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ON SEQUENCES WITHOUT GEOMETRIC PROGRESSIONS

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ABSTRACT. Several papers have investigated sequences which have no k-term arithmetic progressions, finding bounds on their density and looking at sequences generated by greedy algorithms. Rankin in 1960 suggested looking at sequences without k-term geometric progressions, and constructed such sequences for each k with positive density. In this paper we improve on Rankin's results, derive upper bounds, and look at sequences generated by a greedy algorithm.

1. INTRODUCTION

Erdős and Turan [1] defined $r_k(n)$ to be the least r for which any sequence of r numbers less than n must contain a k-term arithmetic progression. Roth [7] showed that $r_3(n) = O(n/\log\log n)$, and Szemerédi [8] showed that $r_k(n) = o(n)$ for all k.

We will denote all sets of nonnegative integers without a k-term arithmetic progression by APF_k (for arithmetic progression-free). Erdős conjectured that the sum of reciprocals of the (nonzero) terms of any such sequence converge, and offered \$3,000 for a proof or disproof.

One way to generate an arithmetic progression-free sequence is to use a greedy algorithm: start with 0, and add the smallest number which does not form a k-term arithmetic progression. Variations on the resulting sequences have been studied by several people [2, 3, 5]. For prime k, greedy sequences are just the integers whose base-k representation has no digits equal to $k-1$. For composite k their behavior is still mysterious.

In [4], the *span* of a set is defined to be the difference of its largest and smallest elements, and $\mathrm{sp}(k,n)$ to be the smallest span of a set in APF_k with n members, and a table of values for $\mathrm{sp}(k,n)$ for small k and n due to Usiskin is given. The value given for $\mathrm{sp}(3,10)$ in that table is wrong; Table 1 corrects it and gives $\mathrm{sp}(k,n)$ for a larger range of k and n.

The corresponding questions for sequences with no geometric progressions have received little attention. Rankin [6] used sequences in APF_k to form sequences with no k-term geometric progressions, and found their density. In §2 we review his methods, and show how sequences coming from a greedy method are superior to his for $k > 3$. In §3 we derive upper bounds for the density of such sequences.

Throughout this paper, A will denote an arbitrary sequence of nonnegative integers, A_k will be an arbitary sequence in APF_k, and A_k^* will be the greedy sequence described above.

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TABLE 1. Smallest span for APF_k

$k\backslash n$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
3	3	4	8	10	12	13	19	23	25	29	31	35	39	40	50
4		4	5	7	8	9	12	14	16	18	21	22	24	26	27
5			5	6	7	8	10	11	12	13	15	16	17	18	23
6				6	7	8	9	11	12	13	14	16	17	18	19

2. GEOMETRIC PROGRESSION-FREE SEQUENCES

Let GPF_k denote all sets of positive integers with no k-term geometric progressions. The only previous consideration of geometric progression-free sequences we know of is by Rankin [6]. An obvious sequence in GPF_3 is the set of squarefree numbers, which have density $6/\pi^2 \approx 0.608$.

Rankin showed that sequences in APF_k can be used to form denser sequences in GPF_k:

For a nonnegative sequence of integers $A = \{a_1, a_2, \ldots\}$, let $G(A)$ be the set of all integers

$$N = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}, \tag{1}$$

where the p_i are distinct primes, r is any nonnegative integer, and $e_i \in A$ for $i = 1, \ldots, r$.

Theorem 1. *If A is in APF_k, then $G(A)$ is in GPF_k.*

Proof. Let $\{a, as, as^2, \ldots, as^{k-1}\}$ be any set of integers in a geometric progression. (Note that, while $a \in \mathbb{Z}$, s may be a rational noninteger, e.g. the progression 9,12,16). Any prime dividing the numerator or denominator of s occurs to powers $c, c+d, c+2d, \ldots, c+(k-1)d$, for some $c \in \mathbb{Z}^+$ and $d \in \mathbb{Z}$. These powers form a k-term arithmetic progression, which cannot be contained in A, and so the numbers in the geometric progression cannot all be in $G(A)$. □

Let G_k^* be the set in GPF_k generated by the greedy algorithm; $g_1 = 1$, and g_i is the smallest integer which does not form a k-term geometric progression with $g_1, \ldots, g_{i-1}$.

Theorem 2. *We have $G_k^* = G(A_k^*)$.*

Proof. Let m be the smallest number in G_k^* which is not in $G(A_k^*)$. We will show that m is in a geometric progression with $k-1$ numbers in $G(A_k^*)$. This contradicts the definition of G_k^*, since G_k^* is equal to $G(A_k^*)$ up to m, proving that no such m exists.

Let $m = \prod_j p_j^{e_j} \prod_l q_l^{f_l}$, where the e_j are in A_k^*, and the f_l are not. Then for each f_l, there is an arithmetic progression $\{f_{l,1}, f_{l,2}, \ldots, f_{l,k} = f_l\}$ with $f_{l,1}, \ldots, f_{l,k-1} \in$

A_k^*. Then

$$N_1 = \prod_j p_j^{e_j} \prod_l q_l^{f_{l,1}},$$
$$N_2 = \prod_j p_j^{e_j} \prod_l q_l^{f_{l,2}},$$
$$\vdots$$
$$N_{k-1} = \prod_j p_j^{e_j} \prod_l q_l^{f_{l,k-1}},$$

together with m would form a geometric progression. All of $N_1, \dots, N_{k-1}$ are less than m and in $G(A_k^*)$, and so are in G_k^*. They form an arithmetic progression with m, contradicting $m \in G_k^*$. □

Rankin also gave a method to compute the density of a sequence $G(A) \in \mathrm{GPF}_k$ of the form (1). The Dirichlet series

$$f_{G(A)}(s) = \sum_{n \in G} n^{-s}$$

has the Euler product

$$f_{G(A)}(s) = \prod_p F_A(p^{-s}),$$

where, for $|x| < 1$,

$$F_A(x) = \sum_{q \in A} x^q. \tag{2}$$

When k is prime, $A = A_k^*$ consists of numbers with no digits equal to $k-1$ base k, and (2) becomes

$$\begin{aligned} F_{A_k^*}(x) &= \prod_{v=0}^{\infty} \left(1 + x^{k^v} + x^{2k^v} + \cdots + x^{(k-2)k^v}\right) \\ &= \prod_{v=0}^{\infty} \frac{1 - x^{(k-1)k^v}}{1 - x^{k^v}}, \end{aligned}$$

which implies

$$f_{G_k^*}(s) = \prod_{v=0}^{\infty} \frac{\zeta(k^v s)}{\zeta((k-1)k^v s)}. \tag{3}$$

The asymptotic density of G equals the residue at $s = 1$ of $f_G(s)$. For $G = G_3^*$, this is 0.7197 (Rankin gave the same sequence). Even for composite k, where there is no known closed form for $f_{G_k^*}(s)$, we may still compute the residue to any desired precision. For example, for $k = 4$, $A_4^* = \{0, 1, 2, 4, 5, \dots\}$, and

$$\begin{aligned} f_{G_4^*}(s) &= \prod_p \left(1 + p^{-s} + p^{-2s} + p^{-4s} + \cdots\right) \\ &= \zeta(s) \prod_p \left(1 - p^{-3s} + p^{-4s} - p^{-6s} + \cdots\right), \end{aligned}$$

which has residue ≈ 0.895.

This is better than the density 0.8626 GPF_4 sequence Rankin found. In fact, we can show that the greedy sequence is the best of the form (1):

Theorem 3. *If $G = G(A_k)$ for $k \geq 3$ and some APF_k sequence A_k, then its density is no greater than the greedy sequence.*

Proof. Any sequence $G = G(A)$ has a Dirichlet series of the form

$$f_G(s) = \prod_p \left(a_0 + a_1 p^{-s} + a_2 p^{-2s} + \cdots\right), \tag{4}$$

where $a_i = 1$ if $i \in A$, and $a_i = 0$ otherwise. As stated above, the residue at $s = 1$ of this function gives the density of the corresponding sequence.

Suppose there is another sequence A' for which $G' = G(A')$ has density greater than the greedy sequence $G(A)$. Let a_i' be the coefficients for the Dirichlet series $f_{G'}(s)$. The density of G' is greater than G if and only if the residue of $f_{G'}(s)$ at $s = 1$ is greater than the residue of $f_G(s)$.

At some point A' diverges from the greedy sequence, and we have $a_i = 1$ and $a_i' = 0$ for some i. Let H be the greedy sequence truncated at i, and H' be the same sequence with i removed and containing all $j > i$. Then H has density less than G and H' has density greater than G', so it suffices to show that

$$f_H(s) = \prod_p \left(a_0 + a_1 p^{-s} + \cdots + a_{i-1} p^{-(i-1)s} + p^{-is}\right) \tag{5}$$

has a larger residue at $s = 1$ than

$$\begin{aligned} f_{H'}(s) &= \prod_p \left(a_0 + \cdots + a_{i-1} p^{-(i-1)s} + p^{i+1)s} + p^{-(i+2)s} + \cdots\right) \\ &= \prod_p \left(a_0 + \cdots + a_{i-1} p^{-(i-1)s} + \frac{p^{-(i+1)s}}{1 - p^{-s}}\right). \end{aligned} \tag{6}$$

This is equivalent to showing that

$$\lim_{s \to 1} \frac{f_H(s)}{f_{H'}(s)} > 1.$$

But this is obvious, since for $p = 2$ the terms in (5) and (6) are equal at $s = 1$, and for all $p > 2$ and $s \geq 1$ the term in (5) is larger. $\square$

This leaves open the question of whether geometric progression-free sequences not of the form (1) have better density than greedy sequences. They can certainly do better over finite ranges; the greedy GPF_3 sequence:

1	2	3	5	6	7	8	10	11	13
14	15	16	17	19	21	22	23	24	26
27	29	30	31	33	34	35	37	38	39
40	41	42	43	46					

may be improved by removing 5 and adding 25 and 45.

3. Upper bounds

It is easy to show that the density of a GPF_k sequence is strictly less than one:

Theorem 4. *For any $k \geq 3$, the density of a sequence in GPF_k is at most $1-2^{-k}$.*

Proof. For any N, let a be an odd number less than $N/2^{k-1}$. Then the k numbers $a, 2a, 4a, \ldots, 2^{k-1}a$ cannot all appear in a GPF_k sequence. There are $N/2^k$ different a's, so this excludes $N/2^k$ numbers less than N from the sequence. □

Theorem 4 can be improved slightly:

Theorem 5. *For any $k \geq 3$, the density of a sequence in GPF_k is at most*

$$1 - 2^{-k} - \frac{5^{-(k-1)} - 6^{-(k-1)}}{2}.$$

Proof. Let b be an odd number, $N/6^{k-1} < b < N/5^{k-1}$. Then the numbers $3^{k-1}b, 3^{k-2}5b, \ldots, 5^{k-1}b$ cannot all appear in the sequence. There are $N/(2\cdot 5^{k-1}) - N/(2\cdot 6^{k-1})$ such b's, and none of them are the numbers $a, 2a, \ldots, 2^{k-1}a$ from Theorem 4, since they are all odd, and $3^{k-1}b > a$ for a and b in the ranges chosen. Moreover, since $6^{k-1}/5^{k-1} < 5/3$, the numbers $3^{k-1}b, 3^{k-2}5b, \ldots, 5^{k-1}b$ are distinct for different b in the range. □

Table 2. Densities for geometric progression-free sequences

k	greedy density	upper bound
3	0.71974	0.868889
4	0.89537	0.935815
5	0.95805	0.968336
6	0.98085	0.984279
7	0.99116	0.992166

The bounds can be further improved by taking fractions of larger primes over smaller ranges, but the improvements become marginal very quickly.

Table 2 gives the best known upper and lower bounds for the density of sequences in GPF_k for $k \leq 7$. For $k = 3$ and 4 they are still far apart, but as k gets large they approach each other.

Theorem 6. *As $k \to \infty$, the optimal density for a sequence in GPF_k is $1 - 2^{-k}(1-o(1))$.*

Proof. From Theorem 4, we have that the density is no greater than $1 - 2^{-k}$. Therefore, it suffices to show that the greedy sequence $G(A_k)$ has the stated density.

It is easy to see that the greedy APF_k sequence A_k starts off

$$\{0, 1, \ldots, k-2, k, k+1, \ldots, 2k-3, 2k-1\}$$

for k even and

$$\{0, 1, \ldots, k-2, k, k+1, \ldots, 2k-2, 2k\}$$

for $k > 3$ odd. For simplicity, we will handle the odd case (the even case is virtually identical). The density of $G(A_k)$ is the residue at $s = 1$ of

$$\prod_p \left(1 + p^{-s} + \cdots + p^{-(k-2)s} + p^{-ks} + \cdots + p^{-(2k-2)s} + p^{-2ks} + \cdots\right)$$
$$= \prod_p \frac{1}{1-p^{-s}} \left(1 - p^{-(k-1)s} + p^{-ks} - p^{-(2k-1)s} + \cdots\right)$$
$$= \zeta(s) \prod_p \left(1 - p^{-(k-1)s} + p^{-ks} - p^{-(2k-1)s} + \cdots\right).$$

The residue of $\zeta(s)$ is one, so the density is

$$\prod_p \left(1 - p^{-(k-1)} + p^{-k} - p^{-(2k-1)} + \cdots\right)$$
$$\geq (1 - 2^{-k} - 2^{-(2k-1)}) \prod_{p>2} \left(1 - p^{-(k-1)}\right)$$
$$= \frac{1 - 2^{-k} - 2^{-(2k-1)}}{(1 - 2^{-(k-1)})\zeta(k-1)}.$$

For large k, we have $\zeta(k-1) \to 1 + 2^{-(k-1)}$, and the density becomes

$$1 - 2^{-k}(1 - o(1)). \quad \square$$

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