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ANALYSIS AND CONVERGENCE OF A COVOLUME METHOD FOR THE GENERALIZED STOKES PROBLEM

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ABSTRACT. We introduce a covolume or MAC-like method for approximating the generalized Stokes problem. Two grids are needed in the discretization; a triangular one for the continuity equation and a quadrilateral one for the momentum equation. The velocity is approximated using nonconforming piecewise linears and the pressure piecewise constants. Error in the L^2 norm for the pressure and error in a mesh dependent H^1 norm as well as in the L^2 norm for the velocity are shown to be of first order, provided that the exact velocity is in H^2 and the true pressure in H^1. We also introduce the concept of a network model into the discretized linear system so that an efficient pressure-recovering technique can be used to simplify a great deal the computational work involved in the augmented Lagrangian method. Given is a very general decomposition condition under which this technique is applicable to other fluid problems that can be formulated as a saddle-point problem.

1. INTRODUCTION

The fundamental field equations of fluid mechanics are expressed in terms of a set of PDEs in the physical unknowns such as pressure, velocity, and/or an appropriate energy variable. Finite element, finite difference, and finite volume methods have been employed to numerically solve them. In particular, the MAC (marker and cell) method of Harlow and Welch [14] on rectangular grids and its variants on unstructured grids have been very popular among the practitioners of the finite volume method due to their proven reliability and robustness in dealing with heat transfer problems. However, unlike in the finite element method the theoretical analysis of a MAC-like method is usually ad hoc. One reason for this might be that the velocity approximants sought in a MAC-like method is often only the normal components of velocities at the interelements or cell interfaces of the partition of the flow domain. Another reason might be that the starting discretization procedure is usually done on the governing PDEs instead of a weak formulation in terms of inner products. A consistent relation between the problem domain partition(s) and the discretization is often not so obvious as in the finite element methodology. Attempts to improve this situation were made in [18, 19] and [4, 5] in which the dual network model approach was adopted to solve two phase fluid problems. The emphasis of these papers was on a conservation of mass or energy through the

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design of primal and dual partitions. However, no convergence analysis was done for the *full* discretized systems. Another approach was taken in [15, 16] where rigorous analysis was given to the so-called covolume methods. The partitions used were the Delaunay-Voronoi mesh systems, which differ from those used in the above papers. Nicolaides's approach represents a major advance since the usual vector operators (div, curl, laplacian, etc.) were generalized to irregular networks. (See also [6]. As for the implementation issues resulting from his methodology, Hall et al. [12, 13] have demonstrated covolume methods can be effectively implemented by their dual variable method (DVM) [1]. For the status of the covolume methods, see the review article by Nicolaides, Porsching, and Hall [17].

The purpose of this paper is to introduce and analyze a covolume method on unstructured triangular grids, along the line mentioned in the first approach above. The primal and dual partitions are a special case of those used in [4, 19]. The corresponding covolume method on unstructured distorted rectangular grids will be reported in a forthcoming paper. We also introduce the concept of a network model into the discretized linear system so that an efficient pressure-recovering technique can be used along with the augmented Lagrangian method (Fortin and Glowinski [8]. Although our methodology is applicable to a wide range of heat transfer problems, in this paper we concentrate on the generalized Stokes problem.

The generalized Stokes problem in two dimensions for steady flow of a heavily viscous fluid is

$$\alpha_0 \mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega \subset R^2, \tag{1.1}$$

$$\text{div } \mathbf{u} = 0, \tag{1.2}$$

$$\mathbf{u} = 0 \qquad \text{on } \partial\Omega, \tag{1.3}$$

where $\alpha_0 \geq 0, \nu > 0$. When $\alpha_0 = 0$ we have the Stokes problem, and the case of $\alpha_0 \neq 0$ usually arises as part of the solution process for the Navier-Stokes equation. We shall assume $\nu = 1$ in this paper, as $\nu\mathbf{u}$ can be used as a transformed variable. Let $H_0^1(\Omega)$ be the space of weakly differentiable functions with zero trace, $H^i(\Omega), i = 1, 2$, be the usual Sobolev spaces, and $L_0^2(\Omega)$ be the set of all L^2 functions over Ω with zero integral mean. Define the bilinear forms

$$\tilde{a}(\mathbf{u},\mathbf{v}) := \sum_{i,j}\left(\frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j}\right) + \alpha_0(\mathbf{u},\mathbf{v}), \qquad \mathbf{u},\mathbf{v} \in \mathbf{H}_0^1 := H_0^1(\Omega)^2, \tag{1.4}$$

$$\tilde{b}(\mathbf{v}, q) := -(q, \text{div}\,\mathbf{v}), \qquad \mathbf{v} \in \mathbf{H}_0^1, q \in L_0^2, \tag{1.5}$$

where $(\cdot,\cdot)$ is the L^2 inner product. The weak formulation associated with (1.1)-(1.3) is: Find $(\mathbf{u}, p) \in \mathbf{H}_0^1 \times L_0^2$ such that

$$\tilde{a}(\mathbf{u},\mathbf{v}) + \tilde{b}(\mathbf{v}, p) = (\mathbf{f},\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0^1, \tag{1.6}$$

$$\tilde{b}(\mathbf{u}, q) = 0 \quad \forall q \in L_0^2. \tag{1.7}$$

The approximation of this system using the mixed finite element method is well documented in [2]. We now describe a MAC-like method. The method is motivated by the MAC technique for incompressible flow problems and will be viewed as a Petrov-Galerkin method as far as error analysis is concerned. First we need to partition the problem domain, which for simplicity, will be assumed to be polygonal. Referring to Fig. 1, let $T_h = \bigcup K_B$ be a partition of the domain Ω into a union of

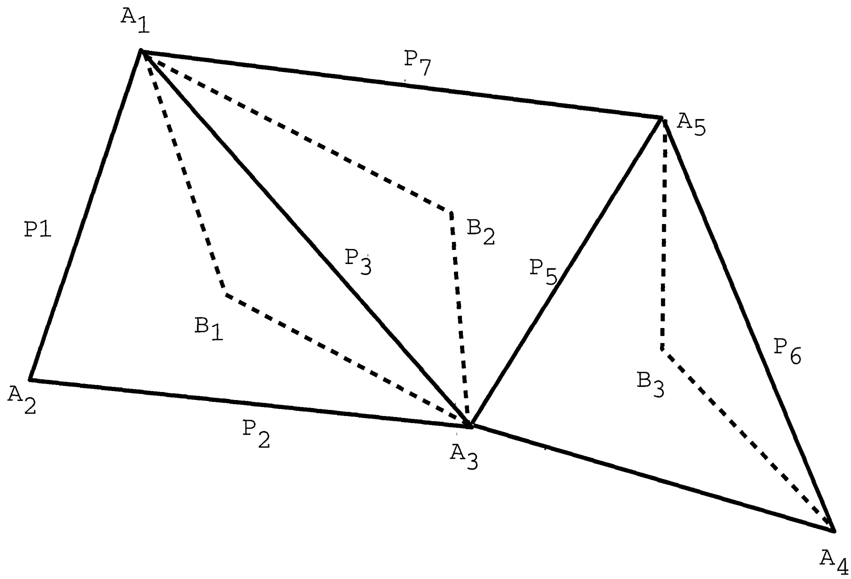

FIGURE 1. Triangulation and its dual

triangular elements, where K_B stands for the triangle whose barycenter is B. (The choice of barycenters is used only once in Lemma 2.3 to show the coerciveness of the bilinear form A of Eq. (1.22).) The nodes of an element are the midpoints of its sides. We denote by $P_1, P_2, \ldots, P_{N_S}$ those nodes belonging to the interior of Ω and $P_{N_S+1}, \ldots, P_N$ those on the boundary. The *trial* function space $\mathbf{H}_h$ associated with the approximation to the fluid velocity space $\mathbf{H}_0^1$ is defined as

(1.8)
$$\mathbf{H}_h = \{\mathbf{v}_h : \mathbf{v}_h|_K \in P_1(K)^2 \qquad \forall K \in T_h; \mathbf{v}_h = \mathbf{0} \text{ at all boundary nodes};$$
$$\mathbf{v}_h \text{ is continuous at all nodes}\},$$

where $P_1(K)$ denotes the space of linears on K. The choice of the degrees of freedom is motivated by the fact that in a MAC-like method the velocity approximant is usually associated with the midpoints of interelements. The choice of a nonconforming space is justified by the fact that the only divergence-free, continuous (conforming) piecewise linear polynomial is the zero vector function [3, p. 208]. Next we construct the dual partition T_h^* and the test function space. The dual grid is a union of interior quadrilaterals and border triangles. Referring to Fig. 1, the interior node P_3 belongs to the common side of the triangles $K_{B_1} = \Delta A_1A_2A_3$ and $K_{B_2} = \Delta A_1A_3A_5$, and the quadrilateral $A_1B_2A_3B_1$ is the dual element with node at P_3. For a boundary node like P_6 the associated dual element is a triangle ($\Delta A_5B_3A_4$ in this case). Physically one can view the dual element based along a common side as a flow subregion where the fluid information is imparted to the interelement or the common side. Obviously carrying out the construction for every node generates a dual partition for the domain. We shall denote the dual partition as $T_h^* = \bigcup K_P^*$ and associate with it the *test function space* $\mathbf{Y}_h$, the space of certain piecewise constant vector functions. That is

$$\mathbf{Y}_h = \{q \in (L^2(\Omega))^2 : q|_{K_P^*} \text{ is a constant vector,}$$

$$\text{and } q|_{K_P^*} = \mathbf{0} \text{ on any boundary dual element } K_P^*\}.$$

Denote by χ_j^* the scalar characteristic function associated with the dual element $K_{P_j}^*$, $j = 1, \ldots, N_S$. We see that for any $\mathbf{v}_h \in \mathbf{Y}_h$

$$\mathbf{v}_h(x) = \sum_{j=1}^{N_S} \mathbf{v}_h(P_j)\chi_j^*(x) \quad \forall x \in \Omega. \tag{1.9}$$

As for the approximate pressure space $L_h \subset L_0^2(\Omega)$, we define it to be the set of all piecewise constants with respect to the primal partition since in MAC-like methods the pressure is assigned at the centers of triangular elements. Finally, our test and trial function spaces should reflect the fact that in a MAC-like method the momentum equation (1.1) is integrated over the dual element and the continuity equation (1.2) over the primal element. This is indeed the case. Define

$$a^S(\mathbf{u}_h, \mathbf{v}_h) := -\sum_{i=1}^{N_S} \int_{\partial K_{P_i}^*} \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}} \cdot \mathbf{v}_h \, d\sigma \tag{1.10}$$

$$= -\sum_{i=1}^{N_S} \mathbf{v}_h(P_i) \cdot \int_{\partial K_{P_i}^*} \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}} \, d\sigma, \tag{1.11}$$

Eq. (1.10) is motivated by integrating the second term of (1.1) against a test function and then formally applying the second Green's identity. Let N_T denote the number of triangles in the primal partition.

$$a^N(\mathbf{u}_h, \mathbf{v}_h) := \alpha_0(\mathbf{u}_h, \mathbf{v}_h), \tag{1.12}$$

$$a(\mathbf{u}_h, \mathbf{v}_h) := a^S(\mathbf{u}_h, \mathbf{v}_h) + a^N(\mathbf{u}_h, \mathbf{v}_h), \tag{1.13}$$

$$b(\mathbf{v}_h, p_h) := \sum_{i=1}^{N_S} \mathbf{v}_h(P_i) \cdot \int_{\partial K_{P_i}^*} p_h \, \mathbf{n} \, d\sigma, \tag{1.14}$$

$$c(\mathbf{u}_h, q_h) := -\sum_{k=1}^{N_T} q_h(B_k) \int_{K_{B_k}} \operatorname{div} \mathbf{u}_h \, dx, \tag{1.15}$$

$$(\mathbf{f}, \mathbf{v}) = \sum_{i=1}^{N_S} \mathbf{v}_h(P_i) \cdot \int_{K_{P_i}^*} \mathbf{f} \, dx. \tag{1.16}$$

The weak formulation of the approximate problem to Eqs. (1.6)-(1.7) is: Find $(\mathbf{u}_h, p_h) \in \mathbf{H}_h \times L_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{Y}_h, \tag{1.17}$$

$$c(\mathbf{u}_h, q_h) = 0 \quad \forall q_h \in L_h. \tag{1.18}$$

Note that there are as many unknowns as equations; the number of unknowns being $2N_S + N_T$. (We did not count the zero-mean pressure condition.)

It turns out we can reformulate this system into a saddle-point problem as Eqs. (1.6)-(1.7). Convergence analysis can thus be done in the frame of the nonconforming mixed finite element method. We outline how the convergence analysis is done.

Introduce the one to one *transfer* operator γ_h from $\mathbf{H}_h$ *onto* $\mathbf{Y}_h$ by

$$\gamma_h \mathbf{u}_h(x) := \sum_{j=1}^{N_S} \mathbf{u}_h(P_j)\chi_j^*(x) \quad \forall x \in \Omega. \tag{1.19}$$

Define the following bilinear forms:

$$A^S(\mathbf{z}_h, \mathbf{w}_h) := a^S(\mathbf{z}_h, \gamma_h \mathbf{w}_h) \quad \forall \mathbf{z}_h, \mathbf{w}_h \in \mathbf{H}_h, \tag{1.20}$$

$$A^N(\mathbf{z}_h, \mathbf{w}_h) := a^N(\mathbf{z}_h, \gamma_h \mathbf{w}_h) \quad \forall \mathbf{z}_h, \mathbf{w}_h \in \mathbf{H}_h, \tag{1.21}$$

$$A(\mathbf{z}_h, \mathbf{w}_h) := A^S(\mathbf{z}_h, \mathbf{w}_h) + A^N(\mathbf{z}_h, \mathbf{w}_h), \tag{1.22}$$

$$B(\mathbf{w}_h, q_h) := b(\gamma_h \mathbf{w}_h, q_h) \quad \forall \mathbf{w}_h \in \mathbf{H}_h; \forall q_h \in L_h. \tag{1.23}$$

It is shown in Section 2 that the bilinear form A^S is symmetric and that the two bilinear forms B and c are identical. Hence the approximation problem (1.17)-(1.18) becomes: Find $(\mathbf{u}_h, p_h) \in \mathbf{H}_h \times L_h$ such that

$$A(\mathbf{u}_h, \mathbf{w}_h) + B(\mathbf{w}_h, p_h) = (\mathbf{f}, \gamma_h \mathbf{w}_h) \qquad \forall \mathbf{w}_h \in \mathbf{H}_h, \tag{1.24}$$

$$B(\mathbf{u}_h, q_h) = 0 \quad \forall q_h \in L_h. \tag{1.25}$$

Since $\mathbf{H}_h$ is nonconforming, the gradient and divergence operator on it must be defined piecewise:

$$(\nabla_h \mathbf{w}_h)|_K := \nabla(\mathbf{w}_h|_K),$$
$$(\operatorname{div}_h \mathbf{w}_h)|_K := \operatorname{div}(\mathbf{w}_h|_K).$$

On the space $\mathbf{H}_h$ we define

$$|\mathbf{w}_h|_{1,h}^2 := (\nabla_h \mathbf{w}_h, \nabla_h \mathbf{w}_h) = \sum_K (\nabla \mathbf{w}_h, \nabla \mathbf{w}_h)_K, \tag{1.26}$$

and

$$(\nabla \mathbf{w}_h, \nabla \mathbf{z}_h)_K := \sum_{i=1}^{2} (D_i \mathbf{w}_h, D_i \mathbf{z}_h)_K,$$

where $(\cdot,\cdot)_K$ is the $L_2(K)^2$ inner product, and D_i denotes the partial derivatives on K;

$$||\mathbf{w}_h||_{1,h}^2 := ||\mathbf{w}_h||_0^2 + |\mathbf{w}_h|_{1,h}^2.$$

Lemma 2.2 shows

$$A^S(\mathbf{w}_h, \mathbf{z}_h) = (\nabla_h \mathbf{w}_h, \nabla_h \mathbf{z}_h).$$

Therefore our covolume method resembles the Crouzeix-Raviart nonconforming method [7] in the case of $\alpha_0 = 0$; the difference being in the right-hand side of (1.24). For nonzero α_0, the A^N term further introduces a nonsymmetric term. Exploiting this similarity and adopting a known error estimate in [7], we derive the main error estimate result in Theorem 3.1, which states that there exists a constant $C > 0$ independent of h such that

$$||\mathbf{u}_h - \mathbf{u}||_{1,h} + ||p - p_h||_0 \le Ch(||\mathbf{u}||_2 + ||p||_1 + 1),$$

provided that $\mathbf{u} \in \mathbf{H}^2(\Omega)$ and $p \in H^1(\Omega)$ and that the triangulation family is quasi-uniform.

This is an improvement, as in the literature the velocity of the covolume schemes is assumed as a constant over each triangle and first order error estimates in the velocity are given only in some discrete L^2 norms with quasi-uniformity restriction on the primal triangulation and more demanding regularity assumptions on the exact velocity and pressure. The same optimal order error estimates also hold for a symmetric version of (1.24)-(1.25). See the remark following Theorem 3.1.

In Section 4, we introduce a "P-recovering" technique for solving the general saddle-point linear system in the unknown $\{U, P\}$ of the form

$$\tilde{A}U + \tilde{B}^t P = F, \tag{1.27}$$

$$\tilde{B}U = 0, \tag{1.28}$$

where $\tilde{A} \in R^{2N_S \times 2N_S}$ is symmetric positive definite, $\tilde{B} \in R^{N_T \times 2N_S}$, $\dim \ker \tilde{B}^t \geq 1$, and $\tilde{B}$ has the decomposition

$$\tilde{B} = \mathcal{A}\tilde{D} \text{ for some } \mathcal{A} \in R^{N_T \times N_S}, \tilde{D} \in R^{N_S \times 2N_S}. \tag{1.29}$$

The dimension of $\ker \tilde{B}^t$ is usually one in fluid mechanics applications in which P is the pressure and hence determined up to a constant vector of identical components. If $\operatorname{Ker} \tilde{B}^t = \{0\}$, the most basic method for solving such a system is the so-called pressure-matrix method and its preconditioned variants. In the pressure-matrix method we write the above system as

$$\tilde{B}\tilde{A}^{-1}\tilde{B}^t P = \tilde{B}\tilde{A}^{-1}F,$$
$$U = \tilde{A}^{-1}(F - \tilde{B}^t P),$$

and solve for P and then for U. Obviously the *pressure matrix* $R := \tilde{B}\tilde{A}^{-1}\tilde{B}^t$ is only semidefinite since $\tilde{B}^t$ in (1.27) is not of full rank. Thus the conjugate gradient method cannot be applied without caution. Of course we could add the zero integral-mean pressure condition to the above system. But that would destroy the symmetric structure of the system. The zero-mean pressure condition for finite element, finite volume or finite difference methods, unlike in the spectral method, is not a lowest Fourier coefficient condition and hence inconvenient to be incorporated into the algebraic system. On the other hand, the augmented Lagrangian method (pp. 49-51, [11]) works well for nonzero $\operatorname{Ker} \tilde{B}^t$ space and produces a sequence of U^k convergent to U and P^k convergent to the minimum norm pressure P_M. By interpreting system (1.27)-(1.28) as a network model, we show how to recover the P once the solution U and $\mathcal{A}^t P$ are known. The augmented Lagrangian method applied to (1.27)-(1.28) reads: Given $P^0 \in R^{N_T}$, with P^n known, calculate U^n, then P^{n+1}, by

$$(\tilde{A} + r\tilde{B}^t\tilde{B})\, U^n + \tilde{B}^t P^n = F, \tag{1.30}$$

$$P^{n+1} = P^n + \rho_n \tilde{B}U^n, \tag{1.31}$$

where r and ρ_n are two parameters. In Theorem 4.1 we show that so long as the definition of B is used in discretizing the divergence-free condition and the pressure-gradient term, the decomposition $\tilde{B} = \mathcal{A}\tilde{D}$ holds where $\mathcal{A}$ is the incidence matrix [cf. (4.10)], and $\tilde{D}$ is rectangular and consists of two diagonal matrices. Upon using the decomposition and introducing a new variable $\mathcal{A}^t P$, the pressure drops, (1.30)-(1.31) reads: Given $\tilde{P}^0 \in R^{N_S}$, with $\tilde{P}^n$ known, calculate U^n, then $\tilde{P}^{n+1}$, by

$$(\tilde{A} + r\tilde{B}^t\tilde{B})\, U^n + \tilde{D}^t \tilde{P}^n = F, \tag{1.32}$$

$$\tilde{P}^{n+1} = \tilde{P}^n + \rho_n (\mathcal{A}^t\mathcal{A})\tilde{D}U^n. \tag{1.33}$$

Since $\mathcal{A}$, $\mathcal{A}^t\mathcal{A}$ are sparse and $\tilde{D}$ is "diagonal", this saves work due to the $\tilde{D}^t\tilde{P}^n$ and $\tilde{D}U^n$ terms. We can delay recovering the solution P untill the last step when we have obtained an acceptable $\tilde{P} = \mathcal{A}^t P$.

2. Saddle-point form and inf-sup condition

In this section we prove several important properties of the following bilinear forms:

$$a^S(\mathbf{z}_h, \mathbf{v}_h) = -\sum_{i=1}^{N_S} \mathbf{v}_h(P_i) \cdot \int_{\partial K^*_{P_i}} \frac{\partial \mathbf{z}_h}{\partial \mathbf{n}}\, d\sigma, \tag{2.1}$$

$$b(\mathbf{v}_h, p_h) = \sum_{i=1}^{N_S} \mathbf{v}_h(P_i) \cdot \int_{\partial K^*_{P_i}} p_h\, \mathbf{n}\, d\sigma, \tag{2.2}$$

$$c(\mathbf{z}_h, q_h) = -\sum_{k=1}^{N} q_h(B_k) \int_{K_{B_k}} \operatorname{div} \mathbf{z}_h\, dx, \tag{2.3}$$

$$(\mathbf{f}, \mathbf{v}) = \sum_{i=1}^{N_s} \mathbf{v}_h(P_i) \cdot \int_{K^*_{P_i}} \mathbf{f}\, dx. \tag{2.4}$$

The following simple lemma of line integral conversion will be used often throughout the paper.

Lemma 2.1. *Suppose we subdivide each triangular element into three subtriangles as in Fig. 2. Let g be a continuous function in the interior of each subtriangle and let N be the number of sides of the triangles in the triangulation. Then*

$$\sum_{i=1}^{N} \int_{\partial K^*_{P_i}} g(x)\, d\sigma = \sum_{K \in T_h} I_K, \tag{2.5}$$

where

$$I_K = \int_{A_2BA_1} g(x)d\sigma + \int_{A_3BA_2} g(x)d\sigma + \int_{A_1BA_3} g(x)d\sigma = \sum_{j=1}^{3} \int_{A_{j+1}BA_j} g(x)d\sigma.$$

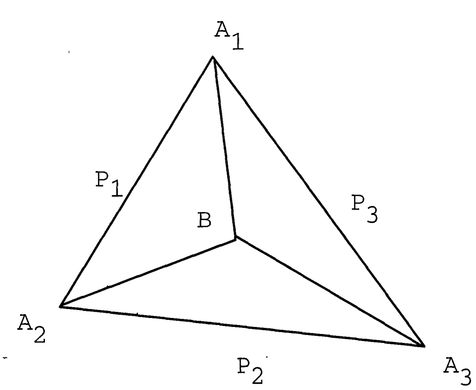

FIGURE 2. Integration conversion from dual elements to triangular ones

Here and below we adopt the convention $Q_{j+3} = Q_j, j = 1, 2, 3,$ when a subindex is out of bound.

Proof. The proof is straightforward. □

Lemma 2.2. *The bilinear form $A^S(\mathbf{z}_h, \mathbf{w}_h) = a^S(\mathbf{z}_h, \gamma_h \mathbf{w}_h)$ is symmetric:*

$$A^S(\mathbf{z}_h, \mathbf{w}_h) = A^S(\mathbf{w}_h, \mathbf{z}_h) = (\nabla_h \mathbf{z}_h, \nabla_h \mathbf{w}_h) \qquad \forall \mathbf{z}_h, \mathbf{w}_h \in \mathbf{H}_h. \tag{2.6}$$

Proof. By (2.1), Lemma 2.1, and the fact that $\mathbf{w}_h$ vanishes at the boundary nodes,

$$a^S(\mathbf{z}_h, \gamma_h \mathbf{w}_h) = \sum_{K \in T_h} I_K,$$

where

$$I_K = -\sum_{j=1}^{3} \mathbf{w}_h(P_j) \cdot \int_{\partial K^*_{P_j}} \frac{\partial \mathbf{z}_h}{\partial \mathbf{n}}\, d\sigma.$$

Letting $\mathbf{n} = (n_1, n_2)$, using $n_1 d\sigma = dx_2$, $n_2 d\sigma = -dx_1$ and the index convention at the end of Lemma 2.1, we have

(2.7)
$$\begin{aligned} I_K &= -\sum_{j=1}^{3} \mathbf{w}_h(P_j) \cdot \left(\int_{A_{j+1}BA_j} \frac{\partial \mathbf{z}_h}{\partial x_1} n_1 d\sigma + \int_{A_{j+1}BA_j} \frac{\partial \mathbf{z}_h}{\partial x_2} n_2 d\sigma \right) \\ &= -\sum_{j=1}^{3} \mathbf{w}_h(P_j) \cdot \Big(\int_{A_{j+1}BA_j} \frac{\partial \mathbf{z}_h}{\partial x_1} dx_2 - \int_{A_{j+1}BA_j} \frac{\partial \mathbf{z}_h}{\partial x_2} dx_1 \Big) \\ &= -\frac{\partial \mathbf{z}_h}{\partial x_1} \cdot \Big[\sum_{j=1}^{3} \mathbf{w}_h(P_j)\, (x_2(A_j) - x_2(A_{j+1}))\Big] \\ &+ \frac{\partial \mathbf{z}_h}{\partial x_2} \cdot \Big[\sum_{j=1}^{3} \mathbf{w}_h(P_j)(x_1(A_j) - x_1(A_{j+1})\Big], \end{aligned}$$

where we have used the fact that the partial derivatives of $\mathbf{z}_h$ are constant within each K. Let $\lambda_i, i = 1, \cdots, 3$, denote the Lagrange nodal basis functions associated with the vertices of K i.e., the barycentric coordinate functions on K. It is easy to see then that the local nodal basis function associated with P_i is $\phi_i = \lambda_i + \lambda_{i+1} - \lambda_{i+2}, i = 1, \cdots, 3,$ and $\phi_i(P_j) = \delta_{ij}$. Hence for a piecewise linear function $\mathbf{w}_h$,

$$\mathbf{w}_h|_K = \sum_{i=1}^{3} \mathbf{w}_h(P_i)(\lambda_i + \lambda_{i+1} - \lambda_{i+2}),$$

and

$$D_j \mathbf{w}_h|_K = \sum_{i=1}^{3} \mathbf{w}_h(P_i) D_j(\lambda_i + \lambda_{i+1} - \lambda_{i+2}), (D_j = \frac{\partial}{\partial x_j}). \tag{2.8}$$

Replacing the two $x_i(A_j) - x_i(A_{j+1})$ expressions in the right side of (2.7) using

$$\frac{\partial \lambda_j}{\partial x_1} = \frac{x_2(A_{j+1}) - x_2(A_{j+2})}{2|K|},$$
$$\frac{\partial \lambda_j}{\partial x_2} = \frac{x_1(A_{j+1}) - x_1(A_{j+2})}{2|K|},$$

and then (2.8), we have

$$I_K = \int_K D_1 \mathbf{z}_h \cdot D_1 \mathbf{w}_h + D_2 \mathbf{z}_h \cdot D_2 \mathbf{w}_h dx.$$

Thus

$$\begin{aligned} a^S(\mathbf{z}_h, \gamma_h \mathbf{w}_h) &= \sum_{K \in T_h} \int_K \nabla_h \mathbf{z}_h \cdot \nabla_h \mathbf{w}_h dx \\ &= (\nabla_h \mathbf{z}_h, \nabla_h \mathbf{w}_h). \quad \square \end{aligned}$$

Remark 2.1. The result of this lemma is not unexpected since an analogous result holds in the covolume solution of Poisson's equation when conforming elements are used (p. 45, [15]).

Lemma 2.3. *The bilinear form A is coercive: there exists a positive constant C independent of h such that*

$$A(\mathbf{w}_h, \mathbf{w}_h) \geq C |\mathbf{w}_h|_{1,h}^2.$$

Also, there exists a constant $C_1 > 0$ independent of h such that

$$C_1 \|\mathbf{w}_h\|_0^2 \leq (\mathbf{w}_h, \gamma_h \mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{H}_h. \tag{2.9}$$

Proof. We prove the coercivity first. By Lemma 2.2, it suffices to show

$$A^N(\mathbf{w}_h, \mathbf{w}_h) = \alpha_0 (\mathbf{w}_h, \gamma_h \mathbf{w}_h) \geq 0. \tag{2.10}$$

Referring to Fig. 2,

$$(\mathbf{w}_h, \gamma_h \mathbf{w}_h) = \sum_K I_K,$$

where

$$\begin{aligned} I_K &= \int_K \mathbf{w}_h \cdot \gamma_h \mathbf{w}_h dx \\ &= \sum_{j=1}^{3} \{ \int_{\Delta A_j A_{j+1} B} \mathbf{w}_h(x) dx \} \cdot \mathbf{w}_h(P_j). \end{aligned}$$

But $\mathbf{w}_h$ is linear and hence

$$\int_{\Delta A_j A_{j+1} B} \mathbf{w}_h(x) dx = \frac{1}{3} \left[\mathbf{w}_h(P_j) + \mathbf{w}_h(m_{A_j B}) + \mathbf{w}_h(m_{A_{j+1} B}) \right] |\Delta A_j A_{j+1} B|,$$

where $m_{A_j B}$ is the midpoint of side $A_j B, j = 1, 2, 3$. Noting that for $j = 1, 2, 3$

$$\mathbf{w}_h(m_{A_jB}) = \frac{1}{2}\mathbf{w}_h(B) + \frac{1}{2}\mathbf{w}_h(A_j),$$
$$\mathbf{w}_h(P_j) = \frac{1}{2}(\mathbf{w}_h(A_j) + \mathbf{w}_h(A_{j+1})),$$

and

$$|\Delta A_j A_{j+1} B| = \frac{1}{3}|\Delta A_1 A_2 A_3| = \frac{1}{3}|K|,$$

we have

$$\int_{\Delta A_j A_{j+1} B} \mathbf{w}_h(x)dx = \frac{|K|}{9}[4\mathbf{w}_h(A_j) + 4\mathbf{w}_h(A_{j+1}) + \mathbf{w}_h(A_{j+2})].$$

Using $A_j = P_j - P_{j+1} + P_{j+2}$ and that $\mathbf{w}_h$ is linear, we have

$$\begin{aligned} I_k &= \sum_j \mathbf{w}_h(P_j) \cdot \int_{\Delta A_j A_{j+1} B} \mathbf{w}_h(x)dx \\ &= \frac{|K|}{27}\sum_j \{\mathbf{w}_h(P_j)[7\mathbf{w}_h(P_j) + \mathbf{w}_h(P_{j+1}) + \mathbf{w}_h(P_{j+2})]\} \\ &= \frac{|K|}{27}\{\sum_j 7|\mathbf{w}_h(P_j)|^2 + (\sum_j \mathbf{w}_h(P_j))^2 - \sum_j \mathbf{w}_h(P_j)^2\} \\ &= |K|\{\frac{6}{27}\sum_j |\mathbf{w}_h(P_j)|^2 + \frac{1}{27}[\sum_j \mathbf{w}_h(P_j)]^2\}. \end{aligned}$$

Thus I_K is nonnegative which proves the first assertion. The last equation implies

$$(\mathbf{w}_h, \gamma_h \mathbf{w}_h) \geq \sum_K (|K|\frac{6}{27}\sum_j |\mathbf{w}_h(P_j)|^2) = \frac{6}{9}||\mathbf{w}_h||_0^2. \quad \square$$

Remark 2.2. Lemma 2.3 is the only place we use the fact that two vertices of a dual element should be from barycenters. Another advantage of choosing barycenters over circumcenters (as in the Delaunay-Voronoi grids for the covolume methods [15]) is that they are inside their corresponding triangles, which is not necessarily the case for the circumcenters.

Lemma 2.4.

$$B(\mathbf{w}_h, q_h) = b(\gamma_h \mathbf{w}_h, q_h) = c(\mathbf{w}_h, q_h) \qquad \forall \mathbf{w}_h \in \mathbf{H}_h, q_h \in L_h.$$

Proof.

$$\begin{aligned} B(\mathbf{w}_h, q_h) = b(\gamma_h \mathbf{w}_h, q_h) &= \sum_1^{N_S} \mathbf{w}(P_i) \cdot \int_{\partial K_{P_i}^*} q_h \mathbf{n}\, d\sigma \\ &= \sum_{K \in T_h} I_K, \end{aligned}$$

where

$$\begin{aligned}
I_K &= \sum_{j=1}^{3} \int_{A_{j+1}BA_j} q_h \mathbf{w}_h(P_j) \cdot \mathbf{n}\, d\sigma \\
&= \sum_{j=1}^{3} \left[\int_{\Delta A_{j+1}BA_j} \operatorname{div}\left(q_h \mathbf{w}_h(P_j)\right) d\sigma - \int_{A_jA_{j+1}} q_h \mathbf{w}_h(P_j) \cdot \mathbf{n}\, d\sigma \right] \\
&= \sum_{j=1}^{3} \left[0 - \int_{A_jA_{j+1}} q_h \mathbf{w}_h(P_j) \cdot \mathbf{n}\, d\sigma \right] \\
&= -\sum_{j=1}^{3} (q_h \mathbf{w}_h(P_j) \cdot \mathbf{n}) |A_jA_{j+1}| \\
&= -\sum_{j=1}^{3} \left(q_h \frac{\mathbf{w}_h(A_j) + \mathbf{w}_h(A_{j+1})}{2} \cdot \mathbf{n} \right) |A_jA_{j+1}| \\
&= -\sum_{j=1}^{3} \int_{A_jA_{j+1}} q_h \mathbf{w}_h(x) \cdot \mathbf{n}\, d\sigma \\
&= -q_h \int_K \operatorname{div} \mathbf{w}_h \, dx. \quad \square
\end{aligned}$$

Remark 2.3. Lemma 2.4 was implicitly used in [4, 19] and is central to the first order accurate network method therein. Part of Section 4 is dedicated to this issue where it is related to the incidence matrix of a network. See also Lemmas 4.1, 4.2 below.

Due to Lemma 2.4, the bilinear form B becomes a well known form in the analysis of a nonconforming mixed method applied to the Stokes problem [7]. The validity of the following inf-sup condition is verified in [7].

Lemma 2.5. *There exists a positive constant β independent of h such that*

$$\sup_{\mathbf{w}_h \neq \mathbf{0}} \frac{B(\mathbf{w}_h, q_h)}{|\mathbf{w}_h|_{1,h}} \geq \beta ||q_h||_0. \tag{2.11}$$

We shall also need the following approximation property of the transfer operator γ_h.

Lemma 2.6. *There exists a positive constant C_0 independent of h such that*

$$||\gamma_h \mathbf{w}_h - \mathbf{w}_h||_0 \leq C_0 h\, |\mathbf{w}_h|_{1,h} \quad \forall \mathbf{w}_h \in \mathbf{H}_h. \tag{2.12}$$

Proof.

$$||\gamma_h \mathbf{w}_h - \mathbf{w}_h||_0^2 = \sum_K \int_K |\gamma_h \mathbf{w}_h(x) - \mathbf{w}_h(x)|^2 dx.$$

Referring to Fig. 2, writing $\mathbf{w}_h = (w_h^1, w_h^2)^t$ and using Taylor's expansion, we have

$$\begin{aligned}\int_K |\gamma_h \mathbf{w}_h(x) - \mathbf{w}_h(x)|^2 dx &= \sum_{j=1}^{3} \int_{\Delta A_j A_{j+1} B} |\gamma_h \mathbf{w}_h(x) - \mathbf{w}_h(x)|^2 dx \\ &= \sum_{j=1}^{3} \int_{\Delta A_j A_{j+1} B} |\mathbf{w}_h(P_j) - \mathbf{w}_h(x)|^2 dx \\ &= \sum_{j=1}^{3} \int_{\Delta A_j A_{j+1} B} \sum_{k=1}^{2} \left(|w_h^k(P_j) - w_h^k(x)|^2\right) dx \\ &= \sum_{j=1}^{3} \int_{\Delta A_j A_{j+1} B} \sum_{k=1}^{2} \left(|\nabla w_h^k(x) \cdot (P_j - x)|^2\right) dx \\ &= \sum_{j=1}^{3} \int_{\Delta A_j A_{j+1} B} \sum_{k=1}^{2} \left(|\nabla w_h^k(x)|^2 |(P_j - x)|^2\right) dx \\ &\leq C_0 h^2 \sum_{j=1}^{3} \int_{\Delta A_j A_{j+1} B} |\nabla \mathbf{w}_h(x)|^2 dx. \quad \square\end{aligned}$$

The following lemma is a special case of Lemma 2.3 in ([9], p. 591).

Lemma 2.7. *There exists a constant $C > 0$ independent of h such that*

$$||\mathbf{z}_h||_0 \leq C|\mathbf{z}_h|_{1,h}, \quad \forall \mathbf{z}_h \in \mathbf{H}_h, \tag{2.13}$$

provided that the triangulation family is quasi-uniform.

Remark 2.4. The quasi-uniformity is essential for this Poincaré inequality on the nonconforming space $\mathbf{H}_h$ to hold since an inverse inequality is used in the proof. For its conforming counterpart no such restriction is needed.

3. Error estimates

We now prove the main theorem of this paper.

Theorem 3.1. *Let the triangulation family of the domain Ω be quasi-uniform, let $\{\mathbf{u}_h, p_h\}$ be the solution of the problem (1.24)-(1.25), and $\{\mathbf{u}, p\}$ solve the problem (1.6)-(1.7). Then there exists a positive constant C independent of h such that*

$$|\mathbf{u} - \mathbf{u}_h|_{1,h} + ||p - p_h||_0 \leq Ch(||\mathbf{u}||_2 + ||p||_1 + 1), \tag{3.1}$$

provided that $\mathbf{u} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega), p \in H^1(\Omega)$. Furthermore,

$$||\mathbf{u} - \mathbf{u}_h||_0 \leq Ch(||\mathbf{u}||_2 + ||p||_1 + 1). \tag{3.2}$$

Proof. Lemmas 2.3 and 2.5 guarantee the existence and uniqueness of the solution $\{\mathbf{u}_h, p_h\}$. We first introduce an auxiliary symmetric Stokes approximation problem to (1.6)-(1.7): Find $(\tilde{\mathbf{u}}_h, \tilde{p}) \in \mathbf{H}_h \times L_h$ such that

$$A^S(\tilde{\mathbf{u}}_h, \mathbf{w}_h) + \alpha_0(\tilde{\mathbf{u}}_h, \mathbf{w}_h) + B(\mathbf{w}_h, \tilde{p}_h) = (\mathbf{f}, \mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{H}_h, \tag{3.3}$$

$$B(\tilde{\mathbf{u}}_h, q_h) = 0 \quad \forall q_h \in L_h. \tag{3.4}$$

This is a well known nonconforming method and we have the following convergence result. (Cf. pp. 231, 246, [3] for the $\alpha_0 = 0$ case whose proof can be easily carried over to the nonzero case; just redefine their $a_h(\mathbf{v}, \mathbf{w})$ as $A^s(\mathbf{v}, \mathbf{w}) + \alpha_0(\mathbf{v}, \mathbf{w})$.)

$$(3.5) \qquad |\mathbf{u} - \tilde{\mathbf{u}}_h|_{1,h} + \alpha_0^{1/2}||\mathbf{u} - \tilde{\mathbf{u}}_h||_0 + ||p - \tilde{p}_h||_0 \le Ch(||\mathbf{u}||_2 + ||p||_1),$$

provided that $\mathbf{u} \in \mathbf{H}^2(\Omega), p \in H^1(\Omega)$. On the other hand,

$$(3.6) \qquad A^S(\mathbf{u}_h, \mathbf{w}_h) + \alpha_0(\mathbf{u}_h, \gamma_h \mathbf{w}_h) + B(\mathbf{w}_h, p_h) = (\mathbf{f}, \gamma_h \mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{H}_h,$$

$$(3.7) \qquad B(\mathbf{u}_h, q_h) = 0 \quad \forall q_h \in L_h.$$

Subtracting (3.4) from (3.7) gives

$$(3.8) \qquad B(\mathbf{u}_h - \tilde{\mathbf{u}}_h, q_h) = 0 \quad \forall q_h \in L_h.$$

Subtracting (3.3) from (3.6) gives

$$(3.9) \qquad \begin{aligned} A^S(\mathbf{u}_h - \tilde{\mathbf{u}}_h, \mathbf{w}_h) + \alpha_0(\mathbf{u}_h, \gamma_h \mathbf{w}_h) - \alpha_0(\tilde{\mathbf{u}}_h, \mathbf{w}_h) + B(\mathbf{w}_h, p_h - \tilde{p}_h) \\ = (\mathbf{f}, \gamma_h \mathbf{w}_h) - (\mathbf{f}, \mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{H}_h. \end{aligned}$$

Define

$$\tilde{\mathbf{e}}_h := \mathbf{u}_h - \tilde{\mathbf{u}}_h.$$

Replace the $\mathbf{w}_h$ in (3.6) with $\tilde{\mathbf{e}}_h$ and use (3.8), Lemma 2.2 to obtain

$$(3.10) \qquad |\tilde{\mathbf{e}}_h|_{1,h}^2 + \alpha_0(\tilde{\mathbf{e}}_h, \gamma_h \tilde{\mathbf{e}}_h) = (\mathbf{f}, \gamma_h \tilde{\mathbf{e}}_h - \tilde{\mathbf{e}}_h) + \alpha_0(\tilde{\mathbf{u}}_h, \tilde{\mathbf{e}}_h - \gamma_h \tilde{\mathbf{e}}_h).$$

Use (2.10) on the second term of the left-hand side, Lemma 2.6, and $||\tilde{\mathbf{u}}_h||_0 \le M$ to obtain

$$(3.11) \qquad |\tilde{\mathbf{e}}_h|_{1,h}^2 \le ||\mathbf{f}||_0 C_0 h |\tilde{\mathbf{e}}_h|_{1,h} + C_0 \alpha_0 M h |\tilde{\mathbf{e}}_h|_{1,h}.$$

Hence

$$(3.12) \qquad |\tilde{\mathbf{e}}_h|_{1,h} \le Ch,$$

where C may depend on $\mathbf{f}, \mathbf{u}$, but not on h. Combining this with (3.5) and using the triangle inequality gives

$$|\mathbf{u} - \mathbf{u}_h|_{1,h} \le Ch(||\mathbf{u}||_2 + ||p||_1 + 1).$$

We can use the inf-sup condition on Eq. (3.6) and the same techniques as above to derive

$$||p_h - \tilde{p}_h||_0 \le C_1 h.$$

An application of the triangle inequality then proves (3.1).

If $\alpha_0 \ne 0$ then (2.9), (3.10) and (3.12) imply

$$C_1 \alpha_0 ||\tilde{\mathbf{e}}_h||_0^2 \le \alpha_0(\tilde{\mathbf{e}}_h, \gamma_h \tilde{\mathbf{e}}_h) \le C_2 h |\tilde{\mathbf{e}}_h|_{1,h} \le C_3 h^2,$$

and hence

$$||\tilde{\mathbf{e}}_h||_0 \le Ch,$$

which upon combining with (3.5) gives (3.2).

If $\alpha_0 = 0$, then setting $\mathbf{z}_h = \tilde{\mathbf{e}}_h$ in Lemma 2.7 and using (3.12) derive the last inequality again. However, this time we cannot use (3.5) directly. Now, let

$\pi_h \mathbf{u} \in \mathbf{H}_0^1$ be the piecewise continuous linear interpolant of $\mathbf{u}$. Using Lemma 2.7, approximation properties of the interpolant, and (3.5), we have

$$\begin{aligned}
||\mathbf{u} - \tilde{\mathbf{u}}_h||_0 &\leq ||\mathbf{u} - \pi_h \mathbf{u}||_0 + ||\pi_h \mathbf{u} - \tilde{\mathbf{u}}_h||_0 \\
&\leq ||\mathbf{u} - \pi_h \mathbf{u}||_0 + C_4 |\pi_h \mathbf{u} - \tilde{\mathbf{u}}_h|_{1,h} \\
&\leq ||\mathbf{u} - \pi_h \mathbf{u}||_0 + C_4 |\pi_h \mathbf{u} - \mathbf{u}|_{1,h} + C_4 |\mathbf{u} - \tilde{\mathbf{u}}_h|_{1,h} \\
&\leq (C_5 h^2 + C_6 h) ||\mathbf{u}||_2 + C_4 |\mathbf{u} - \tilde{\mathbf{u}}_h|_{1,h} \\
&\leq C_7 h (||\mathbf{u}||_2 + ||p||_1).
\end{aligned}$$

A simple application of the triangle inequality now proves (3.2). □

Remark 3.1. Note that we can symmetrize the problem (1.24)-(1.25) by replacing $(\gamma_h \mathbf{v}_h, \mathbf{w}_h)$ by $\frac{1}{2}[(\gamma_h \mathbf{v}_h, \mathbf{w}_h) + (\mathbf{v}_h, \gamma_h \mathbf{w}_h)]$ and still obtain the same optimal error estimate in the above theorem.

4. A FLOW NETWORK MODEL FOR SADDLE-POINT PROBLEMS

In this section we introduce a "P-recovering" technique for solving the general saddle-point linear system in the unknown $\{U, P\}$ of the form

$$\tilde{A} U + \tilde{B}^t P = F, \tag{4.1}$$

$$\tilde{B} U = 0, \tag{4.2}$$

where $\tilde{A} \in R^{2N_S \times 2N_S}$ is symmetric positive definite, $\tilde{B} \in R^{N_T \times 2N_S}$, $\dim \ker \tilde{B}^t \geq 1$, and $\tilde{B}$ has the decomposition

$$\tilde{B} = \mathcal{A} \tilde{D} \text{ for some } \mathcal{A} \in R^{N_T \times N_S}, \tilde{D} \in R^{N_S \times 2N_S}. \tag{4.3}$$

The dimension of $\ker \tilde{B}^t$ is usually one in fluid mechanics applications in which P is the pressure and hence determined up to a constant vector of identical components. If $\operatorname{Ker} \tilde{B}^t = \{0\}$, the most basic method for solving such a system is the so-called pressure-matrix method and its preconditioned variants. In the pressure-matrix method we write the above system as

$$\begin{aligned}
\tilde{B} \tilde{A}^{-1} \tilde{B}^t P &= \tilde{B} \tilde{A}^{-1} F, \\
U &= \tilde{A}^{-1} (F - \tilde{B}^t P),
\end{aligned}$$

and solve for P and then for U. Obviously the *pressure matrix* $R := \tilde{B} \tilde{A}^{-1} \tilde{B}^t$ is only semidefinite since $\tilde{B}^t$ in (4.1) is not of full rank. Thus the conjugate gradient method cannot be used without caution. On the other hand, the augmented Lagrangian method (pp. 49-51, [11]) works well for nonzero $\operatorname{Ker} \tilde{B}^t$ space and produces a sequence of U^k convergent to U and P^k convergent to the minimum norm pressure P_M. By interpreting system (4.1)-(4.2) as a network model we show how to recover the P once the solution U and $\mathcal{A}^t P$ are known. The augmented Lagrangian method applied to (4.1)-(4.2) reads: Given $P^0 \in R^{N_T}$, with P^n known, calculate U^n, then P^{n+1}, by

$$(\tilde{A} + r \tilde{B}^t \tilde{B}) U^n + \tilde{B}^t P^n = F, \tag{4.4}$$

$$P^{n+1} = P^n + \rho_n \tilde{B} U^n, \tag{4.5}$$

where r and ρ_n are two parameters. In Theorem 4.1 we show that so long as the definition of B is used in discretizing the divergence-free condition and the pressure-gradient term, the decomposition $\tilde{B} = \mathcal{A} \tilde{D}$ holds where $\mathcal{A}$ is the incidence

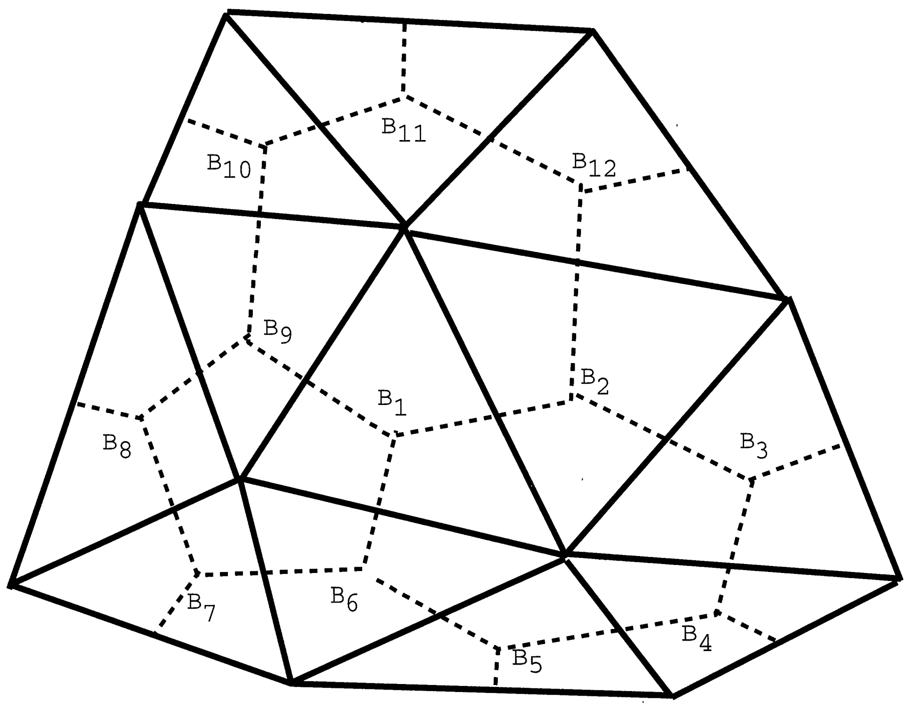

FIGURE 3. A network model for a fluid domain

matrix [cf. (4.10)], and $\tilde{D} = (D_1, D_2)$ is rectangular and consists of two diagonal matrices D_1, D_2 [cf. Eq. (4.20)]. Upon using the decomposition and introducing a new variable $\mathcal{A}^t P$, the pressure drops, (4.4)-(4.5) reads: Given $\tilde{P}^0 \in R^{N_S}$, with $\tilde{P}^n$ known, calculate U^n, then $\tilde{P}^{n+1}$, by

$$(\tilde{A} + r\tilde{B}^t\tilde{B})\,U^n + \tilde{D}^t\tilde{P}^n = F, \tag{4.6}$$

$$\tilde{P}^{n+1} = \tilde{P}^n + \rho_n(\mathcal{A}^t\mathcal{A})\tilde{D}U^n. \tag{4.7}$$

Since $\mathcal{A}$ is sparse and $\tilde{D}$ is "diagonal", this saves work due to the $\tilde{D}^t\tilde{P}^n$ and $\tilde{D}U^n$ terms. Also it is well known (p. 92, Strang [20])and can be easily checked that the matrix $\mathcal{A}^t\mathcal{A}$ is sparse; its jk-th entry being either 0 or -1 depending whether or not the network has a link from node k to j. Finally, we can delay recovering the solution P until the last step when we have obtained an acceptable $\tilde{P} = \mathcal{A}^t P$.

From the discussion above we see that the transformed augmented Lagrangian method is applicable to linear systems resulted from the covolume elements in this paper and the nonconforming Crouzeix-Raviart elements. For ease of exposition we shall illustrate the ideas using (1.24)-(1.25) with $\alpha_0 = 0$. Hence the system resulting from an application of the Crouzeix-Raviart nonconforming method [7] will also be covered. The general case can be handled easily once the underlying principles are presented.

First we need the idea of a network [20]. We view a graph as a set of nodes and links. A network is a digraph with link and node based quantities. We construct a network induced by the problem domain as follows. Globally order the sides of the triangles in the flow domain partition (cf. Fig. 3) as $e_i, i = 1, \cdots, N$, the triangles as $j = 1, \cdots, N_T$, and the midpoints of e_i as $P_i, i = 1, \cdots, N_S$. Assign a unit normal vector $\mathbf{n}_i$ to side e_i. There are two choices, but once chosen the $\mathbf{n_i}$ is fixed. Denote by Γ the digraph whose vertices are $B_i, i = 1, \cdots, N_T$, and whose links are B_iB_j or B_jB_i, depending on which of the two has positive component along $\mathbf{n}_k$ where k is the unique side that intersects the line segment B_iB_j.

Lemma 4.1. *The discrete continuity equation* (1.25)

$$B(\mathbf{u}_h, q_h) = -\int_\Omega \operatorname{div} \mathbf{u}_h q_h \, dx = 0 \quad \forall q_h \in L_h \tag{4.8}$$

is implied by

$$-\mathcal{A}D\mathbf{u}^n = \mathbf{0}, \tag{4.9}$$

where $u_j^n := \mathbf{u}_h(P_j) \cdot \mathbf{n}_j$*, the* $N_S \times N_S$ *matrix* $D := \operatorname{diag}(h_j)$*,* $h_j = |e_j|$*, and the* $N_T \times N_S$ *incidence matrix* $\mathcal{A}$ *is defined as*

$$a_{ij} = \begin{cases} 1, & \text{if } \mathbf{n}_j \text{ is an outward normal on side } j \text{ of triangle } i, \\ -1, & \text{if } \mathbf{n}_j \text{ is an inward normal on side } j \text{ of triangle } i, \\ 0, & \text{if } \mathbf{n}_j \text{ is not associated with triangle } i. \end{cases} \tag{4.10}$$

Before proving the lemma, two remarks are in order.

Remark 4.1. See also (4.31) where the velocity of (4.9) is expressed in terms of the Cartesian coordinate system. Thus $\mathcal{A}\tilde{D}$ can be thought of as the discrete divergence operator.

Remark 4.2. The phrase "is implied by" means essentially "is equivalent to." Recall that L_h, the set of all piecewise constants with the zero mean, is used to guarantee the uniqueness of the pressure approximation. It is inconvenient to derive the corresponding algebraic continuity equation in terms of this space. Rather, the space of all piecewise constants suffices so long as we remember when the algebraic continuity equation is combined with the discrete momentum equation (4.17) below only the pressure drops or differences $\mathcal{A}^t\mathbf{p}$ are unique. Note that the i-th component of $\mathcal{A}^t\mathbf{p}$ is the pressure drop across the side e_i.

Proof. We can represent the discrete continuity equation in terms of the characteristic basis functions of the triangles. For a triangle $K = K_i$ whose barycenter is B_i, (4.8) reads

$$\begin{aligned} \int_K \operatorname{div} \mathbf{u}_h \, dx &= \int_{\partial K} \mathbf{u}_h \cdot \mathbf{n} \, d\sigma \\ &= \sum_{k=1}^{3} \mathbf{u}_h(P_{i_k}) \cdot \tilde{\mathbf{n}}_{i_k} |e_{i_k}| \\ &= \sum_{k=1}^{3} [\mathbf{u}_h(P_{i_k}) \cdot \mathbf{n}_{i_k}] \mathbf{n}_{i_k} \cdot \tilde{\mathbf{n}}_{i_k} |e_{i_k}| = 0, \end{aligned} \tag{4.11}$$

where P_{i_k} are the midpoints; $\tilde{\mathbf{n}}_{i_k}$ the three outward unit normals; i_k the global indices of the sides in the network. Note that the tangential component is irrelevant. Now observe a_{ij} is nonzero only for $j \in \text{Nbd}_i := \{i_1, i_2, i_3\}$ associated with the three sides of K_i and for these three j's

$$\mathbf{n}_j \cdot \tilde{\mathbf{n}}_j = a_{ij}, \tag{4.12}$$

where $\tilde{\mathbf{n}}_j$ is the unit outward normal to side e_j.

On the other hand, the i-th equation in (4.9), ignoring the negative sign, reads

$$\begin{aligned}\sum_{j=1}^{N_S} a_{ij}|e_j|\mathbf{u}_h(P_j)\cdot\mathbf{n}_j &= \sum_{j\in \mathrm{Nbd}_i} a_{ij}|e_j|\mathbf{u}_h(P_j)\cdot\mathbf{n}_j \\ &= \sum_{j\in \mathrm{Nbd}_i} \mathbf{n}_j\cdot\tilde{\mathbf{n}}_j|e_j|\mathbf{u}_h(P_j)\cdot\mathbf{n}_j.\end{aligned}$$

Comparing this with (4.11) proves the assertion. □

Define a basis for $\mathbf{H}_h$:

$$\Phi_j^{(1)} := \begin{pmatrix}\phi_j\\ 0\end{pmatrix}, \qquad \Phi_j^{(2)} := \begin{pmatrix}0\\ \phi_j\end{pmatrix}, \tag{4.13}$$

where $\phi_j, j = 1, \cdots, N_S$, are the global Lagrange basis functions associated with P_j. Hence

$$\mathbf{u}_h(x) = \sum_{1}^{N_S}\sum_{k=1}^{2} u_h^k(P_j)\Phi_j^{(k)} \tag{4.14}$$

where

$$u_h^k(P_j) = \mathbf{u}_h(P_j)\cdot\mathbf{e}^{(k)}, \tag{4.15}$$

$$\mathbf{e}^{(1)} = \begin{pmatrix}1\\ 0\end{pmatrix}, \mathbf{e}^{(2)} = \begin{pmatrix}0\\ 1\end{pmatrix}.$$

Define the gradient related matrices $G^{(k)}$ and $\mathcal{G}^{(k)}, k = 1, 2$, by

$$G_{jl}^{(k)} := (\nabla_h\Phi_l^{(1)}, \nabla_h\Phi_j^{(k)}); \qquad \mathcal{G}_{jl}^{(k)} := (\nabla_h\Phi_l^{(2)}, \nabla_h\Phi_j^{(k)}), \tag{4.16}$$

and hence $G^{(1)} = \mathcal{G}^{(2)}, G^{(2)} = \mathcal{G}^{(1)}$.

Lemma 4.2. *The discrete momentum equation* (1.24) *with* $\alpha_0 = 0$

$$(\nabla_h\mathbf{u}_h, \nabla_h\mathbf{w}_h) + B(\mathbf{w}_h, p_h) = (\mathbf{f}, \gamma_h\mathbf{w}_h) \qquad \forall \mathbf{w}_h \in \mathbf{H}_h \tag{4.17}$$

is equivalent to

$$G^{(1)}\mathbf{u}^{(1)} + G^{(2)}\mathbf{u}^{(2)} + D_1\mathcal{A}^t\mathbf{p} = \mathbf{f}_1, \tag{4.18}$$

$$G^{(2)}\mathbf{u}^{(1)} + G^{(1)}\mathbf{u}^{(2)} + D_2\mathcal{A}^t\mathbf{p} = \mathbf{f}_2, \tag{4.19}$$

where for $k = 1, 2$, $\mathbf{u}^{(k)}$ *is the vector whose* j*-th component is* $\mathbf{u}_h(P_j)\cdot\mathbf{e}^{(k)}, j = 1, \cdots, N_S$, *and* $\mathbf{p}$ *is the vector whose* j*-th component is* $p_h(B_j), j = 1, \cdots, N_T$, *the matrices*

$$D_1 := \mathrm{diag}\left(\mathbf{e}^{(1)}\cdot\mathbf{n}_j|e_j|\right), \qquad D_2 := \mathrm{diag}\left(\mathbf{e}^{(2)}\cdot\mathbf{n}_j|e_j|\right), \tag{4.20}$$

$$\mathbf{f}_i := (\mathbf{f}, \gamma_h\Phi^{(i)}), \qquad i = 1, 2, \tag{4.21}$$

and $G^{(i)}, i = 1, 2$, *are defined in* (4.16).

Proof. We shall express (4.17) in terms of the basis functions of (4.13), with (4.18) from (4.17) in terms of the basis $\Phi_j^{(1)}, j = 1, \cdots, N_S$, and (4.19) from (4.17) in terms of $\Phi_j^{(2)}, j = 1, \cdots, N_S$. We show only how to arrive at those pressure terms. Using the techniques in proving Lemma 4.1 , we have for $\mathbf{w}_h \in \mathbf{H}_h$

$$
\begin{aligned}
B(\mathbf{w}_h, p_h) &= \sum_{j=1}^{N_T} p_h(B_j) \int_{\partial K_{B_j}} \mathbf{w}_h \cdot \mathbf{n}\, dx \\
&= \sum_{j=1}^{N_T} p_h(B_j) \sum_{k \in \mathrm{Nbd}_j} \mathbf{w}_h(P_k) \cdot \tilde{\mathbf{n}}_k |e_k| \\
&= \sum_{j=1}^{N_T} p_h(B_j) \sum_k \{\mathbf{w}_h(P_k) \cdot \mathbf{n}_k |e_k|\}\, \mathbf{n}_k \cdot \tilde{\mathbf{n}}_k, \\
&= \sum_{j=1}^{N_T} \sum_{k=1}^{N_S} p_h(B_j) \{\mathbf{w}_h(P_k) \cdot \mathbf{n}_k |e_k|\}\, a_{jk},
\end{aligned}
\tag{4.22}
$$

where we used (4.12) in the last equality. Setting $\mathbf{w}_h = \Phi_l^{(1)}, 1 \le l \le N_S$, and evaluating derive (4.18). Likewise, setting $\mathbf{w}_h = \Phi_l^{(2)}, 1 \le l \le N_S$, and evaluating derive (4.19). □

On each side $e_j, j = 1, \cdots, N_S$, let

$$\mathbf{e}^{(1)} \cdot \mathbf{n}_j = \cos\theta_j \text{ and hence } \mathbf{e}^{(2)} \cdot \mathbf{n}_j = \sin\theta_j. \tag{4.23}$$

Then on side e_j the velocity $\mathbf{u}_h(P_j)$ in the Cartesian coordinate system $(\mathbf{e}^{(1)}, \mathbf{e}^{(2)})$ and the same velocity in the normal-tangential right-handed coordinate system $(\mathbf{n}_j, \mathbf{t}_j)$ are related by

$$\mathbf{u}_j^{(1)} = \cos\theta_j \mathbf{u}_j^n - \sin\theta_j \mathbf{u}_j^t, \tag{4.24}$$

$$\mathbf{u}_j^{(2)} = \sin\theta_j \mathbf{u}_j^n + \cos\theta_j \mathbf{u}_j^t. \tag{4.25}$$

In matrix form, we have

$$\mathbf{u}^{(1)} = \mathcal{D}_c \mathbf{u}^n - \mathcal{D}_s \mathbf{u}^t, \tag{4.26}$$

$$\mathbf{u}^{(2)} = \mathcal{D}_s \mathbf{u}^n + \mathcal{D}_c \mathbf{u}^t, \tag{4.27}$$

where the N_S by N_S matrices

$$\mathcal{D}_c := \mathrm{diag}(\cos\theta_j), \quad \mathcal{D}_s := \mathrm{diag}(\sin\theta_j). \tag{4.28}$$

Thus with the notation of Lemmas 4.1 and 4.2 the linear system we want to solve is:

Find $(\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{p}) \in R^{N_S} \times R^{N_S} \times R^{N_T}$ such that

$$G^{(1)}\mathbf{u}^{(1)} + G^{(2)}\mathbf{u}^{(2)} + D_1 \mathcal{A}^t \mathbf{p} = \mathbf{f}_1, \tag{4.29}$$

$$G^{(2)}\mathbf{u}^{(1)} + G^{(1)}\mathbf{u}^{(2)} + D_2 \mathcal{A}^t \mathbf{p} = \mathbf{f}_2, \tag{4.30}$$

and

$$\mathcal{A}(D_1 \mathbf{u}^{(1)} + D_2 \mathbf{u}^{(2)}) = 0. \tag{4.31}$$

Setting $U = (\mathbf{u}^{(1)}, \mathbf{u}^{(2)})^t, P = \mathbf{p}, F = (\mathbf{f}_1, \mathbf{f}_2)^t$,

$$\tilde{D} := (D_1, D_2), \tag{4.32}$$

$$\tilde{A} := \begin{pmatrix} G^{(1)} & G^{(2)} \\ G^{(2)} & G^{(1)} \end{pmatrix}$$

and

$$\tilde{B} = \mathcal{A}\tilde{D}.$$

Hence

Theorem 4.1. *With the notation of Lemmas* 4.1 *and* 4.2, *the algebraic system of the covolume method with* $\alpha_0 = 0$ *can be represented as a saddle-point system with a decomposition on the discrete divergence operator:*

(P1) *Find* (U, P) *such that*

$$\tilde{A}U + \tilde{B}^t P = F, \tag{4.33}$$

$$\tilde{B}U = 0, \tag{4.34}$$

where $\tilde{B} = \mathcal{A}\tilde{D}$.

Remark 4.3. The matrix $\tilde{B}$ is really the discrete divergence operator and it has a convenient decomposition. The matrix $-\tilde{B}^t$ is the discrete gradient operator acting on the pressure space and Lemma 2.4 makes the saddle-point form possible. Now as stated in the beginning of this section we can use the transformed augmented Lagrangian method to solve this problem.

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