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DECAY RATE FOR PERTURBATIONS OF STATIONARY DISCRETE SHOCKS FOR CONVEX SCALAR CONSERVATION LAWS

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ABSTRACT. This paper is to study the decay rate for perturbations of stationary discrete shocks for the Lax-Friedrichs scheme approximating the scalar conservation laws by means of an elementary weighted energy method. If the summation of the initial perturbation over $(-\infty, j)$ is small and decays at the algebraic rate as $|j| \to \infty$, then the solution approaches the stationary discrete shock profiles at the corresponding rate as $n \to \infty$. This rate seems to be almost optimal compared with the rate in the continuous counterpart. Proofs are given by applying the weighted energy integration method to the integrated scheme of the original one. The selection of the time-dependent discrete weight function plays a crucial role in this procedure.

1. INTRODUCTION

In this paper, we continue to study the asymptotic stability of the Lax-Friedrichs (LF) scheme

$$u_j^{n+1} - u_j^n + \frac{\lambda}{2}(f(u_{j+1}^n) - f(u_{j-1}^n)) = \frac{\mu}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) \tag{1.1}$$

approximating the convex scalar conservation laws

$$u_t + f(u)_x = 0, \quad u(x,0) = u_0(x) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases} \tag{1.2}$$

The corresponding shock wave solution is

$$u(t,x) = \begin{cases} u_-, & x - st < 0, \\ u_+, & x - st > 0, \end{cases}$$

where the end states $u_\pm$ related shock speed s by the Rankine-Hugoniot condition

$$- s(u_+ - u_-) + f(u_+) - f(u_-) = 0, \tag{1.3a}$$

and Oleinik's shock condition

$$(u_+ - u_-)Q \equiv (u_+ - u_-)[f(u) - f(u_\pm) - s(u - u_\pm)] > 0 \tag{1.3b}$$

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for $u \in (\min(u_-, u_+), \max(u_-, u_+))$. It is noted that when $s \neq f'(u_\pm)$ (1.3b) implies the Lax shock condition

$$f'(u_+) < s < f'(u_-). \tag{1.3c}$$

Let $x_j = jr$ and $t_n = nh$, with $r = \Delta x$ and $h = \Delta t$ being the spatial and the temporal grid sizes. Denote an approximation of $u(x_j, t_n)$ by u_j^n, μ is a constant satisfying $0 < \mu < 1$, and the temporal and spatial grid ratio $\lambda = \frac{\Delta t}{\Delta x}$ satisfies the Courant-Friedrichs-Levy (CFL) condition,

$$\lambda \max |f'| \leq \mu. \tag{1.4}$$

Corresponding to the difference equation (1.1) we have the following viscous conservation law

$$u_t + f(u)_x = \epsilon u_{xx}, \quad \epsilon > 0, \tag{1.5}$$

which has a viscous shock profile solution $u = U(x - st)$ satisfying

$$U(z) \to u_\pm \quad \text{as } z \to \pm\infty.$$

For convenience, we assume that $u_+ < u_-$ and $s \neq f'(u_\pm)$, then $U'(z) < 0$ and $|U - u_\pm| \sim O(1)\exp(-c|z|)$ as $z \to \pm\infty$. Hence the shock profile of (1.5) has the following property

$$u(x, t + \Delta t) = u(x - s\Delta t, t). \tag{1.6}$$

Since the solutions of the difference equation are only defined on the grid nodes, (1.6) does not always make sense. For simplicity, we focus on the stationary discrete shock profile solution ϕ_j of (1.1) i.e.,

$$\lambda(f(\phi_{j+1}) - f(\phi_{j-1})) = \mu(\phi_{j+1} - 2\phi_j + \phi_{j-1}), \tag{1.7a}$$

$$\phi_j \to u_\pm \quad \text{as } j \to \pm\infty. \tag{1.7b}$$

Its existence and properties have been proved by Jennings [7] provided that (u_-, u_+) satisfies (1.3a)–(1.3b).

Consider the initial value problem for (1.1) with the initial condition

$$u_j^0 \to u_\pm \quad \text{as } j \to \pm\infty \tag{1.8}$$

and

$$\sum_j (1 + j^2)^{3/2} |u_j^0 - \phi_j|^2 \leq c_1 \tag{1.9}$$

for some (suitably small) positive constant c_1. Under these assumptions and f being non-convex Liu and Wang [13] successfully proved that as $n \to \infty$, the solution u_j^n of (1.1), (1.8) tends uniformly with respect to $j \in Z$, in maximum norm, to ϕ_j which is uniquely determined by the relation

$$\sum_j (u_j^0 - \phi_j) = 0. \tag{1.10}$$

In [13], we proved that the above discrete shocks for the L-F scheme (1.1) is asymptotically stable in the l^2- and l^∞-norm. But, as far as we know, the decay rate is not known even in the case when f is convex. For this reason, the asymptotic stability theory of discrete shock is not complete as yet. The main contribution of this article is to show the decay rate for a convex flux f under some additional assumptions on initial data. We assume that the summation

$$v_j^0 = \sum_{-\infty}^{j}(u_j^0 - \phi_j) \in l_\alpha^2 \tag{1.11}$$

exists for any $j \in Z$ and denote $|v_j^0|_\alpha = [\sum_j (1 + (jr)^2)^\alpha (v_j^0)^2]^{1/2}$

Then our main theorem is given as follows.

Theorem 1.1. *Suppose* (1.3a)–(1.3c) *and* (1.4) (*CFL condition*) *hold. Let* ϕ_j *be a stationary discrete shock profile defined by* (1.7a) *connecting* u_+ *to* u_-. *Suppose the initial data* u_j^0 *satisfy* (1.10) *and, for some* $\alpha > 0$,

$$\sum_{j=-\infty}^{\infty} (1 + j^2)^\kappa |u_j^0 - \phi_j|^2 \leq \delta_1 \quad (\kappa > \frac{\alpha}{2} + 1) \tag{1.12}$$

for some (*suitably small*) *positive constant* δ_1, *then the unique global solution* u_j^n, *to the L-F scheme* (1.1) *with initial data* u_j^0 *tends in the maximum norm to the shock profile* ϕ_j *at the following rate: If* α *is an integer, then it holds*

$$\sup_j |u_j^n - \phi_j| \leq C(1 + nh)^{-\alpha/2} |v_j^0|_\alpha, \quad n \geq 0, \tag{1.13}$$

while if α *is not an integer, then*

$$\sup_j |u_j^n - \phi_j| \leq C_\varepsilon (1 + nh)^{-\mu/2+\varepsilon} |v_j^0|_\alpha, \quad n \geq 0, \tag{1.14}$$

for any constant $\varepsilon > 0$ *and some constant* C_ε *such that* $C_\varepsilon \to \infty$ *as* $\varepsilon \to 0$.

The study of existence and stability of discrete shocks is important in understanding the convergence behavior of numerical shock computations. Jennings [7] first investigated the existence and stability of discrete shocks for scalar difference equations. But the work is only restricted to the strictly monotone schemes. Engquist and Osher proved the stability of the first-order general monotone scheme for the scalar case [3]. Smyrlis [23] proved stability of a scalar stationary discrete shock wave for the Lax-Wendroff scheme. For scalar conservation laws, the L-F scheme belongs to the class of monotone schemes which have been well understood, see [2], [21], etc. Tadmor [25] studied the large time behavior for the rarefaction waves for some monotone schemes. The earliest important works in the study of the large time behavior for parabolic equations and monotone schemes can be seen in [5].

For the L-F scheme approximating systems, in the case that far field is a constant state, Chern [1] proved stability of the L-F scheme using diffusion waves. Liu and Xin [14] have proved that, for the L-F scheme, the solutions of the Riemann problem are single or multiple shocks; and if the summation of the initial perturbations equal zero, then the scheme solutions are asymptotically stable; they also study the stability of stationary discrete shock in [15]. The existence of discrete shock waves of first-order accurate finite difference methods for systems of conservation laws was established by Majda and Ralston [16] by using the center manifold theorem.

Our stability analysis is strongly motivated by the nonlinear stability of viscous shock profile for a viscous conservation law of the form (1.6) which is one of the hot spots in mathematical physics and interests many mathematicians (see [6], [20], [19], [9]). Studies on systems began with the independent works of Matsumura-Nishihara [17] and Goodman [4]. Important progress for general initial perturbations has been achieved by Liu [12] and Szepessy and Xin [24]. Recently, some interesting papers on the stability of viscous shock profiles in the case of $I(f) \neq \emptyset$, where $I(f)$ is the set of a reflection point of f, appeared (see [10], [8], [22], [11]).

As to the decay rate, Il'in and Oleinik [6] showed in the case of $I(f) = \emptyset$ that if the integral of the initial disturbance over $(-\infty, x]$ decays exponentially $e^{-\alpha|x|}$ (with some $\alpha > 0$) for $|x| \to \infty$, then the solution approaches, in the maximum norm, the viscous shock profiles at an exponential rate $e^{-\beta t}$ (for some $\beta > 0$) as $t \to \infty$. In the particular case $f = u^2/2$, Nishihara [18] showed further properties, if the initial disturbance over $(-\infty, x]$ has an algebraic order $O(|x|^{-\alpha})$ (with some $\alpha > 0$) for $|x| \to \infty$, then the solution converges, in the maximum norm, to the shock profiles at the same algebraic rate $t^{-\alpha}$ as $t \to \infty$. He also notes that this time decay rate is optimal in general. These detailed results are not known for general f with convexity. However, for such f, Kawashima and Matsumura [9] showed that if $(1 + |x|^2)^{\alpha/2}\phi_0 \in H^2$ $(\exists \alpha > 0)$ is suitably small, then it holds that $\sup_{x\in R} |u - U| \leq C(1+t)^{-[\alpha]/2}$, where ϕ_0 is the integral of initial disturbance over $(-\infty, x]$ of viscous shock profile U . Recently, for $I(f) \neq \emptyset$ and $s \neq f'(u_\pm)$, Jones et al. [8] have shown $\sup_R |u - U| \leq C(1+t)^{-[\alpha]/4}$ based on the spectral analysis. In [11] we proved that $\sup_R |u - U| \leq C(1+t)^{-[\alpha]/2}$ for non-convex f by introducing a weight function to overcome the difficulty caused by non-convexity of f.

Our main task is to estimate the time decay rate. To carry out our analysis, we use the weighted energy integration method, with regard to this method we point especially to the works [14], [15] from which we draw ideas in the present work. The specific choice of our time-dependent weight is made to insure that the information can be transferred from spatial decay to temporal decay. In its general approach, our method resembles that of [15], but there are also essential differences between the two methods.

Our plan of this paper is as follows. After stating the notations, we reformulate the original problem and state theorems for the reformulated one. In section 3, we investigate the time decay rate when f is convex, due to the weighted energy method the time-dependent weight $(1 + nh)^\gamma(1 + (jr)^2)^{\beta/2}$ plays a central role in this procedure.

Notations. Let us now define the following weighted l^2 spaces,

$$l_K^2 \doteq \{f_j : \quad ||f_j||_{l_K^2} \equiv |f_j|_K = [\sum_j |f_j|^2 K_j]^{\frac{1}{2}} < \infty\},$$

where K_j is a discrete weight function. When $K_j = \langle jr\rangle^\alpha = (1 + (jr)^2)^{\frac{\alpha}{2}}$, we write $l_K^2 = l_\alpha^2$ and $|\cdot|_K = |\cdot|_\alpha$; when $K_j = \langle jr\rangle^\alpha w_j$, we write $l_K^2 = l_{\alpha,w}^2$ with the norm $|\cdot|_K = |\cdot|_{\alpha,w}$; when $C^{-1} \leq w_j \leq C$, we note that $l^2 = l_w^2$ with the norm $||\cdot|| \sim |\cdot|_w$ and that $l_{\alpha,w}^2 = l_\alpha^2$ with $|\cdot|_{\alpha,w} \sim |\cdot|_\alpha$.

2. Reformulation of the problem

Let ϕ_j be a stationary discrete shock wave for the L-F scheme (1.1), that is , ϕ_j satisfies (1.7a)

$$\lambda(f(\phi_{j+1}) - f(\phi_{j-1})) = \mu(\phi_{j+1} - 2\phi_j + \phi_{j-1}),$$

summing it over j from $-\infty$ to j yields

$$\lambda(f(\phi_{j+1}) - 2f(u_\pm) + f(\phi_j)) = \mu(\phi_{j+1} - \phi_j),$$

that is

$$\mu(\phi_{j+1} - \phi_j) = \lambda(Q_{j+1} + Q_j), \tag{2.1}$$

which has a unique solution ϕ_j up to a shift satisfying $\phi_j(\pm\infty) = u_\pm$. We have

Lemma 2.1. *Suppose* (1.3a)–(1.3b) *and* $u_+ < u_-$ *for* $s = 0$, *then for each* $\bar{u} \in (u_+, u_-)$ *there exists a unique stationary discrete shock profile* ϕ_j *to* (1.1), *i.e.,* (1.7a)–(1.7b) *holds and* ϕ_j *satisfies*

$$\phi_0 = \bar{u}, \quad \phi_j > \phi_{j+1}, \quad \text{for } j \in Z. \tag{2.2}$$

The proof of Lemma 2.1 is a consequence of the fact that a shock profile continuously depends on its value at a point (see [7]), we omit it here.

To obtain the decay rate, let us rewrite the initial value problem (1.1), (1.8) by setting

$$v_j^n = \sum_{k=-\infty}^{j} (u_k^n - \phi_k), \tag{2.3}$$

then $v_j^n \to 0$ as $j \to \pm\infty$. Subtracting (1.7a) from (1.1), and summing up the resulting expression from $-\infty$ to j, we get

$$\begin{aligned} v_j^{n+1} - v_j^n &+ \frac{\lambda}{2}(f(u_{j+1}^n) - f(\phi_{j+1})) + \frac{\lambda}{2}(f(u_j^n) - f(\phi_j)) \\ &= \frac{\mu}{2}(v_{j+1}^n - 2v_j^n + v_{j-1}^n). \end{aligned}$$

Noting $u_j^n - \phi_j = v_j^n - v_{j-1}^n$, we have

(2.4)

$$\begin{aligned} v_j^{n+1} - v_j^n &+ \frac{\lambda}{2} f'(\phi_{j+1}^n)(v_{j+1}^n - v_j^n) + \frac{\lambda}{2} f'(\phi_j^n)(v_j^n - v_{j-1}^n) \\ &+ \frac{\lambda}{2} F(\phi_{j+1}^n, v_{j+1}^n - v_j^n) + \frac{\lambda}{2} F(\phi_j^n, v_j^n - v_{j-1}^n) = \frac{\mu}{2}(v_{j+1}^n - 2v_j^n + v_{j-1}^n), \end{aligned}$$

where

$$F(\phi, u - \phi) = f(u) - f(\phi) - f'(\phi)(u - \phi), \tag{2.5a}$$

satisfies the estimate

$$|F(\phi, u - \phi)| \le O(1)|u - \phi|^2, \tag{2.5b}$$

here $O(1)$ is a positive constant. Using the notations

$$\Lambda_j = f'(\phi_j), \quad \theta_j^n = F(\phi_j, v_j^n - v_{j-1}^n),$$

we may rewrite the equation (2.4) as

$$\begin{aligned} v_j^{n+1} - v_j^n &+ \frac{\lambda}{2}\Lambda_{j+1}(v_{j+1}^n - v_j^n) + \frac{\lambda}{2}\Lambda_j(v_j^n - v_{j-1}^n) \\ &- \frac{\mu}{2}(v_{j+1}^n - 2v_j^n + v_{j-1}^n) = e_j^n, \end{aligned} \tag{2.6}$$

where

$$e_j^n = -\frac{\lambda}{2}(\theta_{j+1}^n + \theta_j^n).$$

Noting that (1.12) implies $|v_j^0|_\alpha$ is suitably small (which will be shown in the next section), then the problem (2.6) with initial data v_j^0 can be solved globally in time as follows.

Theorem 2.1. *Suppose $v_j^0 \in l_\alpha^2$ for some $\alpha \geq 0$ and $f'' > 0$, then there exists a positive constant δ_2 such that if $|v_j^0|_\alpha < \delta_2$, the problem (2.6) with initial data v_j^0 has a unique global solution v_j^n satisfying, for any $n \geq 0$*

$$\sup_n (1+nh)^\gamma ||v_j^n||^2 + \sum_n (1+nh)^\gamma ||v_{j+1}^n - v_j^n||^2 \leq C|v_j^0|_\alpha^2 \tag{2.7}$$

for any γ such that $0 \leq \gamma \leq \alpha$ if α is an integer and that $0 \leq \gamma < \alpha$ if α is not an integer.

From (2.6), v_j^{n+1} can be expressed in terms of v_j^n in the explicit scheme, we can obtain v_j^n step by step from the beginning of $n = 0$. Moreover, we can estimate the l^2-norm of v_j^n as follows

$$\sum_j (v_j^{n+1})^2 \leq C \sum_j (v_j^n)^2. \tag{2.8}$$

Combining (2.8) with the following a priori estimate and a standard continuity argument proves that Theorem 2.1 holds.

Proposition 2.2 (A priori estimate). *Let n_1 be a positive integer. Suppose that the problem (2.6) with initial data v_j^0 has a unique solution $v_j^n \in l_\alpha^2$ for some $\alpha \geq 0$, then v_j^n satisfies (2.7) for $0 \leq n \leq n_1$, provided $\sup_{0 \leq n \leq n_1} ||v_j^n||$ is suitably small.*

The global existence can be proved in a way similar to that in [13]; we omit the details. For the proofs of (2.7) more estimates are necessary.

3. Time-decay estimates

We proceed with more a priori estimates of the solution v_j^n of the problem (2.6) with initial data v_j^0. In order to estimate the time-decay rate, we introduce a time-dependent discrete weight function K_j^n , then multiplying (2.6) by $2v_j^n K_j^n$ and summing over j, we obtain

(3.1)
$$\sum_j 2(v_j^{n+1} - v_j^n) v_j^n K_j^n + \lambda \{ \sum_j \Lambda_{j+1} v_j^n K_j^n (v_{j+1}^n - v_j^n) + \sum_j \Lambda_j v_j^n K_j^n (v_j^n - v_{j-1}^n) \}$$
$$+ \mu \sum_j v_j^n K_j^n (2v_j^n - v_{j+1}^n - v_{j-1}^n) = 2 \sum_j v_j^n K_j^n e_j^n.$$

We now estimate each term denoted by I_i $(i=1,2,3)$ in the sequel on the left-hand side of (3.1), we have

$$\begin{aligned}
I_1 &= \sum_j \{(v_j^{n+1})^2 - (v_j^{n+1} - v_j^n)^2 - (v_j^n)^2\} K_j^n \\
&= \sum_j (v_j^{n+1})^2 K_j^{n+1} - \sum_j (v_j^n)^2 K_j^n \\
&\quad - \sum_j (v_j^{n+1} - v_j^n)^2 K_j^n - \sum_j (v_j^{n+1})^2 (K_j^{n+1} - K_j^n), \\
I_2 &= \lambda\{\sum_j \Lambda_{j+1} K_j^n v_j^n v_{j+1}^n - \sum_j \Lambda_{j+1} K_j^n (v_j^n)^2 \\
&\quad + \sum_j \Lambda_j K_j^n (v_j^n)^2 - \sum_j \Lambda_j K_j^n v_j^n v_{j-1}^n\} \\
&= \lambda\{-\sum_j (\Lambda_{j+1} K_{j+1}^n - \Lambda_j K_j^n)(v_j^n)^2 \\
&\quad + \sum_j \Lambda_{j+1}(K_j^n - K_{j+1}^n) v_j^n (v_{j+1}^n - v_j^n)\},
\end{aligned}$$

$$\begin{aligned}
I_3 &= \mu\{\sum_j v_j^n K_j^n (v_j^n - v_{j+1}^n) - \sum_j v_j^n K_j^n (v_{j-1}^n - v_j^n)\} \\
&= \mu\{\sum_j \frac{K_j^n}{2}[(v_j^n - v_{j+1}^n)^2 + (v_j^n)^2 - (v_{j+1}^n)^2] \\
&\quad + \sum_j \frac{K_{j+1}^n}{2}[(v_j^n - v_{j+1})^2 + (v_{j+1}^n)^2 - (v_j^n)^2]\} \\
&= \mu\{\sum_j (v_j^n - v_{j+1}^n)^2 \frac{K_j^n + K_{j+1}^n}{2} \\
&\quad + \sum_j (K_{j+1}^n - K_j^n) \frac{v_{j+1}^n + v_j^n}{2} (v_{j+1}^n - v_j^n)\}.
\end{aligned}$$

Then we have

(3.2)
$$\begin{aligned}
&\sum_j (v_j^{n+1})^2 K_j^{n+1} - \sum_j (v_j^n)^2 K_j^n + \sum_j A_j^n (v_j^n)^2 + \mu \sum_j \frac{K_j^n + K_{j+1}^n}{2} |v_{j+1}^n - v_j^n|^2 \\
&\quad = \sum_j (v_j^{n+1} - v_j^n)^2 K_j^n - \sum_j B_j^n + \sum_j (K_j^{n+1} - K_j^n)(v_j^{n+1})^2 + 2\sum_j v_j^n K_j^n e_j^n,
\end{aligned}$$

where

$$
\begin{aligned}
A_j^n &= -\lambda(\Lambda_{j+1}K_{j+1}^n - \Lambda_j K_j^n), \\
B_j^n &= (-\lambda\Lambda_{j+1}v_j^n + \mu\tfrac{v_{j+1}^n+v_j^n}{2})(K_j^n - K_{j+1}^n)(v_{j+1}^n - v_j^n).
\end{aligned} \tag{3.3}
$$

Now setting

$$K_j^n = (1+nh)^\gamma \langle jr\rangle^\beta \tag{3.4}$$

and denoting $|v_j^n|_\beta^2 = \sum_j \langle jr\rangle^\beta |v_j^n|^2$. Then

$$\frac{K_j^n + K_{j+1}^n}{2} = (1+nh)^\gamma \frac{\langle jr\rangle^\beta + \langle (j+1)r\rangle^\beta}{2} \geq (1+nh)^\gamma \langle jr\rangle^\beta, \tag{3.5}$$

and

$$
\begin{aligned}
K_j^{n+1} - K_j^n &= \langle jr\rangle^\beta((1+(n+1)h)^\gamma - (1+nh)^\gamma) \\
&= \langle jr\rangle^\beta(\gamma(1+nh)^{\gamma-1}h + \tfrac{\gamma(\gamma-1)(1+\tilde{n}h)^{\gamma-2}}{2}h^2) \\
&= \gamma(1+O(h))\langle jr\rangle^\beta(1+nh)^{\gamma-1}h, \quad (n<\tilde{n}<n+1),
\end{aligned}
$$

which leads to

$$
\begin{aligned}
\sum_j (K_j^{n+1} - K_j^n)(v_j^{n+1})^2 &\leq 2\gamma(1+O(h))\{(1+nh)^{\gamma-1}\sum_j \langle jr\rangle^\beta (v_j^n)^2 h \\
&\quad + \frac{h}{1+nh}((1+nh)^\gamma \sum_j \langle jr\rangle^\beta (v_j^{n+1} - v_j^n)^2\} \\
&\leq C\gamma(1+nh)^{\gamma-1}|v_j^n|_\beta^2 h + O(1)h(1+nh)^\gamma |v_j^{n+1} - v_j^n|_\beta^2,
\end{aligned} \tag{3.6}
$$

where we have used

$$(v_j^{n+1})^2 = (v_j^{n+1} - v_j^n + v_j^n)^2 \leq 2(v_j^{n+1} - v_j^n)^2 + 2(v_j^n)^2.$$

Collecting (3.2), (3.5) and (3.6) we have

$$
\begin{aligned}
&(1+(n+1)h)^\gamma |v_j^{n+1}|_\beta^2 - (1+nh)^\gamma |v_j^n|_\beta^2 \\
&\quad + \alpha(1+nh)^\gamma |v_{j+1}^n - v_j^n|_\beta^2 + \sum_j A_j^n (v_j^n)^2 \\
&\leq (1+O(1)h)(1+nh)^\gamma |v_j^{n+1} - v_j^n|_\beta^2 + \sum_j |B_j^n| \\
&\quad + C_1\gamma(1+nh)^{\gamma-1}h|v_j^n|_\beta^2 + 2\sum_j v_j^n K_j^n e_j^n.
\end{aligned} \tag{3.7}
$$

Next we estimate the terms on the right-hand side of (3.7), we set

$$N(n_1) = \sup_{n\leq n_1} (\sum_j |v_j^n|^2)^{1/2}, \tag{3.8}$$

and assume that $N(n_1)$ is small. Obviously, we have

$$\sup_{n\leq n_1, j} |v_j^n| \leq N(n_1). \tag{3.9}$$

It follows from equation (2.6) that

$$v_j^{n+1} - v_j^n = \{\frac{\mu}{2} - \frac{\lambda}{2}\Lambda_{j+1}\}(v_{j+1}^n - v_j^n) - \{\frac{\mu}{2} + \frac{\lambda}{2}\Lambda_j\}(v_j^n - v_{j-1}^n) + e_j^n,$$

then we have

(3.10)
$$\begin{aligned}|v_j^{n+1} - v_j^n|^2 \le \frac{1}{2}\{(\mu - \lambda\Lambda_{j+1})^2|v_{j+1}^n - v_j^n|^2 + (\mu + \lambda\Lambda_j)^2|v_j^n - v_{j-1}^n|^2\}\\ + O(1)N(n_1)\{|v_{j+1}^n - v_j^n|^2 + |v_j^n - v_{j-1}^n|^2\},\end{aligned}$$

where we have used (2.5b) and (3.9). Consequently, we have

(3.11)
$$\sum_j \langle jr\rangle^\beta |v_j^{n+1} - v_j^n|^2 \le ((\mu + \lambda \max|f'|)^2 + O(1)N(n_1)) \times |v_{j+1}^n - v_j^n|_\beta^2.$$

Next using (2.5b) and (3.9), we get

$$2\sum_j |v_j^n K_j^n e_j^n| \le CN(n_1)(1+nh)^\gamma \sum_j \langle jr\rangle^\beta |v_{j+1}^n - v_j^n|^2. \tag{3.12}$$

To get the desired estimate, we must estimate $\sum_j A_j^n(v_j^n)^2$ and $\sum_j B_j^n$.

Step 1. We estimate $\sum_j A_j^n(v_j^n)^2$. For the estimate, we need some properties of the stationary discrete shock wave. Let u^* be the unique state determined by

$$0 = \frac{f(u_+) - f(u_-)}{u_+ - u_-} = f'(u^*)$$

since the discrete shock profile continuously depends on its value at a point and ϕ_j is strictly decreasing in $j \in Z$. On the other hand, the uniqueness of the shock profile is understood modulo translation; without loss of generality, we assume $\phi_0 = u^*$. Thus $\Lambda_0 = Q'(\phi_0) = 0$ and $Q'(\phi_{-1}) > 0 > Q'(\phi_1)$ due to $Q''(u) > 0$.

With respect to A_j^n we have the following lemma.

Lemma 3.1. *For any $\beta \in [0, \alpha]$, there is a positive constant c_0 independent of β such that*

$$A_j^n \ge c_0\beta(1+nh)^\gamma \langle jr\rangle^{\beta-1}h, \tag{3.13}$$

for any $j \in Z$, provided that λ is suitably small.

Proof. By the Taylor expression formula we have

$$\begin{aligned}\langle (j+1)r\rangle^\beta = \langle jr\rangle^\beta + \beta jr\langle jr\rangle^{\beta-2}r\\ + \beta\langle \tilde{jr}\rangle^{\beta-4}(\langle \tilde{jr}\rangle^2 + (\beta-2)(\tilde{jr})^2)r^2/2\end{aligned} \tag{3.14}$$

and

$$\Lambda_{j+1} = Q'(\phi_{j+1}) = Q'(\phi_j) + Q''(\bar{\phi}_j)(\phi_{j+1} - \phi_j). \tag{3.15}$$

Together with (3.14) and (3.15), we have

$$
\begin{aligned}
A_j^n &= -\lambda(1+nh)^\gamma[\langle (j+1)r\rangle^\beta \Lambda_{j+1} - \langle jr\rangle^\beta \Lambda_j] \\
&= -\lambda(1+nh)^\gamma[\beta jr\langle jr\rangle^{\beta-2} Q'(\phi_j) r \\
&\quad + Q''(\bar\phi_j)\langle jr\rangle^\beta(\phi_{j+1}-\phi_j) + \langle jr\rangle^{\beta-1} O(1) r^2] \\
&= (1+nh)^\gamma \langle jr\rangle^{\beta-1} h[-\beta \tfrac{jr}{\langle jr\rangle} Q'(\phi_j) - \langle jr\rangle Q''(\bar\phi_j)\tfrac{\phi_{j+1}-\phi_j}{r} + O(r)],
\end{aligned}
\tag{3.16}
$$

where we have used $|\phi_{j+1}-\phi_j| = |\frac{\phi_{j+1}-\phi_j}{r}| r \le O(1)r$ and $\lambda r = h$.

Due to (3.15), we have $Q'(\phi_{j+1}) - Q'(\phi_j) < 0$, by virtue of $f'(u_+) < 0 < f'(u_-)$, so $-\frac{jr}{\langle jr\rangle}Q'(\phi_j) \to \mp Q'(u_\pm) = \mp f'(u_\pm) > 0$, as $j \to \pm\infty$, so

$$
-\frac{jr}{\langle jr\rangle} Q'(\phi_j) \ge c, \quad j \ne 0, \tag{3.17}
$$

for some $c > 0$. On the other hand,

$$
-\langle jr\rangle Q''(\bar\phi_j)\frac{\phi_{j+1}-\phi_j}{r} = \frac{\phi_0-\phi_1}{r} Q''(\bar\phi_0) = c_1 > 0, \quad j = 0. \tag{3.18}
$$

Combining (3.17) with (3.18), we obtain (3.13), where $c_0 > \min\{c, \frac{c_1}{\alpha}\}$, provided λ is suitably small. □

Step 2. We estimate $\sum_j |B_j^n|$.

First we compute

$$
\begin{aligned}
K_{j+1}^n - K_j^n &= (1+nh)^\gamma(\langle (j+1)r\rangle^\beta - \langle jr\rangle^\beta) \\
&= (1+nh)^\gamma \beta \tilde{j} r\langle \tilde{j} r\rangle^{\beta-2} r \\
&\le \beta(1+nh)^\gamma \langle \tilde{j} r\rangle^{\beta-1} r \\
&\le \beta c_r (1+nh)^\gamma \langle jr\rangle^{\beta-1} r,
\end{aligned}
\tag{3.19}
$$

where $c_r = c_{r,\beta} = \sup_j \frac{\langle \tilde{j} r\rangle^{\beta-1}}{\langle jr\rangle^{\beta-1}}$ $(j < \tilde{j} < j+1)$,

$$
\begin{aligned}
&[-\lambda\Lambda_{j+1} v_j^n + \mu\frac{v_j^n + v_{j+1}^n}{2}](v_{j+1}^n - v_j^n) \\
&= [(\mu - \lambda\Lambda_{j+1})v_j^n + \frac{\mu}{2}(v_{j+1}^n - v_j^n)](v_{j+1}^n - v_j^n) \\
&\le \frac{(\mu + \lambda \max|f'|)^2}{4\varepsilon}|v_j^n|^2 + (\varepsilon + \frac{\mu}{2})|v_{j+1}^n - v_j^n|^2
\end{aligned}
\tag{3.20}
$$

for any $\varepsilon > 0$. Combining (3.3), (3.19) with (3.20), we obtain

(3.21)
$$\begin{aligned}\sum_j |B_j^n| &\le c_r\beta(1+nh)^\gamma \sum_j \langle jr\rangle^{\beta-1} \\ &\quad\times \left[\frac{(\mu+\lambda\max|f'|)^2}{4\varepsilon}|v_j^n|^2 + (\varepsilon+\frac{\mu}{2})|v_{j+1}^n - v_j^n|^2\right] r \\ &\le \beta(1+nh)^\gamma \left[\frac{c_r(\mu+\lambda\max|f'|)^2}{4\varepsilon} r|v_j^n|_{\beta-1}^2 \right. \\ &\quad \left. + c_r(\varepsilon+\frac{\mu}{2})|v_{j+1}^n - v_j^n|_{\beta-1}^2 r\right].\end{aligned}$$

But

$$\begin{aligned}|v_{j+1}^n - v_j^n|_{\beta-1}^2 &= \sum_{|j|\le J}\langle jr\rangle^{\beta-1}|v_{j+1}^n - v_j^n|^2 + \sum_{|j|\ge J}\frac{\langle jr\rangle^\beta}{\langle jr\rangle}|v_{j+1}^n - v_j^n|^2 \\ &\le C(J)\sum_j |v_{j+1}^n - v_j^n|^2 + \frac{1}{Jr}\sum_j \langle jr\rangle^\beta |v_{j+1}^n - v_j^n|^2 \\ &\le C(J)\|v_{j+1}^n - v_j^n\|^2 + \frac{1}{Jr}|v_{j+1}^n - v_j^n|_\beta^2\end{aligned}$$

for some large fixed number $J > 0$, we have

(3.22)
$$\begin{aligned}\sum_j |B_j^n| &\le \beta(1+nh)^\gamma[\tfrac{c_0 h}{2}|v_j^n|_{\beta-1}^2 \\ &\quad + C(J)c_r(\varepsilon+\tfrac{\mu}{2})r\|v_{j+1}^n - v_j^n\|^2 + \tfrac{c_r(2\varepsilon+\mu)}{2J}|v_{j+1}^n - v_j^n|_\beta^2] \\ &\le \tfrac{c_0\beta h}{2}(1+nh)^\gamma|v_j^n|_{\beta-1}^2 + C\beta(1+nh)^\gamma\|v_{j+1}^n - v_j^n\|^2 \\ &\quad + \tfrac{\beta c_r(2\varepsilon+\mu)}{2J}(1+nh)^\gamma|v_{j+1}^n - v_j^n|_\beta^2,\end{aligned}$$

here we have chosen

$$\varepsilon \ge \frac{c_r(\mu+\lambda\max|f'|)^2}{2c_0\lambda},$$

and $C \ge \max\{C(J)c_r(\varepsilon+\frac{\alpha}{2})r, C_1\}$. □

Assuming Lemma 3.1, we obtain the following basic a priori estimate:

Proposition 3.2. *Let v_j^n be a solution of (2.6) for $n \le n_1$. Then there exists a positive constant C independent of n_1 such that for all $n \le n_1$*

(3.23)
$$\begin{aligned}&(1+nh)^\gamma|v_j^n|_\beta^2 + \beta\sum_{i<n}(1+ih)^\gamma|v_j^i|_{\beta-1}^2 h + \sum_{i<n}(1+ih)^\gamma|v_{j+1}^i - v_j^i|_\beta^2 \\ &\quad\le C\{|v_j^0|_\beta^2 + \gamma\sum_{i<n}(1+ih)^{\gamma-1}|v_j^i|_\beta^2 h + \beta\sum_{i<n}(1+ih)^\gamma\|v_{j+1}^i - v_j^i\|^2\},\end{aligned}$$

provided λ and $N(n_1)$ are suitably small.

Proof. By Lemma 3.1, we collect (3.7), (3.11)–(3.13) and (3.22) to obtain

$$
\begin{aligned}
&(1+(n+1)h)^\gamma |v_j^{n+1}|_\beta^2 - (1+nh)^\gamma |v_j^n|_\beta^2 \\
&\quad + \{\mu - (1+O(h))[(\mu+\lambda \max|f'|)^2 + O(1)N(n_1)] - CN(n_1) \\
&\quad - \frac{\beta c_r(2\varepsilon+\mu)}{2J}\}(1+nh)^\gamma |v_{j+1}^n - v_j^n|_\beta^2 + \frac{c_0\beta h}{2}(1+nh)^\gamma |v_j^n|_{\beta-1}^2 \\
&\leq C\{\beta(1+nh)^\gamma ||v_{j+1}^n - v_j^n||^2 + \gamma(1+nh)^{\gamma-1} h |v_j^n|_\beta^2\},
\end{aligned} \tag{3.24}
$$

here we have used $A_j^n \geq c_0\beta h(1+nh)^\gamma \langle jr\rangle^{\beta-1}$. On one hand, since $\mu < 1$, we take suitably small λ and take J suitably large, then

$$
\begin{aligned}
&\mu - (1+O(h))[(\mu+\lambda\max|f'|)^2 - O(1)N(n_1)] \\
&\quad - CN(n_1) - \frac{\beta c_r(2\varepsilon+\mu)}{2J} > \nu,
\end{aligned} \tag{3.25}
$$

here $0<\nu<\mu$, provided $N(n_1)$ is suitably small. Finally, summing the two sides of (3.24) from 0 to $n-1$ with respect to n, by virtue of (3.25), we have

$$
\begin{aligned}
&(1+nh)^\gamma |v_j^n|_\beta^2 + \nu\sum_{i<n}(1+ih)^\gamma |v_{j+1}^i - v_j^i|_\beta^2 + \frac{c_0\beta}{2}\sum_{i<n}(1+ih)^\gamma |v_j^i|_{\beta-1}^2 h \\
&\leq C\{|v_j^0|_\beta^2 + \beta\sum_{i<n}(1+ih)^\gamma ||v_{j+1}^i - v_j^i||^2 + \gamma\sum_{i<n}(1+ih)^{\gamma-1}|v_j^i|_\beta^2 h\},
\end{aligned} \tag{3.26}
$$

then (3.23) follows immediately. □

We proceed to estimate the solution of the problem (2.6). First, taking $\beta=\gamma=0$ in (3.23), it is easy to obtain the following lemma:

Lemma 3.3. *There is a positive constant C independent of n_1, it holds for $n \in [0, n_1]$ that*

$$||v_j^n||^2 + \sum_{i<n} ||v_{j+1}^i - v_j^i||^2 \leq C||v_j^0||^2, \tag{3.27}$$

provided $N(n_1)$ and λ are suitably small.

Applying the induction to (3.23) we have

Lemma 3.4. *Let $\gamma \in [0,\alpha]$ be an integer. Then it holds for $n \in [0, n_1]$ that*

$$
\begin{aligned}
&(1+nh)^\gamma |v_j^n|_{\alpha-\gamma}^2 + (\alpha-\gamma)\sum_{i<n}(1+ih)^\gamma |v_j^i|_{\alpha-\gamma-1}^2 h \\
&\quad + \sum_{i<n}(1+ih)^\gamma |v_{j+1}^i - v_j^i|_{\alpha-\gamma}^2 \leq C|v_j^0|_\alpha^2.
\end{aligned} \tag{3.28}
$$

Consequently, if α is an integer, then the following estimate holds for $0 \leq \gamma \leq \alpha$

$$(1+nh)^\gamma ||v_j^n||^2 + \sum_{i<n}(1+ih)^\gamma ||v_{j+1}^i - v_j^i||^2 \leq C|v_j^0|_\alpha^2. \tag{3.29}$$

Similar to the argument in continuous counterpart in [16], we prove this lemma as follows.

Proof. Step 1. We take $0 \le \alpha < 1$, letting $\beta = \alpha$ and $\gamma = 0$ in (3.23), we have

$$\begin{aligned} &|v_j^n|_\alpha^2 + \alpha \sum_{i<n} |v_j^i|_{\alpha-1}^2 h + \sum_{i<n} |v_{j+1}^i - v_j^i|_\alpha^2 \\ &\le C\{|v_j^0|_\alpha^2 + \alpha \sum_{i<n} ||v_{j+1}^i - v_j^i||^2\}. \end{aligned} \tag{3.30}$$

Due to (3.27),

$$\sum_{i<n} ||v_{j+1}^i - v_j^i||^2 \le C||v_j^0||^2 \le C|v_j^0|_\alpha^2, \quad \text{for} \quad \alpha \ge 0;$$

combining this with (3.30) we obtain (3.28) with $\gamma = 0$. Therefore Lemma 3.4 is proved for $0 \le \alpha < 1$.

Step 2. we take $1 \le \alpha < 2$. First, letting $\beta = 0$ and $\gamma = 1$ in (3.23), we have

$$(1 + nh)|v_j^n|_0^2 + \sum_{i<n}(1 + ih)|v_{j+1}^i - v_j^i|_0^2 \le C\{|v_j^0|_0^2 + \sum_{i<n} |v_j^i|_0^2 h\},$$

and with (3.28) ($\gamma = 0$) to obtain (3.29) with $\gamma = 1$, where we have used $|v_j^n|_0^2 \le |v_j^0|_{\alpha-1}^2 \le |v_j^0|_\alpha^2$ for $1 \le \alpha < 2$. Secondly, letting $\beta = \alpha - 1$ and $\gamma = 1$ in (3.23), we have

$$\begin{aligned} &(1 + nh)|v_j^n|_{\alpha-1}^2 + (\alpha - 1)\sum_{i<n}(1 + ih)|v_j^i|_{\alpha-1-1}^2 h + \sum_{i<n}(1 + ih)|v_{j+1}^i - v_j^i|_{\alpha-1}^2 \\ &\le C\{|v_j^0|_{\alpha-1}^2 + \sum_{i<n} |v_j^i|_{\alpha-1}^2 h + (\alpha - 1)\sum_{i<n}(1 + ih)||v_{j+1}^i - v_j^i||^2\}, \end{aligned}$$

together with (3.29) with $\gamma = 1$ and (3.28) ($\gamma = 0$) to obtain (3.28) with $\gamma = 1$. Therefore the proof is completed for $\alpha < 2$.

Step 3. We repeat the same procedure as in Step 2. The estimate (3.23) (with $\beta = 0$, $\gamma = 2$) together with (3.28) ($\gamma = 1$) yields (3.29) (with $\gamma = 2$), where $\alpha \ge 2$ is assumed. Also, (3.23) (with $\beta = \alpha - 2$, $\gamma = 2$) together with (3.28) ($\gamma = 1$) and (3.29) (with $\gamma = 2$) yields (3.28) (with $\gamma = 2$), which proves the lemma for $\alpha < 3$.

Repeating the same procedure, we can get the desired estimate (3.28) for any $\alpha \ge 0$. This completes the proof of Lemma 3.4.

Further we show sharper estimate. Let α be not an integer and γ be $[\alpha] < \gamma < \alpha$. Taking $\beta = 0$ in (3.23) we have

$$\begin{aligned} &(1 + nh)^\gamma |v_j^n|_0^2 + \sum_{i<n}(1 + ih)^\gamma |v_{j+1}^i - v_j^i|_0^2 \\ &\le C\{|v_j^0|_0^2 + \gamma \sum_{i<n}(1 + ih)^{\gamma-1} |v_j^i|_0^2 h\}. \end{aligned} \tag{3.31}$$

Using (3.28) with $\gamma = [\alpha]$,

$$\begin{aligned} &(1 + nh)^{[\alpha]} |v_j^n|_{\alpha-[\alpha]}^2 + (\alpha - [\alpha])\sum_{i<n}(1 + ih)^{[\alpha]} |v_j^i|_{\alpha-[\alpha]-1}^2 h \\ &+ \sum_{i<n}(1 + ih)^{[\alpha]} |v_{j+1}^i - v_j^i|_{\alpha-[\alpha]}^2 \le C|v_j^0|_\alpha^2, \end{aligned} \tag{3.32}$$

we estimate the final term in (3.31):

$$\sum_{i<n}(1+ih)^{\gamma-1}|v_j^i|_0^2 h$$

$$=\sum_{i<n}(1+ih)^{\gamma-1}\sum_j \langle jr\rangle^{(\alpha-[\alpha])([\alpha]+1-\alpha)-(\alpha-[\alpha])([\alpha]+1-\alpha)}(|v_j^i|^2)^{([\alpha]+1-\alpha)+(\alpha-[\alpha])}h$$

$$\leq\sum_{i<n}(1+ih)^{\gamma-1}(\sum_j \langle jr\rangle^{\alpha-[\alpha]}|v_j^i|^2)^{[\alpha]+1-\alpha}(\sum_j \langle jr\rangle^{-([\alpha]+1-\alpha)}|v_j^i|^2)^{\alpha-[\alpha]}h$$

$$=\sum_{i<n}(1+ih)^{-([\alpha]+1-\gamma)}((1+ih)^{[\alpha]}|v_j^i|^2_{\alpha-[\alpha]})^{[\alpha]+1-\alpha}((1+ih)^{[\alpha]}|v_j^i|^2_{\alpha-[\alpha]-1})^{\alpha-[\alpha]}h,$$

where we have used the Hölder inequality

$$\sum ab \leq (\sum a^p)^{1/p}(\sum b^{p'})^{1/p'}, \quad \frac{1}{p}+\frac{1}{p'}=1.$$

Here $p=\frac{1}{[\alpha]+1-\alpha}$ and $p'=\frac{1}{\alpha-[\alpha]}$. Further, using this Hölder inequality and (3.32) we obtain

$$\sum_{i<n}(1+ih)^{\gamma-1}|v_j^i|_0^2 h$$

$$\leq C|v_j^0|_\alpha^{2([\alpha]+1-\alpha)}\sum_{i<n}(1+ih)^{-([\alpha]+1-\gamma)}((1+ih)^{[\alpha]}|v_j^i|^2_{\alpha-[\alpha]-1})^{\alpha-[\alpha]}h$$

$$\leq C|v_j^0|_\alpha^{2([\alpha]+1-\alpha)}[\sum_{i<n}(1+ih)^{-\frac{[\alpha]+1-\gamma}{[\alpha]+1-\alpha}}]^{[\alpha]+1-\alpha}[\sum_{i<n}(1+ih)^{[\alpha]}|v_j^i|^2_{\alpha-[\alpha]-1})]^{\alpha-[\alpha]}h$$

$$\leq C|v_j^0|_\alpha^2 h,$$

where $[\alpha]<\gamma<\alpha$ implies $\frac{[\alpha]+1-\gamma}{[\alpha]+1-\alpha}>1$. Thus we have the following from (3.31). □

Lemma 3.5. *If α is not an integer, then it holds for any $\gamma<\alpha$*

$$(1+nh)^\gamma\|v_j^n\|^2+\sum_{i<n}(1+ih)^\gamma\|v_{j+1}^i-v_j^i\|^2\leq C|v_j^0|_\alpha^2. \tag{3.33}$$

Combining the latter part of Lemma 3.4 with Lemma 3.5, we complete the proof of Proposition 2.2.

Thus, assuming that $N(0)$ is suitably small, by a standard continuity argument, the problem (2.7) with the initial value v_j^0 has a unique global solution v_j^n satisfying (2.8) for any $n\geq 0$. Since $N(0)=|v_j^0|\leq|v_j^0|_\alpha$ for $\alpha>0$, Theorem 2.1 follows.

We now turn to prove our main Theorem 1.1.

Proof of Theorem 1.1. First we prove that the condition (1.12) on initial data implies $|v_j^0|_\alpha$ is small. Here we give a proof under the condition

$$\sum_{j=-\infty}^{\infty}(1+j^2)^\kappa|u_j^0-\phi_j|^2\leq\delta$$

for any given constant $\kappa > \frac{\alpha}{2} + 1$ and δ is a suitably small constant. Applying the Hölder inequality to

$$v_j^0 = \sum_{k=-\infty}^{j} (u_k^0 - \phi_k),$$

we have

$$\begin{aligned}|v_j^0|^2 &\le \sum_{k=-\infty}^{j} (1+k^2)^{\kappa} |u_k^0 - \phi_k|^2 \sum_{k=-\infty}^{j} (1+k^2)^{-\kappa} \\ &\le \delta \sum_{k=-\infty}^{j} (1+k^2)^{-\kappa}.\end{aligned}$$

Therefore,

$$\begin{aligned}|v_j^0|_\alpha^2 \le \sum_j \langle j\rangle^\alpha |v_j^0|^2 \le \delta \sum_j (1+j^2)^{\frac{\alpha}{2}} &\sum_{k=-\infty}^{j} (1+k^2)^{-\kappa} \\ &\le \delta \int_{-\infty}^{+\infty} (1+x^2)^{\frac{\alpha}{2}} \int_{-\infty}^{x} (1+y^2)^{-\kappa} dy dx \\ &\le \frac{\delta}{\kappa - \frac{\alpha}{2} - 1} O(1),\end{aligned}$$

so

$$N(0) \le |v_j^0|_\alpha \le O(1)\sqrt{\delta}. \tag{3.34}$$

Thus the hypothesis in Theorem 2.1 is fulfilled under the condition (1.12). It follows from Theorem 2.1 that there exists a unique global solution, u_j^n, to the L-F scheme (1.1) due to the relation

$$u_j^n = \phi_j + v_j^n - v_{j-1}^n,$$

which follows from (2.3). Next we study the asymptotic behavior of the solution u_j^n to (1.1). It follows from (2.7) that

$$\sum_{n=0}^{\infty} (1+nh)^\gamma ||v_j^n - v_{j+1}^n||^2 < C|v_j^0|_\alpha^2, \tag{3.35}$$

which implies

$$\lim_{n\to\infty} (1+nh)^\gamma \sum_j |u_j^n - \phi_j|^2 = \lim_{n\to\infty} (1+nh)^\gamma \sum_j |v_j^n - v_{j-1}^n|^2 = 0,$$

and

$$||v_{j+1}^n - v_j^n|| \le C(1+nh)^{-\frac{\gamma}{2}} |v_j^0|_\alpha. \tag{3.36}$$

Moreover, by virtue of (3.36),

$$|u_j^n - \phi_j| = |v_j^n - v_{j-1}^n| \le ||v_{j+1}^n - v_j^n||.$$

Combining these facts, we obtain

$$\sup_j |u_j^n - \phi_j| \le C(1+nh)^{-\frac{\gamma}{2}} |v_j^0|_\alpha,$$

which proves Theorem 1.1. □

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