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MINIMAL CUBATURE FORMULAE OF TRIGONOMETRIC DEGREE

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ABSTRACT. In this paper we construct minimal cubature formulae of trigonometric degree: we obtain explicit formulae for low dimensions of arbitrary degree and for low degrees in all dimensions. A useful tool is a closed form expression for the reproducing kernels in two dimensions.

1. INTRODUCTION

This paper is concerned with the identification and discovery of cubature formulae of trigonometric degree with the lowest possible number of points for the evaluation of integrals of the form

$$I[f] = \int_0^1 \cdots \int_0^1 f(x_1, \dots, x_n) dx_1 \cdots dx_n = \int_{[0,1]^n} f(\mathbf{x}) d\mathbf{x}. \tag{1}$$

A trigonometric monomial in the variable $\mathbf{x} = (x_1, x_2, \dots, x_n)$ is a function of the form

$$f : \mathbb{R}^n \to \mathbb{C} : (x_1, x_2, \dots, x_n) \mapsto e^{2\pi i \alpha_1 x_1} e^{2\pi i \alpha_2 x_2} \dots e^{2\pi i \alpha_n x_n}, \tag{2}$$

where $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{Z}$ and $i^2 = -1$. The degree of this monomial is $\sum_{l=1}^n |\alpha_l|$. The set of all finite linear combinations of trigonometric monomials is the space of trigonometric polynomials, denoted by $\mathbb{T}^n$. The degree of a trigonometric polynomial is the maximum of the degrees of the monomials used in the linear combination. The subspace of all trigonometric polynomials of degree at most d is denoted by $\mathbb{T}^n_d$.

A cubature formula

$$Q[f] = \sum_{j=1}^N w_j f(\mathbf{x}^{(j)}) \ , \ \ w_j \in \mathbb{R} \ , \ \ \mathbf{x}^{(j)} \in [0,1)^n, \tag{3}$$

is of trigonometric degree d if $Q[f] = I[f] \ \forall f \in \mathbb{T}^n_d$, and if there exists a polynomial of degree $d+1$ for which $Q[f] \neq I[f]$. We are interested in cubature formulae of trigonometric degree d for which the number of points N is as small as possible.

As in the study of cubature formulae of algebraic degree, a useful tool is the *reproducing kernel* (see §2). The important feature of the reproducing kernel is that it allows minimal cubature rules to be definitively characterized. The concept of "reproducing kernel" was first used for the construction of cubature formulae of

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algebraic degree by Mysovskikh in 1968 [5]. It has been applied to derive properties of cubature formulae and to construct low-degree formulae (see, e.g., [4, 6]). It was again Mysovskikh who introduced it into the field of constructing cubature formulae of trigonometric degree, in 1985 for the one-dimensional case [7] and in 1990 for the multivariate case [10]. The use of the reproducing kernel has been limited because no easy expression for it was available, and perhaps also because for the algebraic case the lower bound to which it is related is seldom attained.

A key result in the present paper is a closed-form expression for the 2-dimensional reproducing kernel. With the aid of this expression we are able to find new families of 2-dimensional minimal formulae of odd degree. Interestingly, for $d \geq 3$ the cubature formulae contain a number of real-valued free parameters (see §6). A consequence is that we are able to construct 2-dimensional minimal cubature formulae which are neither lattice rules nor translations of lattice rules. For a discussion of lattice rules see [18, 19].

We conclude this section by collecting some known results concerning the dimension of the space $\mathbb{T}_d^n$ and related matters. Let $\tau(n, d)$ denote the number of monomials (2) in n variables of degree d. Using combinatorics, one can show (see [8]) that

$$\tau(n, d) = \sum_{l=1}^{n} \binom{n}{l} \binom{d-1}{l-1} 2^l,$$

with

$$\tau(n, 0) = 1,$$

where we have used the convention

$$\binom{a}{b} = 0 \text{ if } b > a.$$

A very useful expression is

$$d\tau(n, d) = n\tau(d, n).$$

Example 1.1.

$$\begin{aligned}
\tau(1, d) &= 2, & \tau(n, 1) &= 2n, \\
\tau(2, d) &= 4d, & \tau(n, 2) &= 2n^2, \\
\tau(3, d) &= 4d^2 + 2, & \tau(n, 3) &= \tfrac{2}{3}n(2n^2 + 1), \\
\tau(4, d) &= \tfrac{8}{3}d(d^2 + 2), & \tau(n, 4) &= \tfrac{2}{3}n^2(n^2 + 2). \quad \diamond
\end{aligned}$$

Let $t(n, d)$ denote the number of monomials (2) in n variables of degree at most d. Then (see [8])

$$t(n, d) = \dim \mathbb{T}_d^n = \sum_{l=0}^{d} \tau(n, l) = \sum_{l=0}^{n} \binom{n}{l} \binom{d}{l} 2^l.$$

Note that $t(n, d) = t(d, n)$.

Example 1.2.

$$\begin{aligned}
t(n, 1) &= 2n + 1, \\
t(n, 2) &= 2n^2 + 2n + 1, \\
t(n, 3) &= \tfrac{1}{3}(2n + 1)(2n^2 + 2n + 3), \\
t(n, 4) &= \tfrac{1}{3}(2n^4 + 4n^3 + 10n^2 + 8n + 3). \quad \diamond
\end{aligned}$$

Lemma 1.1. *The number of monomials* (2) *of degree at most d with the same parity as d is* $\frac{\tau(d+1,n)}{2}$.

Proof. A combinatorial proof is given in [9]. A proof by induction is given in [1]. □

2. Theory

In this section we cover the present theory. Much of this theory is well known to Russian authors. A side effect of our effort to make this paper self-contained is that these results now become more accessible. We have provided some shorter proofs in places and included some links to the better-known algebraic case. This theory is needed as background to our new result in §4.

Let

$$\Lambda_d = \{\mathbf{k} \in \mathbb{Z}^n : 0 \leq \sum_{l=1}^{n} |k_l| \leq \left\lfloor \frac{d}{2} \right\rfloor\},$$

so that $|\Lambda_d| = \dim \mathbb{T}^n_{\lfloor \frac{d}{2} \rfloor} = t(n, \lfloor \frac{d}{2} \rfloor)$. An important role is played by the $t(n, \lfloor \frac{d}{2} \rfloor) \times N$ matrix M with elements

$$M_{\mathbf{k}j} = \sqrt{w_j} e^{2\pi i \mathbf{k}\cdot\mathbf{x}^{(j)}}, \quad \mathbf{k} \in \Lambda_d, \quad 1 \leq j \leq N. \tag{4}$$

Because the product of two trigonometric polynomials from $\mathbb{T}^n_{\lfloor \frac{d}{2} \rfloor}$ is a polynomial of degree $\leq d$, demanding that the cubature formula Q has trigonometric degree d implies

$$\forall\, \mathbf{k}, \mathbf{k}' \in \Lambda_d \;:\; Q[e^{2\pi i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}] = \sum_{j=1}^{N} w_j e^{2\pi i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}^{(j)}} = \sum_{j=1}^{N} M_{\mathbf{k}j}\overline{M}_{\mathbf{k}'j} = \delta_{\mathbf{k}\mathbf{k}'}, \tag{5}$$

which can be written as

$$MM^{\star} = I, \tag{6}$$

where $M^{\star}$ denotes the Hermitian conjugate of M and I is the identity matrix.

Theorem 2.1. *A cubature formula* (3) *for the integral* (1) *which is of trigonometric degree d has its number of points N bounded below by*

$$N \geq \dim \mathbb{T}^n_{\lfloor \frac{d}{2} \rfloor} = t(n, \left\lfloor \frac{d}{2} \right\rfloor).$$

Proof. Since the rows of the $t(n, \lfloor \frac{d}{2} \rfloor) \times N$ matrix M are orthogonal, it follows immediately that $N \geq t(n, \lfloor \frac{d}{2} \rfloor)$. □

This almost evident result was given by Mysovskikh [8, 9] who also derived the corresponding result for the case where the integral (1) contains a general, nonnegative weight function. The resemblance with the corresponding result for cubature formulae of algebraic degree, whose history can be traced back to [14], is striking.

The following theorem appears as a corollary in [1].

Theorem 2.2. *If the cubature formula* (3) *for the integral* (1) *is of trigonometric degree d and $N = t(n, \lfloor \frac{d}{2} \rfloor)$, then the weights are given by $w_j = \frac{1}{N}$, $j = 1, \ldots, N$.*

Proof. The hypothesis makes the matrix M in (4) square, and (6) makes it orthogonal. Therefore its columns are orthogonal:

$$\sum_{\mathbf{k}\in\Lambda_d} \overline{M}_{\mathbf{k}j'} M_{\mathbf{k}j} = \delta_{jj'} \text{ for } 1 \le j, j' \le N,$$

or

$$\sqrt{w_j}\sqrt{w_{j'}} \sum_{\mathbf{k}\in\Lambda_d} e^{2\pi i \mathbf{k}\cdot(\mathbf{x}^{(j)}-\mathbf{x}^{(j')})} = \delta_{jj'}. \tag{7}$$

Let $j = j'$. Then (7) becomes

$$w_j \sum_{\mathbf{k}\in\Lambda_d} 1 = w_j t(n, \left\lfloor \frac{d}{2} \right\rfloor) = w_j N = 1,$$

thus $w_j = \frac{1}{N}$ for $j = 1, 2, \dots, N$. □

Let

$$K(\mathbf{x}, \mathbf{x}') = \sum_{\mathbf{k}\in\Lambda_d} e^{2\pi i \mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}. \tag{8}$$

Then $K(\mathbf{x}, \mathbf{x}')$ is a reproducing kernel in the space $\mathbb{T}^n_{\lfloor \frac{d}{2} \rfloor}$: for if $f \in \mathbb{T}^n_{\lfloor \frac{d}{2} \rfloor}$, then f coincides with its Fourier series, so that

$$\begin{aligned} f(\mathbf{a}) &= \sum_{\mathbf{k}\in\Lambda_d} I[f(\mathbf{x})e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}]e^{2\pi i \mathbf{k}\cdot\mathbf{a}} \\ &= I[f(\mathbf{x})K(\mathbf{a}, \mathbf{x})]. \end{aligned}$$

The reproducing kernel $K(\mathbf{x}, \mathbf{x}')$ plays an important role if d is even.

Theorem 2.3. *If d is even, a necessary and sufficient condition for points $\mathbf{x}^{(j)}$, $1 \le j \le t(n, \frac{d}{2})$, lying in $[0, 1]^n$ to be the points of an equal-weight cubature formula of trigonometric degree d is*

$$K(\mathbf{x}^{(j)}, \mathbf{x}^{(j')}) = N\delta_{jj'}\ ,\ 1 \le j, j' \le N, \tag{9}$$

where $N = t(n, \frac{d}{2})$.

Proof. Assume that d is even and let $w_j = \frac{1}{N}$ for $j = 1, 2, \dots, N$. Then (9) is equivalent to

$$M^\star M = I.$$

Because M is here a square matrix, this is equivalent to (5). Because d is even, (5) is equivalent to the statement that Q is of degree d, since for d even every trigonometric monomial of degree $\le d$ can be written in the form $e^{2\pi i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}$ with $\mathbf{k}, \mathbf{k}' \in \Lambda_d$. □

Example 2.1. The 1-dimensional case.

In this case

$$\Lambda_d = \{k \in \mathbb{Z} : -\left\lfloor \frac{d}{2} \right\rfloor \le k \le \left\lfloor \frac{d}{2} \right\rfloor\} \tag{10}$$

and

$$|\Lambda_d| = t(1, \left\lfloor \frac{d}{2} \right\rfloor) = 2\left\lfloor \frac{d}{2} \right\rfloor + 1\ .$$

The reproducing kernel (8) is given by

$$K(x,x') = \sum_{k=-\lfloor \frac{d}{2} \rfloor}^{\lfloor \frac{d}{2} \rfloor} e^{2\pi i k(x-x')} = \frac{\sin \pi(2\lfloor \frac{d}{2} \rfloor + 1)x - x')}{\sin \pi(x-x')} , \; x - x' \notin \mathbb{Z}.$$

It has zeros at

$$x - x' = \frac{j}{2\lfloor \frac{d}{2} \rfloor + 1} , \; j \in \mathbb{Z}, \tag{11}$$

provided $x - x' \notin \mathbb{Z}$. The equal-weight quadrature formula with $N = 2\lfloor \frac{d}{2} \rfloor + 1$ points

$$\frac{\varepsilon}{2\lfloor \frac{d}{2} \rfloor + 1}, \frac{1+\varepsilon}{2\lfloor \frac{d}{2} \rfloor + 1}, \ldots, \frac{2\lfloor \frac{d}{2} \rfloor + \varepsilon}{2\lfloor \frac{d}{2} \rfloor + 1}$$

with $\varepsilon \in [0, \frac{1}{2\lfloor \frac{d}{2} \rfloor + 1}]$ has differences between its points satisfying (11). This quadrature formula, which is the (shifted) rectangle rule, is of trigonometric degree $2\lfloor \frac{d}{2} \rfloor$. If d is even, the rule is of degree d, in conformity with Theorem 2.3. If d is odd, note that the rule is *not* of degree d, because the monomial $e^{2\pi i d x} = e^{2\pi i (2\lfloor \frac{d}{2} \rfloor + 1)x}$ is not integrated exactly.

Note that this result was—with the use of the reproducing kernel—obtained by Mysovskikh [7], who has also derived the corresponding result for a general nonnegative weight function. In [7] it is proven that in the case of a constant weight function there is no quadrature formula of highest trigonometric degree other than the rectangle rule. $\diamond$

Given $\mathbf{a} \in \mathbb{R}^n$, let $\{\mathbf{a}\} \in [0,1)^n$ denote the vector each of whose components is the fractional part of the corresponding component of $\mathbf{a}$. A cubature formula maintains its trigonometric degree if all points are shifted by a constant vector $\mathbf{c}$, with points that are moved out of the region of integration replaced by the related point inside $[0,1)^n$, i.e., if each point $\mathbf{x}^{(j)}$ is replaced by $\{\mathbf{x}^{(j)} + \mathbf{c}\}$. We shall generally take advantage of this degree of freedom by choosing $\mathbf{0}$ to be one of the points of our cubature formulae.

Definition 2.1. A cubature formula (3) for the integral (1) is shift symmetric if whenever $(x_1^{(j)}, \ldots, x_n^{(j)})$ is a point of the formula so is $\{(x_1^{(j)} + \frac{1}{2}, \ldots, x_n^{(j)} + \frac{1}{2})\}$, with both points having the same weight.

Thus, N is even for a shift symmetric cubature formula, and in a formula of this kind the points can be relabelled so that the formula can be written as

$$Q[f] = \sum_{j=1}^{\frac{N}{2}} w_j (f(\mathbf{x}^{(j)}) + f(\{\mathbf{x}^{(j)} + (\frac{1}{2}, \ldots, \frac{1}{2})\})). \tag{12}$$

In a similar way we may speak of a set of points in $[0,1)^n$ as being shift symmetric. In any such set we shall assume that the points are labelled as in (12), so that among the first $\frac{N}{2}$ entries no pair differs by $(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2})$. This symmetry has similar consequences as the more familiar central symmetry for cubature formulae of algebraic degree [1].

Lemma 2.1. *Every monomial of odd degree is integrated exactly by a shift symmetric cubature formula.*

Proof. Using the shift symmetric cubature formula (12) we obtain

$$
\begin{aligned}
Q[e^{2\pi i\mathbf{k}\cdot\mathbf{x}}] &= \sum_{j=1}^{\frac{N}{2}} w_j\left(e^{2\pi i\mathbf{k}\cdot\mathbf{x}^{(j)}} + e^{2\pi i\mathbf{k}\cdot(\mathbf{x}^{(j)}+(\frac{1}{2},\dots,\frac{1}{2}))}\right) \\
&= \sum_{j=1}^{\frac{N}{2}} w_j e^{2\pi i\mathbf{k}\cdot\mathbf{x}^{(j)}}\left(1+e^{\pi i(k_1+\cdots+k_n)}\right) \\
&= \sum_{j=1}^{\frac{N}{2}} w_j e^{2\pi i\mathbf{k}\cdot\mathbf{x}^{(j)}}\left(1+(-1)^{k_1+\cdots+k_n}\right).
\end{aligned}
$$

Thus

$$Q[e^{2\pi i\mathbf{k}\cdot\mathbf{x}}] = 0 \text{ if } \sum_{j=1}^{n}|k_j| \text{ is odd.} \quad \square$$

Let

(13)
$$\widetilde{\Lambda}_m := \{\mathbf{k}\in\mathbb{Z}^n : 0 \le \sum_{j=1}^{n}|k_j| \le m\ ,\ \sum_{j=1}^{n}|k_j| \text{ of the same parity as } m\},$$

so that, according to Lemma 1.1, $|\widetilde{\Lambda}_m| = \frac{\tau(m+1,n)}{2}$.

In the search for shift symmetric cubature formulae, an important role is played by the $\frac{\tau(m+1,n)}{2} \times \frac{N}{2}$ matrix L with elements

$$L_{\mathbf{k}j} = \sqrt{2w_j}e^{2\pi i\mathbf{k}\cdot\mathbf{x}^{(j)}}\ ,\ \mathbf{k}\in\widetilde{\Lambda}_m\ ,\ 1\le j\le\frac{N}{2}. \tag{14}$$

The product of two trigonometric monomials whose degrees have the same parity is of even degree. Moreover, every even-degree monomial of degree $\le 2m$ can be written in the form $e^{2\pi i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}$ with $\mathbf{k},\mathbf{k}'\in\widetilde{\Lambda}_m$. Remembering that a shift symmetric cubature formula is exact for all odd-degree monomials, we see that a shift symmetric cubature formula has trigonometric degree $2m+1$ if and only if for all $\mathbf{k},\mathbf{k}'\in\widetilde{\Lambda}_m$

$$
\begin{aligned}
Q[e^{2\pi i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}] &= \sum_{j=1}^{N/2} w_j\left(e^{2\pi i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}^{(j)}} + e^{2\pi i(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{x}^{(j)}+(\frac{1}{2},\dots,\frac{1}{2}))}\right) \\
&= \sum_{j=1}^{N/2} 2w_j e^{2\pi i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}^{(j)}} \\
&= \sum_{j=1}^{N/2} L_{\mathbf{k}j}\overline{L}_{\mathbf{k}'j} \\
&= \delta_{\mathbf{k}\mathbf{k}'},
\end{aligned}
$$

which can be written as

$$LL^* = I. \tag{15}$$

The following theorem also holds without the assumption that the cubature formula is shift symmetric. A proof of this is presented by Mysovskikh [8]. This result is also given by Noskov [11] who simply refers to an analogous result for the algebraic-degree case proven by Möller [4]. We prefer to present it here to keep the paper self-contained and because with the shift symmetric assumption the proof becomes very easy.

Theorem 2.4. *A shift symmetric cubature formula* (12) *for the integral* (1) *which is of trigonometric degree* $2m+1, m \in \mathbb{N}$, *has its number of points* N *bounded below by*

$$N \geq \tau(m+1,n) = 2|\widetilde{\Lambda}_m|.$$

Proof. Since the rows of the $\frac{\tau(m+1,n)}{2} \times \frac{N}{2}$ matrix L given by (14) are orthogonal, it follows immediately that

$$\frac{N}{2} \geq \frac{\tau(m+1,n)}{2} = |\widetilde{\Lambda}_m|. \quad \square$$

The following theorem appears as a corollary in [1].

Theorem 2.5. *If* $N = \tau(m+1,n)$ *in a shift symmetric cubature formula* (12) *of degree* $2m+1$ *for the integral* (1), *then* $w_j = \frac{1}{N}, j = 1, 2, \dots, N$.

Proof. Analogous to the proof of Theorem 2.2. $\square$

Let

$$\widetilde{K}(\mathbf{x}, \mathbf{x}') = \sum_{\mathbf{k} \in \widetilde{\Lambda}_m} e^{2\pi i \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')}.$$

Then $\widetilde{K}$ is a reproducing kernel in the space spanned by the monomials $e^{2\pi i \mathbf{k} \cdot \mathbf{x}}$, $\mathbf{k} \in \widetilde{\Lambda}_m$.

Theorem 2.6. *A necessary and sufficient condition for a shift symmetric set of points* $\mathbf{x}^{(j)}, 1 \leq j \leq \tau(m+1,n)$, *to be the points of an equal-weight cubature formula of trigonometric degree* $2m+1$ *is*

$$\widetilde{K}(\mathbf{x}^{(j)}, \mathbf{x}^{(j')}) = \frac{N}{2}\delta_{jj'}\ , \quad 1 \leq j, j' \leq \frac{N}{2},$$

where $N = \tau(m+1,n)$.

Proof. Let $w_j = \frac{1}{N}$ for $j = 1, 2, \dots, N$. Then the condition in this theorem is equivalent to

$$L^\star L = I,$$

where L is given by (14). Because L is a square matrix, this is equivalent to

$$LL^\star = I,$$

which is the necessary and sufficient condition (15) for a shift symmetric cubature formula to be of trigonometric degree $d = 2m+1$. $\square$

Example 2.2. The 1-dimensional case with d odd.

Putting $d = 2m+1$, we have

$$\widetilde{\Lambda}_m = \{-m, -m+2, \dots, m-2, m\}\ , \ |\widetilde{\Lambda}_m| = m+1,$$

and

$$\widetilde{K}(x, x') = \sum_{\kappa \in \widetilde{\Lambda}_m} e^{2\pi i \kappa (x - x')} = \frac{\sin 2\pi(m+1)(x-x')}{\sin 2\pi(x-x')}\ , \ 2(x-x') \notin \mathbb{Z}.$$

TABLE 1. Minimal number of points

	dimension				
degree	1	2	3	4	5
1	2	2	2	2	2
2	3	5	7	9	11
3	4	8	12	16	20
4	5	13	25	41	61
5	6	18	38	66	102
6	7	25	63	129	231
7	8	32	88	192	360
8	9	41	129	321	681
9	10	50	170	450	1002
10	11	61	231	681	1683
11	12	72	292	912	2364
12	13	85	377	1289	3653

This has zeros at $x - x' = \frac{j}{2(m+1)}, j \in \mathbb{Z}$, provided $2(x - x') \notin \mathbb{Z}$, corresponding to the fact that the equal-weight quadrature formula with the $N = 2(m+1)$ points

$$0\,,\ \frac{1}{2(m+1)}\,,\ \frac{2}{2(m+1)}\,,\ \dots\,,\ \frac{1}{2}\,,\ \dots\,,\ \frac{2m+1}{2(m+1)}$$

is a shift symmetric quadrature formula of degree $2m+1 = d$. (Note that the 1-dimensional formula in Example 2.1 has an odd number of points, and so is not shift symmetric.)

The note added at the end of Example 2.1 is also relevant for this case. $\diamond$

The lower bound for the number of points in an odd-degree formula from Theorem 2.4 is higher than the lower bound of Theorem 2.1. Based on these theorems, we introduce the following definition:

Definition 2.2. A cubature formula of degree d is called minimal if its number of points N satisfies

$$\begin{aligned} N &= t(n,m) \quad \text{if} \quad d = 2m, \\ N &= \tau(m+1,n) \quad \text{if} \quad d = 2m+1. \end{aligned}$$

Table 1 gives the minimal number of points for various values of the dimension and the degree.

A simplifying aspect of the trigonometric case is that the reproducing kernel is a function of one variable:

$$K(\mathbf{x}, \mathbf{x}') = \mathcal{K}(\mathbf{x} - \mathbf{x}')$$

with

$$\mathcal{K}(\mathbf{x}) = \sum_{\mathbf{k}\in\Lambda_d} e^{2\pi i \mathbf{k}\cdot\mathbf{x}}, \tag{16}$$

and

$$\widetilde{K}(\mathbf{x},\mathbf{x}') = \widetilde{\mathcal{K}}(\mathbf{x}-\mathbf{x}')$$

with

$$\widetilde{\mathcal{K}}(\mathbf{x}) = \sum_{\mathbf{k}\in\widetilde{\Lambda}_m} e^{2\pi i\mathbf{k}\cdot\mathbf{x}}. \tag{17}$$

3. Low-degree minimal formulae in all dimensions

In this section we collect known results. However, the original proofs of the theorems do not use the reproducing kernel.

From Lemma 2.1 it follows immediately that the cubature formula

$$Q[f] = \frac{1}{2}\sum_{j=1}^{2} f\left(\frac{j}{2},\dots,\frac{j}{2}\right)$$

is a minimal formula of degree 1, because it is shift symmetric. This cubature formula is mentioned in [9, 10].

Theorem 3.1 ([13]). *The cubature formula*

$$Q[f] = \frac{1}{2n+1}\sum_{j=1}^{2n+1} f\left(\left\{\frac{j}{2n+1},\frac{2j}{2n+1},\dots,\frac{nj}{2n+1}\right\}\right)$$

is a minimal cubature formula of degree 2.

Proof. With $d=2$ the number of points in a minimal formula is $t(n,1)=2n+1$. Also, $\Lambda_2 = \{(0,\dots,0),(\pm1,0,\dots,0),\dots,(0,\dots,0,\pm1)\}$, thus the reproducing kernel (16) gives

$$\begin{aligned}\mathcal{K}(\mathbf{x}) &= 1+\sum_{l=1}^{n} e^{2\pi i x_l} + \sum_{l=1}^{n} e^{-2\pi i x_l}\\ &= 1+2\sum_{l=1}^{n}\cos(2\pi x_l).\end{aligned}$$

For the formula under investigation the lth component of $\mathbf{x}^{(j)}$ is

$$x_l^{(j)} = \left\{\frac{lj}{2n+1}\right\},\quad 1\le j\le 2n+1\,,\quad 1\le l\le n\,,$$

and so, for $j\neq j'$,

$$\mathcal{K}(\mathbf{x}^{(j)}-\mathbf{x}^{(j')}) = 1+2\sum_{l=1}^{n}\cos\frac{2\pi l(j-j')}{2n+1} = 0. \quad \square$$

Theorem 3.2 ([12]). *The cubature formula*

$$Q[f] = \frac{1}{4n}\sum_{j=1}^{4n} f\left(\left\{\frac{j}{4n},\frac{3j}{4n},\dots,\frac{(2n-1)j}{4n}\right\}\right)$$

is a shift symmetric minimal cubature formula of degree 3.

Proof. With $d=3$, the number of points in a minimal formula is $\tau(2,n)=4n$. The rule is shift symmetric because for $j=1,2,\dots,2n$ the cubature point corresponding to $j+2n$ is the cubature point corresponding to j shifted by $\left(\pm\frac{1}{2},\pm\frac{1}{2},\dots,\pm\frac{1}{2}\right)$. For the proof that the rule is of degree 3 see [12], or use Theorem 2.6. $\square$

All of the explicit minimal cubature formulae quoted above are lattice rules of rank 1 (see [18, 19]). That is to say, they can all be written as a single sum of the form

$$Q[f] = \frac{1}{N}\sum_{j=1}^{N} f\left(\left\{j\frac{\mathbf{z}}{N}\right\}\right), \tag{18}$$

with $\mathbf{z} \in \mathbb{Z}^n$. More generally, any lattice rule having N distinct points may be expressed as

$$\frac{1}{N_1 N_2 \cdots N_t}\sum_{j_1=1}^{N_1} \cdots \sum_{j_t=1}^{N_t} f\left(\left\{\sum_{m=1}^{t} \frac{j_m}{N_m}\mathbf{z}_m\right\}\right),$$

where $t \in \mathbb{Z}$, $N = N_1 N_2 \cdots N_t$ and $\mathbf{z}_m \in \mathbb{Z}^n$; its rank is the minimum possible value of t in such an expression.

4. Closed forms for the 2D reproducing kernel

In this section we obtain closed-form expressions for the reproducing kernels $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ in two dimensions. Such closed-form expressions provide a useful tool for the discovery and analysis of minimal cubature formulae.

Let A be a nonsingular linear transformation acting on $\mathbb{R}^n$. Then we can write (16) as

$$\mathcal{K}(\mathbf{x}) = \sum_{\mathbf{k}\in\Lambda_d} e^{2\pi i (A\mathbf{k})\cdot((A^T)^{-1}\mathbf{x})} = \sum_{\kappa\in A\Lambda_d} e^{2\pi i \kappa\cdot((A^T)^{-1}\mathbf{x})},$$

where the final sum is over the transformed set $A\Lambda_d$. The summation task can sometimes be simplified by choosing A so as to rotate the coordinate system. In particular, for the 2-dimensional case let A be the linear operation which rotates $\mathbf{k}$ by $\pi/4$ clockwise and reduces its Euclidean magnitude by a factor $\frac{1}{\sqrt{2}}$. Thus,

$$A = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}. \tag{19}$$

Then

$$\mathcal{K}(\mathbf{x}) = \sum_{\kappa\in A\Lambda_d} e^{2\pi i \kappa\cdot\mathbf{y}}, \tag{20}$$

with

$$\mathbf{y} = (A^T)^{-1}\mathbf{x}. \tag{21}$$

This is illustrated in Figure 1 for the case $d = 10$ or $d = 11$. The left-hand diagram shows the original summation region with respect to $\mathbf{k}$, while the right-hand diagram shows the summation region with respect to $\kappa = A\mathbf{k}$.

Let $\kappa = (\kappa_1, \kappa_2)^T$. The sum in (20) can be expressed as the aggregate of separate sums over integer- and half-integer lattices:

$$\mathcal{K}(\mathbf{x}) = F_1(\mathbf{y}) + F_2(\mathbf{y}),$$

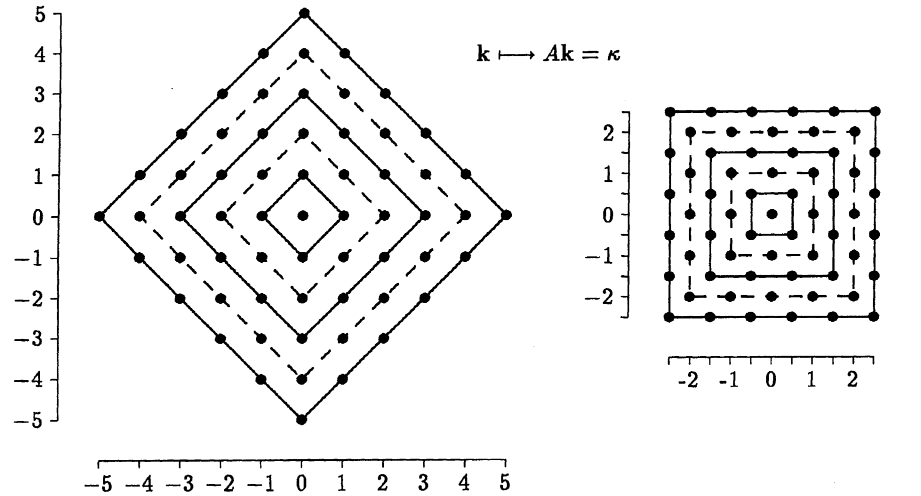

FIGURE 1. Summation lattices

where

$$F_1(\mathbf{y}) = \sum_{\kappa_1,\kappa_2=-\frac12\lfloor\frac d2\rfloor}^{\frac12\lfloor\frac d2\rfloor} e^{2\pi i \boldsymbol{\kappa}\cdot\mathbf{y}} = \sum_{\kappa_1=-\frac12\lfloor\frac d2\rfloor}^{\frac12\lfloor\frac d2\rfloor} e^{2\pi i\kappa_1 y_1} \sum_{\kappa_2=-\frac12\lfloor\frac d2\rfloor}^{\frac12\lfloor\frac d2\rfloor} e^{2\pi i\kappa_2 y_2}$$
$$= \frac{\sin(\pi(\lfloor\frac d2\rfloor+1)y_1)}{\sin(\pi y_1)}\frac{\sin(\pi(\lfloor\frac d2\rfloor+1)y_2)}{\sin(\pi y_2)},$$

$$F_2(\mathbf{y}) = \sum_{\kappa_1,\kappa_2=-\frac12\lfloor\frac d2\rfloor+\frac12}^{\frac12\lfloor\frac d2\rfloor-\frac12} e^{2\pi i\boldsymbol{\kappa}\cdot\mathbf{y}} = \frac{\sin(\pi\lfloor\frac d2\rfloor y_1)}{\sin(\pi y_1)}\frac{\sin(\pi\lfloor\frac d2\rfloor y_2)}{\sin(\pi y_2)}\,.$$

Hence,

$$\mathcal{K}(\mathbf{x}) = \frac{\begin{array}{c}\cos(\pi(\lfloor\frac d2\rfloor+1)(y_1-y_2)) - \cos(\pi(\lfloor\frac d2\rfloor+1)(y_1+y_2)) \\ + \cos(\pi\lfloor\frac d2\rfloor(y_1-y_2)) - \cos(\pi\lfloor\frac d2\rfloor(y_1+y_2))\end{array}}{2\sin(\pi y_1)\sin(\pi y_2)}$$
$$= \frac{\begin{array}{c}\cos(\pi(\lfloor\frac d2\rfloor+\frac12)(y_1-y_2))\cos(\frac{\pi}{2}(y_1-y_2)) \\ - \cos(\pi(\lfloor\frac d2\rfloor+\frac12)(y_1+y_2))\cos(\frac{\pi}{2}(y_1+y_2))\end{array}}{\sin(\pi y_1)\sin(\pi y_2)}$$
$$= \frac{g\left(\frac{y_1-y_2}{2}\right) - g\left(\frac{y_1+y_2}{2}\right)}{\sin(\pi y_1)\sin(\pi y_2)}$$

where $g(z) = \cos(\pi(2\lfloor\frac d2\rfloor+1)z)\cos(\pi z)$.

From (19) and (21) we know that

$$y_1 = x_1 + x_2 \text{ and } y_2 = -x_1 + x_2\ ,$$

so

$$\mathcal{K}(\mathbf{x}) = \frac{g(x_1) - g(x_2)}{\sin(\pi(x_2+x_1))\sin(\pi(x_2-x_1))}\,. \tag{22}$$

During the calculation of $\mathcal{K}(\mathbf{x})$ we already obtained an expression for $\widetilde{\mathcal{K}}(\mathbf{x})$. Indeed, for $d = 2m+1$

$$
\begin{aligned}
\widetilde{\mathcal{K}}(\mathbf{x}) &= \sum_{\mathbf{k}\in\widetilde{\Lambda}_m} e^{2\pi i \mathbf{k}\cdot\mathbf{x}} = F_1(\mathbf{y}) \\
&= \frac{\sin(\pi(m+1)y_1)\sin(\pi(m+1)y_2)}{\sin(\pi y_1)\sin(\pi y_2)} \\
&= \frac{\cos(\pi(m+1)(y_1-y_2)) - \cos(\pi(m+1)(y_1+y_2))}{2\sin(\pi y_1)\sin(\pi y_2)} \\
&= \frac{\cos(2\pi(m+1)x_1) - \cos(2\pi(m+1)x_2)}{2\sin(\pi(x_2+x_1))\sin(\pi(x_2-x_1))} .
\end{aligned}
\tag{23}
$$

We will also use (22) and (23) for $x_1 \pm x_2 = 0$: the limit of the right-hand side of these expressions is well defined.

Note that in the context of summability of multivariate Fourier series, the expression (16) is called a Dirichlet kernel. An expression equivalent to (22) was obtained by Herriot [3].

5. The 2D even-degree case

For $n = 2$ and even degrees $d = 2m, m \in \mathbb{N}_0 = \mathbb{N} - \{0\}$, the lower bound from Theorem 2.1 becomes

$$N \geq t(2,m) = 1 + 2m + 2m^2 = \frac{(d+1)^2+1}{2}.$$

Known minimal formulae are the Fibonacci lattices with 5 and 13 points:
$\frac{1}{5}\sum_{j=1}^{5} f\left(\left\{\frac{j}{5}, \frac{3j}{5}\right\}\right)$ has degree 2 and $N = 5$,
$\frac{1}{13}\sum_{j=1}^{13} f\left(\left\{\frac{1}{13}, \frac{8j}{13}\right\}\right)$ has degree 4 and $N = 13$.
Note that the above formula of degree 2 is geometrically equivalent to the one presented in Theorem 3.1 with n set equal to 2.

The formulae given above are the only minimal formulae of even trigonometric degree among the Fibonacci lattice rules [1]. However, they belong also to another class of formulae discovered by Noskov [13]:

Theorem 5.1. *The cubature formula*

$$\frac{1}{N}\sum_{j=1}^{N} f\left(\left\{\frac{j}{N}, \frac{\alpha j}{N}\right\}\right) \quad \textit{with } N = \frac{(d+1)^2+1}{2},$$

and with $\alpha = d+1$ or $\alpha = N-(d+1)$, is a minimal cubature formula of even degree d.

Proof. The number of points N is equal to the minimum number of points for cubature formulae of even trigonometric degree d. According to Theorem 2.3 it is then sufficient to check whether the difference between every pair of different points of the cubature formula is a zero of (22). Now for $j \neq j'$ and $l = j - j'$

$$\begin{aligned}\mathcal{K}(\mathbf{x}^{(j)} - \mathbf{x}^{(j')}) &= \mathcal{K}\left(\frac{j-j'}{N}, \frac{\alpha(j-j')}{N}\right) = \mathcal{K}\left(\frac{l}{N}, \frac{\alpha l}{N}\right) \\ &= \frac{\cos\frac{\pi(d+1)l}{N}\cos\frac{\pi l}{N} - \cos\frac{\pi(d+1)\alpha l}{N}\cos\frac{\pi\alpha l}{N}}{\sin\frac{\pi l(\alpha+1)}{N}\sin\frac{\pi l(\alpha-1)}{N}} \\ &= \frac{\cos\frac{\pi(d+1)l}{N}\cos\frac{\pi l}{N} - \cos\frac{\pi(d+1)^2 l}{N}\cos\frac{\pi(d+1)l}{N}}{\sin\frac{\pi l(d+2)}{N}\sin\frac{\pi l d}{N}}.\end{aligned}$$

Because

$$\begin{aligned}\cos\frac{\pi(d+1)^2 l}{N} &= \cos\frac{\pi(2N-1)l}{N} \\ &= \cos\left(2\pi l - \frac{\pi l}{N}\right) \\ &= \cos\frac{\pi l}{N},\end{aligned}$$

the numerator of $\mathcal{K}(\mathbf{x}^{(j)} - \mathbf{x}^{(j')})$ is 0. The denominator of $\mathcal{K}(\mathbf{x}^{(j)} - \mathbf{x}^{(j')})$ is always different from 0 when $j \neq j'$, by the following argument. The factor $\sin\frac{\pi l d}{N}$ vanishes only if

$$ld = sN \quad \text{for some } s \in \mathbb{Z}.$$

Since d is even and N is odd, s must be even. Putting $s = 2m$, the condition becomes

$$l = \frac{2mN}{d} = m\left(d + 2 + \frac{2}{d}\right).$$

As l is an integer, we see that $\frac{d}{2}$ must divide m, from which it follows, since $l \neq 0$, that $|m| \geq \frac{d}{2}$, and hence

$$|l| \geq \frac{d}{2}\left(d + 2 + \frac{2}{d}\right) = N.$$

By a similar argument, $\sin\frac{\pi l(d+2)}{N}$ vanishes for $l \neq 0$ only if $|l| \geq N$. Taking these together, the denominator does not vanish for $|l|$ in the range $1, 2, \ldots, N-1$. $\square$

Remark. Noskov proved this theorem [13] by direct application of the cubature formula to the trigonometric monomials of degree $\leq d$, with d even. This approach, although very useful to check formulae, cannot be used to construct them.

6. The 2D odd-degree case

For $n = 2$ and odd degrees $d = 2m + 1$, $m \in \mathbb{N}_0$, the lower bound from Theorem 2.4 becomes

$$N \geq \tau(m+1, 2) = 2(m+1)^2 = \frac{(d+1)^2}{2}.$$

Theorem 6.1. *The points*

$$\left\{\left(C_p + \frac{j}{2(m+1)}, C_p + \frac{j+2p}{2(m+1)}\right)\right\} \begin{array}{lcl} j &=& 0, \ldots, 2m+1, \\ p &=& 0, \ldots, m, \end{array}$$

with $C_0 = 0$ and $C_1, \ldots, C_m$ arbitrary, are the points of a shift symmetric minimal cubature formula of trigonometric degree $2m+1$.

Proof. According to Theorem 2.4 the number of points

$$N = (m+1)(2m+2) = 2(m+1)^2 = \tau(m+1,2)$$

is the minimal number of points for a cubature formula of degree $2m+1$. The set of points is shift symmetric, since, if j is replaced by $j \pm (m+1)$, then the point is shifted by $\left(\frac{1}{2}, \frac{1}{2}\right)$ modulo 1. According to Theorem 2.6 it is then sufficient to check whether the difference between every pair of different points satisfying $0 \le j, j' \le m$ is a zero of $\widetilde{\mathcal{K}}$. We write $\mathbf{x} - \mathbf{x}' = (\Delta x, \Delta y)$, so (23) becomes

$$\widetilde{\mathcal{K}}(\Delta x, \Delta y) = \frac{\cos(2\pi(m+1)\Delta x) - \cos(2\pi(m+1)\Delta y)}{2\sin(\pi(\Delta y + \Delta x))\sin(\pi(\Delta y - \Delta x))}. \tag{24}$$

If $\Delta x = \pm \Delta y$, then we can use

$$\lim_{\pm\Delta y \to \Delta x} \widetilde{\mathcal{K}}(\Delta x, \Delta y) = (m+1)\frac{\sin(2\pi(m+1)\Delta x)}{\sin(2\pi \Delta x)}. \tag{25}$$

From (24) there follows a necessary condition for the points:

$$\widetilde{\mathcal{K}}(\Delta x, \Delta y) = 0$$
$$\Rightarrow$$
$$\cos(2\pi(m+1)\Delta x) = \cos(2\pi(m+1)\Delta y)$$
$$\Leftrightarrow$$
$$2\pi(m+1)\Delta y = 2\pi l \pm 2\pi(m+1)\Delta x \ , \ l \in \mathbb{Z}$$
$$\Leftrightarrow$$
$$\Delta y = \frac{l}{m+1} \pm \Delta x \ , \ l \in \mathbb{Z} \ . \tag{26}$$

It is obvious that all pairs of points satisfy this condition since $\Delta x = \Delta C_p + \frac{\Delta j}{2(m+1)}$, $\Delta y = \Delta C_p + \frac{\Delta j + 2\Delta p}{2(m+1)}$, and so

$$\Delta y - \Delta x = \frac{\Delta p}{m+1}. \tag{27}$$

Condition (26) is not sufficient because the denominator of (24) can also be zero. Now the denominator vanishes if and only if $\Delta y = l \pm \Delta x, l \in \mathbb{Z}$. If $\Delta y = l \pm \Delta x$, then (25) gives a necessary and sufficient condition:

$$\widetilde{\mathcal{K}}(\Delta x, l \pm \Delta x) = \widetilde{\mathcal{K}}(\Delta x, \pm\Delta x) = 0$$
$$\Leftrightarrow$$
$$\frac{\sin(2\pi(m+1)\Delta x)}{\sin(2\pi\Delta x)} = 0$$
$$\Leftrightarrow$$
$$\Delta x = \frac{l}{2(m+1)}, \ l \in \mathbb{Z} \backslash \{l'(m+1) \ : \ l' \in \mathbb{Z}\}. \tag{28}$$

Now from (27), $\Delta y - \Delta x \in \mathbb{Z}$ only if $\Delta p = 0$ (since $|\Delta p| < m+1$), which, if the points are distinct, implies $\Delta j \neq 0$. In this situation, $\Delta x = \frac{\Delta j}{2(m+1)}$ satisfies (28), since $0 < |\Delta j| < m+1$. On the other hand,

$$\Delta y + \Delta x = 2\Delta C_p + \frac{\Delta j + \Delta p}{m+1},$$

which can be an integer only if $2(m+1)\Delta C_p \in \mathbb{Z}$, implying in turn $2(m+1)\Delta x \in \mathbb{Z}$, so that Δx satisfies (28) so long as $2\Delta x \notin \mathbb{Z}$. But if $2\Delta x \in \mathbb{Z}$ and $\Delta y + \Delta x \in \mathbb{Z}$, then $\Delta y - \Delta x \in \mathbb{Z}$, which is the case already considered above. □

Remark. Again, the theorem can also be proved by direct application of the cubature formula to the trigonometric monomials of degree $\leq 2k+1$.

Second proof of Theorem 6.1. Using $f_{\alpha\beta}(x,y) = e^{2\pi i(\alpha x+\beta y)}$, we consider

$$\begin{aligned} Q[f_{\alpha\beta}] &= \frac{1}{N}\sum_{j=0}^{2m+1}\sum_{p=0}^{m} f_{\alpha\beta}(C_p + \frac{j}{2m+2}, C_p + \frac{j+2p}{2m+2}) \\ &= \frac{1}{N}\sum_{j=0}^{2m+1}\sum_{p=0}^{m} e^{2\pi i(\alpha(C_p+\frac{j}{2m+2})+\beta(C_p+\frac{j+2p}{2m+2}))} \\ &= \frac{1}{N}\sum_{p=0}^{m} e^{2\pi i(\alpha C_p+\beta(C_p+\frac{2p}{2m+2}))}\sum_{j=0}^{2m+1}\left(e^{2\pi\frac{i(\alpha+\beta)}{2m+2}}\right)^j. \end{aligned}$$

The sum over j is 0, *unless* $\alpha+\beta$ is a multiple of $2m+2$. Since we are concerned only with showing that the rule is of degree $2m+1$, we may assume $|\alpha+\beta| \leq 2m+1$. Thus, we have shown

$$Q[f_{\alpha\beta}] = 0 \text{ unless } \alpha = -\beta.$$

So now assume $\alpha = -\beta$. Then

$$\begin{aligned} Q[f_{-\beta\beta}] &= \frac{2m+2}{N}\sum_{p=0}^{m} e^{2\pi i\beta\frac{2p}{2m+2}} \\ &= \frac{2m+2}{N}\sum_{p=0}^{m}\left(e^{2\pi i\frac{\beta}{m+1}}\right)^p \\ &= 0 \text{ unless } \beta \text{ is a multiple of } m+1. \end{aligned}$$

Now since degree $f_{\alpha\beta} \leq 2m+1$, we have

$$|\alpha| + |\beta| \leq 2m+1 \Rightarrow 2|\beta| \leq 2m+1 \Rightarrow |\beta| \leq m.$$

Hence, $Q[f_{-\beta\beta}] = 0$ unless $\beta = \alpha = 0$. So Q is of degree $2m+1$. □

Theorem 6.1 gives us an infinite number of minimal cubature formulae of odd degree $d = 2m+1$, $m \in \mathbb{N}$, with m real parameters C_p, $p = 1, 2, \ldots, m$. Interesting special cases for these parameters are:

(1) $C_p = \frac{p}{2(m+1)^2}$, $p = 1, \ldots, m$.
In this case the cubature formula can be written

$$Q[f] = \frac{1}{N}\sum_{j=1}^{N} f\left(\left\{\frac{j}{N}, \frac{j(2m+3)}{N}\right\}\right);$$

this is a rank-1 lattice rule with generator $\left(\frac{1}{N}, \frac{2m+3}{N}\right)$.
In [1] this result was obtained using the relation between the trigonometric degree and the dual lattice.

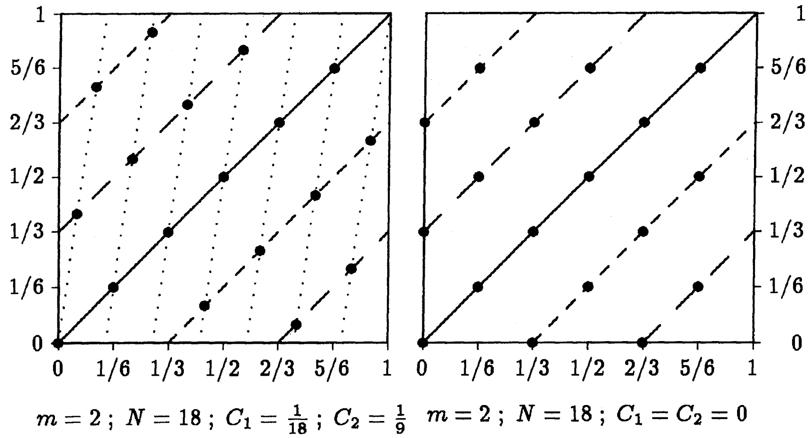

FIGURE 2. Minimal formulae of degree 5

(2) $C_p = 0, p = 1, \dots, m$.
In this case the formula can be written as

$$Q[f] = \frac{1}{N} \sum_{l,k=0}^{m} \left(f\left(\frac{l}{m+1}, \frac{k}{m+1}\right) + f\left(\frac{l+\frac{1}{2}}{m+1}, \frac{k+\frac{1}{2}}{m+1}\right)\right),$$

the body-centered cubic rule [17]. This is a lattice rule of rank 2; that is to say, it cannot be written as a single sum of the form (18). The fact that the body-centered cubic rule is a minimal cubature formula was established in, e.g., [11, 15, 16].

Example 6.1. A shift symmetric minimal cubature formula of trigonometric degree 5 (i.e., $m = 2$) has 18 points. Selecting $C_1 = \frac{1}{18}$ and $C_2 = \frac{2}{18}$ in the formula described in Theorem 6.1 gives the rank-1 rule

$$\frac{1}{18} \sum_{j=1}^{18} f\left(\left\{\frac{j}{18}, \frac{7j}{18}\right\}\right).$$

Selecting $C_1 = C_2 = 0$ in the formula described in Theorem 6.1 gives the body-centered cubic lattice

$$\frac{1}{18} \sum_{l=0}^{2} \sum_{k=0}^{2} \left[f\left(\frac{l}{3}, \frac{k}{3}\right) + f\left(\frac{l}{3} + \frac{1}{6}, \frac{k}{3} + \frac{1}{6}\right)\right].$$

These formulae are pictured in Figure 2.

In both cases (and indeed with any choice of $C_1, \dots, C_m$) all points lie on $m+1 = 3$ lines having unit slope, with each line containing $2m + 2 = 6$ equidistant points. It should be understood that lines that leave the unit square at the top re-enter at the bottom, because we have to look at the square modulo 1. Similarly, lines that leave the unit square at the right re-enter at the left. In the diagrams the $m+1 = 3$ such lines are identified by dashes of different lengths. Changing the value of C_p corresponds to sliding all of the points on the pth line along that line.

This last example serves as a useful counterexample to the otherwise plausible hypothesis that all minimal cubature formulae of trigonometric degree are necessarily lattice rules, or translations of lattice rules. The two particular special choices of

C_1, C_2 considered above, and illustrated in the figures, do indeed give lattice rules, but most other choices of C_1, C_2 clearly will not yield lattice rules. For example, a necessary condition that the rule in Theorem 6.1 be a lattice rule is (see [18, 19]) $2(m+1)^2 C_p \in \mathbb{Z}$ for $p = 1, 2 \ldots, m$. $\diamond$

7. Conclusion

In this paper we exploited the reproducing kernel to construct minimal cubature formulae of trigonometric degree 1,2 and 3 for arbitrary dimensions, as well as minimal cubature formulae of arbitrary degree for 1 and 2-dimensional integrals. The results for odd-degree formulae for two dimensions are new. We also obtained closed-form expressions for the reproducing kernels for one- and two-dimensional minimal cubature formulae.

We know of only one additional minimal cubature formula. Noskov [12] constructed a cubature formula of degree 5 for three dimensions with 38 knots:

$$Q[f] = \frac{1}{38} \sum_{j=1}^{N} f\left(\left\{\frac{j}{38}, \frac{7j}{38}, \frac{27j}{38}\right\}\right) .$$

His formula is geometrically equivalent with the one constructed by Frolov [2], published with some typographical errors.

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