indeed, in that case we can compute the exact solution of the Transport Collapse operator, with $u^n \geq 0$ a piecewise linear function. It is a continuous function given by the formula

$$\begin{aligned} [T(\Delta t)u^n](x) &= \frac{u_i^n + s_i^n(x - x_i)}{1 + \Delta t s_i^n} - \frac{(x - d_{i,2})_-}{\Delta t(1 + \Delta t s_i^n)} + \frac{(x - d_{i,1})_-}{\Delta t(1 + \Delta t s_{i-1}^n)}, \\ d_{i,1} &= x_{i-1/2} + \Delta t(u_{i-1}^n + s_{i-1}^n \Delta x_{i-1}/2), \\ d_{i,2} &= x_{i-1/2} + \Delta t(u_i^n - s_i^n \Delta x_i/2). \end{aligned} \tag{4.5}$$

In each cell C_i, this function is composed of, at most, three linear pieces. In order to compute its projection, we need to identify Du in the following case:

Lemma 4.2. *With the notations of Lemma 3.2, let $a < d_1 < d_2 < b$. Assume that u is continuous on $[a,b]$ and linear in each subinterval $(a,d_1), (d_1,d_2)$ and (d_2,b), with respective slopes u'_l, u'_m, u'_r satisfying the condition*

$$\operatorname{minmod}(u'_l, u'_m, u'_r) = \operatorname{minmod}(u'_l, u'_r). \tag{4.6}$$

Then

$$Du = \frac{2}{b-a} \operatorname{minmod}(\underline{u} - u(a), u(b) - \underline{u}). \tag{4.7}$$

Remark 4.3. (1) Notice that, by Remark 3.3 (3), we have for some $0 \leq \theta \leq 1$,

$$\operatorname{minmod}(u'_l, u'_m, u'_r) = \theta Du.$$

Therefore, the expression (4.7) gives a better slope than (4.6), i.e., the $\underline{Du}$ of (3.17).

(2) In the limit case $d_1 = d_2$, $u'_m = \pm\infty$, we recover the case of Lemma 4.1 and the condition (4.6) is nothing but the no sawtooth condition in Lemma 4.1. Hence, we indeed recover the conclusion (4.3).

Proof of Lemma 4.2. Since u is continuous, ζ is C^1, and one computes

$$\zeta(y) = \begin{cases} \frac{2}{b-y}(\underline{u} - u(\frac{a+y}{2})) & \text{if } a \leq y \leq d_1, \\ \frac{2}{y-a}(u(\frac{b+y}{2}) - \underline{u}) & \text{if } d_2 \leq y \leq b, \\ \frac{\alpha}{y-a} + \frac{\beta}{b-y} + \gamma & \text{if } d_1 \leq y \leq d_2, \end{cases} \tag{4.8}$$

for some real numbers α, β, γ, which are uniquely defined so that $\zeta \in C^1([a,b])$. The function ζ is homographic, hence monotone, on $[a, d_1] \cup [d_2, b]$. We have to prove that $Du = m := \operatorname{minmod}(\zeta(a), \zeta(b))$. Three cases occur: if $m = 0$, then clearly $Du = 0$, and we are done. Next, we treat the case $m > 0$, for instance (the case $m < 0$ is similar and we do not repeat the proof). We are going to prove that

$$\zeta'(a) \geq 0 \quad \text{or} \quad \zeta'(b) \leq 0. \tag{4.9}$$

Indeed, one has

$$\zeta(a) = \frac{2}{b-a}(\underline{u} - u(a)), \quad \zeta'(a) = \frac{1}{b-a}(\zeta(a) - u'(a)),$$

$$\zeta(b) = \frac{2}{b-a}(u(b) - \underline{u}), \quad \zeta'(b) = \frac{1}{b-a}(u'(b) - \zeta(b)).$$

If we had $\zeta'(a) < 0$ and $\zeta'(b) > 0$, since $\zeta(a) > 0$, $\zeta(b) > 0$ $(m > 0)$, we would have (remember that $c = (a+b)/2$)

$$u(a) < \underline{u} < u(b), \quad u(b) + (c-b)u'(b) < \underline{u} < u(a) + (c-a)u'(a). \tag{4.10}$$

Hence, $u'(a) > 0$, $u'(b) > 0$, i.e., $u'_l > 0$, $u'_r > 0$. By the condition (4.6) this implies $u'_m \geq \min(u'_l, u'_r)$. Now, if $u'_m \in [u'_l, u'_r]$, then u is either convex or concave, and we are done thanks to Remark 3.3 (2). On the other hand, if $u'_m > \max(u'_l, u'_r)$, then it is geometrically obvious that this yields

$$u(a) + (y-a)u'_l \leq u(y) \leq u(b) + (y-b)u'_r, \ d_1 \leq y \leq d_2.$$

From (4.10), we deduce that c does not belong to $[d_1, d_2]$. Using (4.10) again, we obtain, in the case $c > d_2$ for example, that $u'_l > u'_r$ and thus, $u(y) \leq u(b) + (y-b)u'_r$. Integrating this over $y \in [a,b]$ gives $\underline{u} \leq u(b) + (c-b)u'_r$, which contradicts (4.10). The case $c < d_1$ is similar and we always obtain that (4.9) holds.

Now that (4.9) is proved (still in the case $m > 0$), we deduce the result $Du = m$. Indeed, notice that ζ is monotone on $[a, d_1]$, $[d_2, b]$; and on $[d_1, d_2]$, ζ is either nondecreasing, nonincreasing, convex or concave. Since ζ is C^1, the only nontrivial case is when ζ is first nonincreasing, then convex, then nondecreasing, which of course contradicts (4.9), and Lemma 4.2 is proved. □

Now, we may complete the proof of Theorem 2.2. On each cell, $T(\Delta t)u^n$ satisfies the conditions of Lemma 4.2. Indeed, the no sawtooth assumption on u^n gives exactly (4.6), as is readily proved computing the three derivatives of $T(\Delta t)u^n$. And the point values referred to as $u(a)$, $u(b)$ are exactly the values $u^{n+1}_{i\pm1/2}$ in (2.9), so that the slope of $[Q^1(\Delta t)u^n]$ in C_i is exactly s_i^{n+1} in (2.11). Therefore, Theorem 2.2 is again nothing but Theorem 3.6 in this case and the no sawtooth condition propagates thanks to Proposition 3.4(vii).

4.3. **Sonic Burgers' equation.** Now, we treat the general case of Burgers' equation without any sign assumption on the initial data. Then, a simple identification of Q^1 is not possible because the exact solution of the TC operator is more complicated. Under the conditions (2.3), (2.4) it is still continuous but composed, at most, of five linear pieces,

$$[T(\Delta t)u^n](x) = \frac{u_i^n + s_i^n(x-x_i)}{1+\Delta t s_i^n} - \frac{(x-d_{i,2})_-}{\Delta t(1+\Delta t s_i^n)} + \frac{(x-d_{i,1})_-}{\Delta t(1+\Delta t s_{i-1}^n)}$$

$$+\frac{(x-d_{i+1,1})_+}{\Delta t(1+\Delta t s_i^n)} - \frac{(x-d_{i+1,2})_+}{\Delta t(1+\Delta t s_{i+1}^n)}. \tag{4.11}$$

Here, $d_{i,1}$, $d_{i,2}$ are still given by (4.5). In principle, it is possible to test where the minmod is attained in the definition of Du. But the resulting effective algorithm is not very simple. Instead, it is simpler to introduce a new projection P^{1*}, and a new scheme $Q^{1*}(\Delta t) = P^{1*}\cdot T(\Delta t)$, with

$$P^{1*} = P^1 \cdot P^1_{h/2}, \tag{4.12}$$

where $P^1_{h/2}$ denotes the second-order projection associated with the grid whose cells are half of the original ones. Of course, the properties of Theorem 3.6 are still valid for P^{1*}, because they are deduced from properties which hold for $T(\Delta t)$, P^1. But $P^1_{h/2}$ introduces some discontinuities at the points $x_{i+1/2}$, therefore item (vii) in Proposition 3.4 does not apply and thus, $u^{n+1} = Q^{1*}(\Delta t)u^n$ does not satisfy the no sawtooth condition.

In order to compute P^{1*} in the above situation, we first prove a preliminary result.

Lemma 4.4. *Let $u \in L^1(a,b)$, and $c=(a+b)/2$. Define*

$$\underline{u} = \frac{1}{b-a}\int_a^b u, \quad \underline{u_l} = \frac{1}{c-a}\int_a^c u, \quad \underline{u_r} = \frac{1}{b-c}\int_c^b u, \tag{4.13}$$

and let Du_l, Du_r be the approximate derivatives of u given by (3.12), corresponding to the intervals (a,c), (c,b), respectively. Set

$$Du^* = \frac{2}{b-a}\,\mathrm{minmod}\Big(\underline{u} - \underline{u_l} + \frac{b-a}{4}Du_l,\ \underline{u_r} + \frac{b-a}{4}Du_r - \underline{u},\ \underline{u_r} - \underline{u_l}\Big). \tag{4.14}$$

Then, for any convex function S and any $0 \le \theta \le 1$, we have

$$\int_a^b S(\underline{u} + \theta Du^*(x-c))dx \ \le\ \int_a^b S(u(x))dx. \tag{4.15}$$

Moreover, if u is continuous at c, then the last argument $\underline{u_r} - \underline{u_l}$ can be omitted in (4.14) and it does not change the value of Du^*.

Again, it is easy to see that Du^* is consistent with the value of the derivatives, because for C^1 functions u, we have $Du^* = u'(\eta)$ for some $\eta \in [a,b]$, and thus $u' = Du^*$ for linear functions. Also, both Du and Du^* belong to $[0, \frac{2}{b-a}(\underline{u_r} - \underline{u_l})]$.

Proof of Lemma 4.4. Define the function

$$v(x) = \begin{cases} \underline{u_l} + Du_l(x - \frac{a+c}{2}) & \text{if } a < x < c, \\ \underline{u_r} + Du_r(x - \frac{c+b}{2}) & \text{if } c < x < b. \end{cases}$$

We have $\underline{v} = \underline{u}$, $\underline{v_l} = \underline{u_l}$, $\underline{v_r} = \underline{u_r}$, and by Lemma 3.2, for any convex function S,

$$\int_a^c S(v) \le \int_a^c S(u), \quad \int_c^b S(v) \le \int_c^b S(u).$$

Hence, using Lemma 3.2 again, for any $0 \le \theta \le 1$ we find

$$\int_a^b S(\underline{v} + \theta Dv(x-c))dx \le \int_a^b S(v) \le \int_a^b S(u).$$

Now, we may compute the approximate derivative of v using Lemma 4.1:

$$\begin{aligned} Dv &= \frac{2}{b-a}\,\mathrm{minmod}(\underline{v} - v(a),\ v(b) - \underline{v},\ \underline{v_r} - \underline{v_l}) \\ &= \frac{2}{b-a}\,\mathrm{minmod}(\underline{u} - \underline{u_l} + \frac{b-a}{4}Du_l,\ \underline{u_r} + \frac{b-a}{4}Du_r - \underline{u},\ \underline{u_r} - \underline{u_l}). \end{aligned}$$

Hence, $Dv = Du^*$, and we obtain (4.15). Finally, if u is continuous at c, we have

$$v(c-0) \in [\underline{u_l}, u(c)], \quad v(c+0) \in [\underline{u_r}, u(c)],$$

and the no sawtooth condition (4.2) is met for v. Then, in view of Lemma 4.1, the last argument in the above minmod can be omitted. □

In the cases when u satisfies either the conditions of Lemma 4.2 or 4.1, with the nosawtooth condition (4.2) fulfilled, it is possible to prove that $Du = \theta Du^*$ for some $0 \le \theta \le 1$. This means that the slope reconstruction using Du is not optimal to realize the minimal entropy dissipation.

Notice that, as is evident from the above proof, the derivative Du^* yields the operator P^{1*} in (4.12). Now, we can complete the proof of Theorem 2.4. We just apply the above lemma to compute $D^*u_i^{n+1}$ on each cell. The formula (2.15) gives the averages in the half-meshes of $T(\Delta t)u^n$, $u_{i,c}^{n+1}$ is its exact value at x_i, and in (2.16) we deduce the right and left derivatives (Du_r, Du_l in the above lemma) by applying Lemma 4.2, since the nosawtooth condition on u^n ensures (4.6) for $u^{n+1,-}$. In the slope reconstruction (2.17) we have just added the two first arguments in the minmod to ensure the preservation of the nosawtooth condition (1.8). This is just a variant of Q^1, which does not affect its properties stated in Theorem 3.6.

5. Convergence rate

In this section, we prove the convergence rate estimate, which has been announced in Theorem 3.6 (vi). The difficulty we have to face is that the averaged entropy inequality (1.4) does not seem strong enough to obtain it. We need to go further and derive a differential form for this entropy inequality (§5.1). Then, the convergence rate follows from a general result that we present in §5.2. In §5.3, we conclude the proof of the convergence rate.

Throughout this section, we use the notations of the introduction, §2.1 and §3.

5.1. **Improved entropy inequality.** In order to study Q^1, we define the functions $f(t,x,\xi) := f_h(t,x,\xi)$, $u(t,x) := u_h(t,x)$ by using the free transport equation (3.3), with discontinuities at times t_n, from $f(t_n^-,x,\xi)$ to $\chi(u^{n,-}(x),\xi)$ and then to $\chi(u^n(x),\xi)$. We set

$$u(t,x) = \int_{\mathbb{R}} f(t,x,\xi)d\xi. \tag{5.1}$$

Of course, this means that u also has a discontinuity at times t_n, but not its cell averages. At these times, the jump from $u^{n,-}$ to u^n is defined by $u^n = P^1 u^{n,-} = Q^1 u^{n-1}$.

We can state a global inequality on the macroscopic entropies.

Lemma 5.1. *For convex and Lipschitz continuous functions S, the scheme Q^1 satisfies*

$$\partial_t S(u) + \partial_x \eta(u) \leq \partial_t G(t,x) + \partial_x [H_0(t,x) + \sum_{n=1}^{\infty} \delta(t-t_n) H_n(x)], \tag{5.2}$$

where the error terms G, H_n are estimated for some measures α_G, α_{H_n} by

$$|G| \leq |S'|_\infty \alpha_G, \quad |H_n| \leq |S'|_\infty \alpha_{H_n},$$

$$|\alpha_G(t,\cdot)|_{L^1(\mathbb{R})} \leq 2\, a_\infty TV(u^\circ)\Delta t, \tag{5.3}$$

$$|\alpha_{H_0}(t,\cdot)|_{L^1(\mathbb{R})} \leq 2\, (a_\infty)^2 TV(u^\circ)\Delta t, \tag{5.4}$$

$$|\alpha_{H_n}|_{L^1(\mathbb{R})} \leq \frac{3}{4} h^2 TV(u^\circ), \quad n \geq 1. \tag{5.5}$$

Remark 5.2. As we will see, the projection P^1 only enters in the estimate of the term H_n. Moreover, it only uses two properties of P^1: in-cell entropy dissipation and error estimate from Proposition 3.4 (vi). It is very clear that they are also true for P^{1*} and the variant used in the nonsonic case. Therefore, our proof holds also in the case of Theorem 2.4. As we will see, these properties are also true for the projection on piecewise constant functions, and thus we recover also, in a very particular case, the rate of convergence for the Engquist-Osher scheme.

Proof of Lemma 5.1. Taking into account the discontinuities on f, recalled above, we may write

$$\begin{aligned}\partial_t f + a(\xi)\partial_x f = \sum_{n=1}^{\infty} \delta(t-t_n) \Big(& \big(\chi(u^{n,-}(x),\xi) - f(t_n^-,x,\xi)\big) \\ & + \big(\chi(u^n(x),\xi) - \chi(u^{n,-}(x),\xi)\big)\Big).\end{aligned} \tag{5.6}$$

We mutiply this equation by $S'(\xi)$ and integrate it in ξ. This yields

$$\partial_t S(u) + \partial_x \eta(u) \leq \partial_t G(t,x) + \partial_x H_0(t,x) + \sum_{n=1}^{\infty} \delta(t-t_n) K_n(x), \tag{5.7}$$

where the error terms are defined as follows:

$$G(t,x) = \int_{\mathbb{R}} S'(\xi)\, (\chi(u(t,x),\xi) - f(t,x,\xi))\, d\xi, \tag{5.8}$$

$$H_0(t,x) = \int_{\mathbb{R}} S'(\xi)a(\xi)\left(\chi(u(t,x),\xi) - f(t,x,\xi)\right) d\xi, \tag{5.9}$$

$$K_n(x) = S(u^n(x)) - S(u^{n,-}(x)). \tag{5.10}$$

Indeed, the first jump term in (5.6) gives a nonpositive contribution, which is the only reason for the '$\leq$' in (5.7), thanks to Brenier's kinetic entropy dissipation inequality (3.9). Next, we estimate separately these three error terms on the time interval $[t_n, t_{n+1})$, using (3.7):

$$\begin{aligned}
|G(t,x)| &\leq |S'|_\infty \int_{\mathbb{R}} \left(|\chi(u(t,x),\xi) - \chi(u^n(x),\xi)| + |f(t,x,\xi) - \chi(u^n(x),\xi)|\right) d\xi \\
&\leq |S'|_\infty \left[|u(t,x) - u^n(x)| + \int_{\mathbb{R}} |f(t,x,\xi) - f(t_n,x,\xi)| d\xi\right].
\end{aligned}$$

Taking α_G as the above bracket, we obtain, thanks to Lemma 3.1 (v) and to the similar direct estimate on f,

$$|\alpha_G(t,\cdot)|_{L^1(\mathbb{R})} \leq 2\, a_\infty TV(u^\circ)\Delta t.$$

Next, we can treat H_0 in exactly the same way and obtain (5.4). Finally, for any nonnegative test function $\Phi \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned}
\int_{\mathbb{R}} \Phi(x)K_n(x)dx &= \sum_{i\in\mathbb{Z}} \int_{x=x_{i-1/2}}^{x_{i+1/2}} K_n(x)\left(\Phi(x_{i-1/2}) + \int_{x_{i-1/2}}^{x} \Phi'(y)dy\right) dx \\
&\leq \sum_{i\in\mathbb{Z}} \int_{y=x_{i-1/2}}^{x_{i+1/2}} \Phi'(y) \int_y^{x_{i+1/2}} K_n(x)dx\, dy \\
&= -\int_{-\infty}^{\infty} \Phi'(y)H_n(y)dy,
\end{aligned}$$

where, for $x_{i-1/2} \leq y < x_{i+1/2}$,

$$H_n(y) = -\int_y^{x_{i+1/2}} K_n(x)dx,$$

$$|H_n(y)| \leq |S'|_\infty \int_y^{x_{i+1/2}} |u^n(x) - u^{n,-}(x)|dx.$$

Above, we have only used that, on each cell C_i, the operator P^1 dissipates entropy thanks to Proposition 3.4 (iii) (and $\Phi(x_{i-1/2}) \geq 0$). We have now obtained all the terms of the equation (5.2), and it remains to estimate

$$\alpha_{H_n}(y) := \int_y^{x_{i+1/2}} |u^n(x) - u^{n,-}(x)|dx \quad \text{for } x_{i-1/2} \leq y < x_{i+1/2},$$

$$\begin{aligned}
\int_{\mathbb{R}} \alpha_{H_n}(y)dy &\leq h \int_{\mathbb{R}} |u^n(x) - u^{n,-}(x)|dx \\
&\leq \frac{3}{4} h^2 TV(u^\circ).
\end{aligned} \tag{5.11}$$

The last inequality is just the error estimate on the projection P^1, Proposition 3.4 (vi). This gives the last result (5.5), and Lemma 5.1 is proved. □

5.2. **A general convergence rate estimate.** We now consider an approximate solution u of the scalar conservation law (1.1), in the sense that it satisfies the approximate entropy inequalities (5.2), and we deduce an error estimate which is nearly that on Q^1.

More generally, suppose we are given $v \in C(\mathbb{R}_+; L^1_{\mathrm{loc}}(\mathbb{R}))$, the exact entropy solution to (1.1) with initial data $v(t=0,\cdot) = u^\circ \in BV(\mathbb{R})$, and let $u \in L^\infty(\mathbb{R}_+; L^1_{\mathrm{loc}}(\mathbb{R}))$ satisfy, in the distribution sense and for the same initial data, the entropy inequalities

$$\left\{ \begin{array}{l} \partial_t S(u) + \partial_x \eta(u) \le \partial_t G(t,x) + \partial_x H(t,x), \\ S(u)(t=0,\cdot) = S(u^\circ), \ G(t=0,x) = 0, \end{array} \right. \tag{5.12}$$

for Lipschitz continuous convex functions S. We assume that the distributions G and H satisfy

$$|G| \le |S'|_\infty \alpha_G(t,x), \quad |H| \le |S'|_\infty \alpha_H(t,x),$$

for some locally bounded measures α_G, α_H.

Theorem 5.3. *With the above notations, and for any $\delta, \Delta > 0, T \ge \delta$, we have for some absolute constant C*

$$\begin{aligned} &\frac{1}{\delta}\int_0^\delta \int_{\mathbb{R}} |u(T+s,x) - v(T+s,x)|dxds \\ &\le \; CTV(u^\circ)(\Delta \, + \, a_\infty \delta) + C\int_0^{T+2\delta} \int_{\mathbb{R}} (\frac{\alpha_G}{\delta} + \frac{\alpha_H}{\Delta})dxds. \end{aligned} \tag{5.13}$$

Proof. We follow the classical approach of Kružkov [14] and just insist on the new point: the treatment of G and H. We choose a test function of the form

$$\Phi(s,t,x,y) = \varphi_1(s+t)\varphi_2(x+y)\zeta_1(s-t)\zeta_2(x-y),$$

where φ_1, φ_2, ζ_1, ζ_2 are smooth nonnegative functions with compact support. We assume, moreover, that

$$\varphi_1 \le 1, \quad \varphi_1(t) = 1 \quad \text{for } 0 \le t \le 2T, \quad \varphi_1(t) = 0 \quad \text{for } t \ge 2T + 4\delta, \quad \varphi_2 \le 1,$$

and $\zeta_1(\sigma) = \frac{1}{\delta}\underline{\zeta}_1(\frac{\sigma}{\delta})$, $\zeta_2(z) = \frac{1}{\Delta}\underline{\zeta}_2(\frac{z}{\Delta})$, with

$$\underline{\zeta}_1 \le 1, \quad \underline{\zeta}_1(\sigma) = 0 \quad \text{for } \sigma \ge 0 \text{ or } \sigma \le -6, \quad \underline{\zeta}_1(\sigma) = 1 \quad \text{for } -4 \le \sigma \le -2,$$

$$\underline{\zeta}_2 \le 1, \quad \underline{\zeta}_2(z) = 0 \quad \text{for } |z| \le 1, \quad \underline{\zeta}_2(z) = 1 \quad \text{for } |z| \ge 2, \quad \int \underline{\zeta}_2(z)dz = 3.$$

We classically introduce the entropies $S(s,t,x,y) = |u(s,x) - v(t,y)|$ and the entropy fluxes $\eta(s,t,x,y) = \mathrm{sign}(u(s,x) - v(t,y))\,(A(u(s,x)) - A(v(t,y)))$. Then, the equation (5.12) in the distributional sense gives

$$\begin{aligned} &-\int_{s=0}^\infty \int_{x\in\mathbb{R}} [S(s,t,x,y)\Phi_s + \eta(s,t,x,y)\Phi_x]dsdx \\ &\qquad \le \int_{x\in\mathbb{R}} |u^\circ(x) - v(t,y)|\Phi(0,t,x,y)dx \\ &\qquad\quad - \int_{s=0}^\infty \int_{x\in\mathbb{R}} [G(s,x)\Phi_s + H(s,x)\Phi_x]dsdx. \end{aligned}$$

Also, since our assumptions imply that $\Phi = 0$ for $t \le 0$, we have

$$-\int_{t=0}^{\infty}\int_{y\in\mathbb{R}}[S(s,t,x,y)\Phi_t + \eta(s,t,x,y)\Phi_y]dtdy = 0.$$

We integrate both equalities in the extra variables t, y and s, x, and we add up the results. This gives

$$-2\int_{s,t=0}^{\infty}\int_{x,y\in\mathbb{R}}[S(s,t,x,y)\varphi_1'(t+s)\varphi_2(x+y)\zeta_1(s-t)\zeta_2(x-y)$$
$$+\eta(s,t,x,y)\varphi_1(t+s)\varphi_2'(x+y)\zeta_1(s-t)\zeta_2(x-y)]\,dxdydsdt$$

$$= \int_{t=0}^{\infty}\int_{x,y\in\mathbb{R}}|u^\circ(x) - v(t,y)|\varphi_1(t)\varphi_2(x+y)\zeta_1(-t)\zeta_2(x-y)dxdydt + R_1$$

$$\le \frac{1}{\delta\Delta}\int_{t=0}^{6\delta}\int_{|x-y|\le\Delta}|u^\circ(x) - v(t,y)|dtdydx + R_1, \tag{5.14}$$

$$R_1 = -\int_{s,t=0}^{\infty}\int_{x,y\in\mathbb{R}}[G(s,x)\Phi_s + H(s,x)\Phi_x]dsdtdxdy. \tag{5.15}$$

Before estimating R_1, let us notice that it is possible to choose a sequence φ_2 which converges to 1 with a derivative which converges uniformly to 0. And we may also choose a sequence φ_1 which converges to the indicator function of $[0, 2T + 4\delta]$, and φ_1' converges to a Dirac mass of weight -1 at the point $\tau = 2T + 4\delta$. We may pass to the limit in the above formula, which gives exactly the first line of (5.13) after some very standard calculations. It remains to estimate R_1:

$$R_1 \le \int_{s,t=0}^{\infty}\int_{x,y\in\mathbb{R}}[\alpha_G(s,x)(|\varphi_1'(t+s)|\zeta_1(s-t) + \varphi_1|\zeta_1'|)\varphi_2(x+y)\zeta_2(x-y)$$
$$+\alpha_H(s,x)(|\varphi_2'(x+y)\zeta_2(x-y) + \varphi_2|\zeta_2'|)\varphi_1(t+s)\zeta_2(s-t)]\,dxdydsdt.$$

Here, using the limits on φ_1, φ_2, we obtain for some constant C

$$R_1 \le 6\int\int_{x\in\mathbb{R}}\alpha_G(T + 2\delta + \frac{\sigma}{2}, x)\,\zeta_1(\sigma)d\sigma dx$$
$$+\frac{C}{\delta}\int_0^{T+2\delta}\int_{x\in\mathbb{R}}\alpha_G(s,x)\,dxds + \frac{C}{\Delta}\int_0^{T+2\delta}\int_{x\in\mathbb{R}}\alpha_H(s,x)\,dxds.$$

This completes the proof of Theorem 5.3. □

5.3. **Proof of the convergence rate.** We are now ready to conclude the proof of the convergence rate in Theorem 3.6 (vi). We can bound the right-hand side of (5.13), using Lemma 5.1 and the fact that v is TVD. We find (using T in place of $T + \delta$ and choosing $T \ge \delta$) that the left-hand side of (5.13) is bounded by

$$C\,TV(u^\circ)\left[(\Delta + a_\infty\delta) + Ta_\infty\frac{\Delta t}{\delta} + T(a_\infty)^2\frac{\Delta t}{\Delta} + \frac{T}{\Delta t}\frac{h^2}{\Delta}\right]. \tag{5.16}$$

The optimal choice of the free parameters δ, Δ reduces (5.16) to

$$C\,TV(u^\circ)\left(a_\infty(T\Delta t)^{1/2} + h(\frac{T}{\Delta t})^{1/2}\right), \tag{5.17}$$

which gives exactly the result (3.23). It remains to notice that this bound for the left-hand side of (5.13) causes a difficulty because u does not belong to $C(\mathbb{R}_+; L^1_{\text{loc}}(\mathbb{R}))$. But the same terms are involved, writing

$$\begin{aligned}
&\int_{\mathbb{R}} |u(T,x) - v(T,x)|dx \; - \; \frac{1}{\delta}\int_0^\delta \int_{\mathbb{R}} |u(T+s,x) - v(T+s,x)|dxds \\
&\quad \le \frac{1}{\delta}\int_0^\delta \int_{\mathbb{R}} |v(T+s,x) - v(T,x)|dxds \\
&\quad\quad + \frac{1}{\delta}\int_0^\delta \int_{\mathbb{R}} |u(T+s,x) - u(T,x)|dxds \\
&\quad \le a_\infty TV(u^\circ)\delta + \frac{1}{\delta}\int_{s=0}^\delta \int_{t=0}^s \int_{\mathbb{R}} |u_t(T+t,x)|dx\, dt\, ds \\
&\quad \le 2a_\infty TV(u^\circ)\delta + \sum_{n=p}^{p+n_*} |u^n - u^{n,-}|_{L^1(\mathbb{R})} \\
&\quad \le 2a_\infty TV(u^\circ)\delta + C\,TV(u^\circ)h\frac{\delta}{\Delta t},
\end{aligned}$$

where $n_*\Delta t = \delta$ and $p\Delta t = T$. The last inequality is obtained by using again the estimate of Proposition 3.4 (vi). This proves the estimate (3.23), and the proof of Theorem 3.6 is complete.

Remark 5.4. In the linear case, we can only improve the transport step which is exact. This means that $G = H_0 = 0$ in Lemma 5.1, in which case the estimate (5.16) becomes

$$\int_{\mathbb{R}} |u(T,x) - v(T,x)| \le C\,TV(u^\circ)\;[(\Delta + a_\infty\delta) + \frac{T}{\Delta t}\frac{h^2}{\Delta} + \; h\;\frac{\delta}{\Delta t}];$$

the last term comes from (5.3). We let δ tend to zero and, optimizing the choice of Δ, we obtain (3.24).

6. Appendix. Convergence rate for the Engquist-Osher scheme

This appendix is devoted to another application of the error estimates developed in §5. In a very simple case, when we consider the projection P^0 on piecewise constant functions instead of P^1, it generalizes the known rates of convergence for the first-order Engqüist-Osher scheme. The general scheme can be written, with our previous notations,

$$Q^0(\Delta t) \;=\; P^0 \cdot T(\Delta t),$$

which is the Engquist-Osher scheme under the CFL condition (2.3). Our theory allows us to give convergence rates without any restriction on the time step. We do not need any inverse CFL condition $\Delta t \ge \alpha h$. Nor do we need the CFL condition; we must use a multipoint extension to identify the numerical fluxes in formula (3.25).

Theorem 6.1. *The first-order scheme Q^0 satisfies the error estimate*

$$|v(t_n,\cdot) - u^n(\cdot)|_{L^1(\mathbb{R})} \le C\,TV(u_h^\circ)\sqrt{t_n}(\sqrt{a_\infty h} + a_\infty\sqrt{\Delta t}) + |v^\circ - u_h^\circ|_{L^1(\mathbb{R})}.$$

This holds in particular for the Engquist-Osher scheme under the CFL condition (2.3).

Proof. We use the proof given in §5, but we change the estimate (5.11) to

$$\int_{\mathbb{R}} \alpha_{H_n}(y)dy \le C\,TV(u^\circ)a_\infty h\ \Delta t.$$

Indeed, instead of the estimate (vi) in Proposition 3.4, we can use

$$|u^{n+1} - u^{n+1,-}|_{L^1(\mathbb{R})} \le TV(u^\circ)a_\infty \Delta t,$$

which is obtained as follows, see Lemma 3.1 (v):

$$\begin{aligned}|u^{n+1} - u^{n+1,-}|_{L^1(\mathbb{R})} &\le |u^{n+1} - u^n|_{L^1(\mathbb{R})} + |u^{n+1,-} - u^n|_{L^1(\mathbb{R})} \\ &\le |u^{n+1} - u^n|_{L^1(\mathbb{R})} + C\,TV(u^\circ)a_\infty \Delta t.\end{aligned}$$

It remains to compute, using the fact that P^0 diminishes in-cell entropies,

$$\begin{aligned}\int_{C_i} |u^{n+1}(x) - u^n(x)|dx &= \int_{C_i} |P^0(u^{n+1,-})(x) - u_i^n|dx \\ &\le \int_{C_i} |u^{n+1,-}(x) - u_i^n|dx,\end{aligned}$$

hence

$$|u^{n+1} - u^n|_{L^1(\mathbb{R})} \le C\,TV(u^\circ)a_\infty \Delta t. \quad \square$$

Remark 6.2. Such a result also holds for any variant $\underline{Q}^1$ of Q^1 which is time Lipschitz continuous, i.e.,

$$|\underline{Q}^1(\Delta t)u^n - u^n|_{L^1(\mathbb{R})} \le C\,TV(u^n)a_\infty \Delta t.$$

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