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ON SOME INEQUALITIES FOR THE GAMMA AND PSI FUNCTIONS

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ABSTRACT. We present new inequalities for the gamma and psi functions, and we provide new classes of completely monotonic, star-shaped, and super-additive functions which are related to Γ and ψ.

1

Euler's gamma function

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt \qquad (x > 0)$$

is one of the most important functions in analysis and its applications. The history and the development of this function are described in detail in a paper by P. J. Davis [10].

There exists a very extensive literature on the gamma function. In particular, numerous remarkable inequalities involving Γ and its logarithmic derivative $\psi = \Gamma'/\Gamma$ have been published by different authors; see, e.g., [2], [3], [6], [7], [9], [12], [13], [18]–[27], [29]–[33], [35]–[46], [50]. Many of these inequalities follow immediately from the monotonicity properties of functions which are closely related to Γ and ψ. In several recent papers [2], [9], [24], [39] it is proved that these functions are not only monotonic, but even completely monotonic. We recall that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I which alternate successively in sign, that is,

$$(1.1) \qquad (-1)^n f^{(n)}(x) \ge 0$$

for all $x \in I$ and for all $n \ge 0$. If inequality (1.1) is strict for all $x \in I$ and all $n \ge 0$, then f is said to be strictly completely monotonic.

It is known that completely monotonic functions play an eminent role in areas like probability theory [15], numerical analysis [49], physics [11], and the theory of special functions. For instance, M. E. Muldoon [39] showed how the notation of complete monotonicity can be used to characterize the gamma function. An interesting exposition of the main results on completely monotonic functions is given in [48].

"In view of the importance of completely monotonic functions ... it may be of interest to add to the available list of such functions" [24, p. 1]. It is the main

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purpose of this paper to present new classes of completely monotonic functions which are all closely related to the gamma and psi functions. Applications of our monotonicity theorems lead to new inequalities for Γ and ψ. Furthermore, we extend and sharpen known inequalities due to W. Gautschi, H. Minc and L. Sathre, and others, and we provide new classes of star-shaped and super-additive functions. In the final section we apply one of our results to present functions which are Laplace transforms of infinitely divisible probability measures.

2

In a recently published article G. D. Anderson et al. [3] proved that the function $f(x) = x(\log(x) - \psi(x))$ is strictly decreasing and strictly convex on $(0, \infty)$. Moreover, the authors presented (complicated) proofs for

$$\lim_{x\to 0} f(x) = 1 \quad \text{and} \quad \lim_{x\to\infty} f(x) = 1/2. \tag{2.1}$$

We note that the limits (2.1) follow immediately from the representations

$$f(x) = x\log(x) - x\psi(x+1) + 1$$

and

$$f(x) = \frac{1}{2} + \frac{1}{12x} - \frac{\theta}{120x^3} \qquad (0 < \theta < 1);$$

see [16, p. 824].

From (2.1) and the monotonicity of f we conclude

$$\frac{1}{2x} < \log(x) - \psi(x) < \frac{1}{x} \qquad (x > 0). \tag{2.2}$$

This extends a result of H. Minc and L. Sathre [37], who established (2.2) for $x > 1$, and used it to prove several discrete inequalities involving the geometric mean of the first n positive integers. Refinements of (2.2) were given by L. Gordon [22]. Our first theorem provides an extension of the result given by Anderson et al.; we prove that f is not only decreasing and convex, but even completely monotonic.

Theorem 1. *Let α be a real number. The function*

$$f_\alpha(x) = x^\alpha(\log(x) - \psi(x))$$

is strictly completely monotonic on $(0, \infty)$ if and only if $\alpha \le 1$.

Proof. First, we show that f_1 is strictly completely monotonic on $(0, \infty)$. Using Binet's formula [14, p. 18] we obtain the representation

$$f_1(x) = x\int_0^\infty \varphi(t)e^{-tx}\,dt, \tag{2.3}$$

where

$$\varphi(t) = 1/(1-e^{-t}) - 1/t.$$

Easy computations reveal that the function φ is strictly increasing on $(0, \infty)$ with $\lim_{t\to 0}\varphi(t) = 1/2$ and $\lim_{t\to\infty}\varphi(t) = 1$.

Let $n \geq 1$; from (2.3) we get

(2.4)

$$\begin{aligned}(-1)^n f_1^{(n)}(x) &= x(-1)^n \frac{d^n}{dx^n} \int_0^\infty \varphi(t) e^{-tx}\, dt \\ &\qquad - n(-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}} \int_0^\infty \varphi(t) e^{-tx}\, dt \\ &= x \int_0^\infty \varphi(t) e^{-tx} t^n dt - n \int_0^\infty \varphi(t) e^{-tx} t^{n-1}\, dt \\ &= \int_0^{n/x} \varphi(t) e^{-tx} t^{n-1}(tx-n)\, dt + \int_{n/x}^\infty \varphi(t) e^{-tx} t^{n-1}(tx-n)\, dt.\end{aligned}$$

If $0 < t < n/x$, then we obtain $\varphi(t) < \varphi(n/x)$; and if $n/x < t$, then we have $\varphi(n/x) < \varphi(t)$. Hence, from (2.4) we get

$$\begin{aligned}(-1)^n f_1^{(n)}(x) &> \varphi(n/x) \int_0^{n/x} e^{-tx} t^{n-1}(tx-n)\, dt \\ &\qquad + \varphi(n/x) \int_{n/x}^\infty e^{-tx} t^{n-1}(tx-n)\, dt \\ &= \varphi(n/x) \int_0^\infty e^{-tx} t^{n-1}(tx-n)\, dt.\end{aligned} \tag{2.5}$$

Using

$$\int_0^\infty e^{-tx} t^m\, dt = (m!)/x^{m+1} \qquad (x > 0; m = 0, 1, 2, \dots),$$

we conclude

$$\int_0^\infty e^{-tx} t^{n-1}(tx-n)\, dt = 0,$$

so that (2.5) implies

$$(-1)^n f_1^{(n)}(x) > 0 \quad \text{for } x > 0 \text{ and } n = 0, 1, 2, \dots.$$

From Leibniz' rule

$$(-1)^n (u(x)v(x))^{(n)} = \sum_{i=0}^n \binom{n}{i} (-1)^i u^{(i)}(x)(-1)^{n-i} v^{(n-i)}(x),$$

it follows that the product of two strictly completely monotonic functions is also strictly completely monotonic. Since $u_\alpha(x) = x^{\alpha-1}$ ($\alpha < 1$) is strictly completely monotonic on $(0, \infty)$, we conclude that $f_\alpha(x) = u_\alpha(x) f_1(x)$ ($\alpha \leq 1$) has the same property.

Next, we assume that f_α is strictly completely monotonic on $(0, \infty)$. Then we have for all $x > 0$:

$$f'_\alpha(x) = x^{\alpha-1}[\alpha(\log(x) - \psi(x)) + 1 - x\psi'(x)] < 0,$$

which implies

$$\alpha < \frac{x^2\psi'(x) - x}{x(\log(x) - \psi(x))}.$$

If we let x tend to 0, then we get $\alpha \leq 1$. The proof of Theorem 1 is complete. □

Anderson et al. [3] used the monotonicity of f_1 to prove that the function $g_1(x) = x^{1/2}(e/x)^x\Gamma(x)$ is decreasing on $(0,\infty)$, and that $g_2(x) = x(e/x)^x\Gamma(x)$ is increasing on $(0,\infty)$. The following theorem provides a slight extension of these results.

Theorem 2. *Let $a \geq 0$, r and s be real numbers. The function*

$$F_r(x) = x^r(e/x)^x\Gamma(x)$$

is decreasing on (a,∞) if and only if $r \leq 1/2$; and the function

$$G_s(x) = x^s(e/x)^x\Gamma(x)$$

is increasing on (a,∞) if and only if

$$s \geq \begin{cases} a(\log(a) - \psi(a)) & \text{if } a > 0, \\ 1 & \text{if } a = 0. \end{cases}$$

Proof. Since $F_r'(x) \leq 0$ is equivalent to

$$r \leq x(\log(x) - \psi(x)) = f_1(x),$$

the first part of Theorem 2 follows from the fact that f_1 is decreasing on $(0,\infty)$ and tends to $1/2$ if x tends to ∞. The second part can be proved similarly. We omit the details. □

Remark. Let g be a strictly completely monotonic function on $(0,\infty)$, and let c be a real number. From Theorem 1 we conclude that the function

$$x \mapsto g(x)(f_1(x) - c) \tag{2.6}$$

is strictly completely monotonic on $(0,\infty)$ if and only if $c \leq 1/2$. This extends a result of M. E. Muldoon [39], who proved the complete monotonicity of (2.6) for the special case $g(x) = 1/x$.

3

In 1974, C. H. Kimberling [28] established the following property of completely monotonic functions: If f is continuous on $[0,\infty)$ and completely monotonic on $(0,\infty)$ and satisfies $0 < f(x) \leq 1$ for all $x \geq 0$, then $\log(f)$ is super-additive on $[0,\infty)$.

We recall that a function g is said to be super-additive on an interval I if

$$g(x) + g(y) \leq g(x+y) \quad \text{for all } x, y \in I \text{ with } x + y \in I.$$

In the previous section we have proved that $f(x) = x(\log(x) - \psi(x))$ is continuous on $[0,\infty)$, completely monotonic on $(0,\infty)$, and $1/2 < f(x) \leq 1$ for all $x \geq 0$, so that Kimberling's theorem implies

$$1 \leq \frac{f(x+y)}{f(x)f(y)} \qquad (x, y \geq 0).$$

This leads to the problem to determine sharp upper and lower bounds for the ratio $f(x+y)/(f(x)f(y))$.

Theorem 3. *Let $f(x) = x(\log(x) - \psi(x))$. Then we have for all real $x, y \geq 0$:*

$$1 \leq \frac{f(x+y)}{f(x)f(y)} < 2. \tag{3.1}$$

Both bounds are best possible.

Proof. To prove the second inequality of (3.1) we define

$$g(x,y) = f(x+y)/f(x).$$

Partial differentiation yields

$$\frac{\partial g(x,y)}{\partial x} = \frac{f(x+y)}{f(x)}\left[\frac{f'(x+y)}{f(x+y)} - \frac{f'(x)}{f(x)}\right]. \tag{3.2}$$

Let

$$h(x,y) = f'(x+y)/f(x+y);$$

then we have

$$\frac{\partial h(x,y)}{\partial y} = [f''(x+y)f(x+y) - (f'(x+y))^2]/(f(x+y))^2. \tag{3.3}$$

Since completely monotonic functions are log-convex (see [17]), we conclude from (3.3) and Theorem 1 that $\partial h(x,y)/\partial y \geq 0$. This implies

$$h(x,y) \geq h(x,0), \tag{3.4}$$

so that (3.2) and (3.4) lead to

$$\frac{\partial g(x,y)}{\partial x} \geq 0 \quad \text{and} \quad g(x,y) \leq \lim_{x\to\infty} g(x,y) = 1.$$

Thus, we have

$$\frac{f(x+y)}{f(x)} \leq 1 < 2f(y) \quad \text{for } x,y \geq 0.$$

From

$$\lim_{y\to 0} \frac{f(x+y)}{f(x)f(y)} = 1$$

and

$$\lim_{y\to\infty}\lim_{x\to\infty} \frac{f(x+y)}{f(x)f(y)} = \lim_{y\to\infty} \frac{1}{f(y)} = 2,$$

we conclude that both bounds in (3.1) are sharp.

Remark. If we set

$$Q_\alpha(x,y) = f_\alpha(x+y)/(f_\alpha(x)f_\alpha(y)),$$

where $f_\alpha(x) = x^\alpha(\log(x) - \psi(x))$ and $\alpha \neq 1$, then we conclude from the limit relations

$$\lim_{y\to 0}\lim_{x\to\infty} Q_\alpha(x,y) = \lim_{y\to 0}\frac{1}{f_\alpha(y)} = \begin{cases} \infty & \text{if } \alpha > 1, \\ 0 & \text{if } \alpha < 1, \end{cases}$$

and

$$\lim_{y\to\infty}\lim_{x\to\infty} Q_\alpha(x,y) = \lim_{y\to\infty}\frac{1}{f_\alpha(y)} = \begin{cases} 0 & \text{if } \alpha > 1, \\ \infty & \text{if } \alpha < 1, \end{cases}$$

that the inequalities $0 \leq Q_\alpha(x,y) < \infty$ $(x,y > 0; \alpha \neq 1)$ cannot be refined.

4

In 1974, W. Gautschi [20] proved that the function $x \mapsto x\psi(x)$ is convex on $(0,\infty)$, and applied this result to establish some mean value inequalities involving the gamma function. Our next theorem provides an extension of Gautschi's proposition.

Theorem 4. *Let $n \geq 2$ be an integer. Then we have for all real $x > 0$:*

$$0 < (-1)^n x^{n-1}[x\psi(x)]^{(n)} < (n-2)!. \tag{4.1}$$

Both bounds are best possible.

Proof. Let $f(x) = x(\log(x) - \psi(x))$ and let $n \geq 2$. From Theorem 1 we obtain

$$\begin{aligned} 0 < (-1)^n f^{(n)}(x) &= (-1)^n (x\log(x))^{(n)} - (-1)^n (x\psi(x))^{(n)} \\ &= \frac{(n-2)!}{x^{n-1}} - (-1)^n (x\psi(x))^{(n)}, \end{aligned}$$

which leads to the second inequality of (4.1). Since

$$\psi^{(m)}(x) = (-1)^{m+1} m! \sum_{i=0}^{\infty} \frac{1}{(x+i)^{m+1}} \qquad (m = 1, 2, \dots),$$

we get

$$\begin{aligned} (-1)^n (x\psi(x))^{(n)} &= (-1)^n [x\psi^{(n)}(x) + n\psi^{(n-1)}(x)] \\ &= n! \sum_{i=1}^{\infty} \frac{i}{(x+i)^{n+1}} > 0, \end{aligned} \tag{4.2}$$

which implies the left-hand inequality of (4.1).

It remains to show that the bounds in (4.1) cannot be refined. Using $\psi^{(m)}(x) = \psi^{(m)}(x+1) + (-1)^{m+1} m!/x^{m+1}$ $(m = 0, 1, \dots)$, we get

$$(-1)^n x^{n-1} (x\psi(x))^{(n)} = (-1)^n x^{n-1} [x\psi^{(n)}(x+1) + n\psi^{(n-1)}(x+1)].$$

Hence, we have

$$\lim_{x \to 0} (-1)^n x^{n-1} (x\psi(x))^{(n)} = 0.$$

Let $m \geq 1$ be an integer; from

$$\begin{aligned} \frac{1}{mx^m} = \int_0^{\infty} \frac{dt}{(x+t)^{m+1}} \leq \sum_{i=0}^{\infty} \frac{1}{(x+i)^{m+1}} &\leq \frac{1}{x^{m+1}} + \int_0^{\infty} \frac{dt}{(x+t)^{m+1}} \\ &= \frac{1}{x^{m+1}} + \frac{1}{mx^m}, \end{aligned}$$

we conclude

$$\begin{aligned} (m-1)! \leq m! x^m \sum_{i=0}^{\infty} \frac{1}{(x+i)^{m+1}} &= -(-1)^m x^m \psi^{(m)}(x) \\ &\leq \frac{m!}{x} + (m-1)!, \end{aligned}$$

which implies

$$\lim_{x \to \infty} (-1)^m x^m \psi^{(m)}(x) = -(m-1)! \qquad (m \geq 1). \tag{4.3}$$

From (4.2) and (4.3) we obtain

$$\lim_{x\to\infty}(-1)^n x^{n-1}(x\psi(x))^{(n)} = (n-2)!.$$

Hence, both bounds in (4.1) are best possible. □

5

A function f is said to be star-shaped on $(0,\infty)$ if

$$f(ax) \le af(x) \tag{5.1}$$

is valid for all $x > 0$ and for all $a \in (0,1)$. These functions have been investigated intensively by A. M. Bruckner and E. Ostrow [8]. It is well known that star-shaped functions are super-additive. Indeed, from (5.1) we obtain $f(x) \le (x/(x+y))f(x+y)$ and $f(y) \le (y/(x+y))f(x+y)$; summing leads to $f(x)+f(y) \le f(x+y)$. In this section we answer the questions: For which real β is

$$x \mapsto \frac{(-1)^{k+1}x^\beta}{\psi^{(k)}(x) - (\log(x))^{(k)}} \qquad (0 \le k \in \mathbb{Z})$$

star-shaped; and for which β is this function super-additive?

Theorem 5. *Let $k \ge 0$ be an integer and let β be a real number. The function*

$$x \mapsto g_\beta(k;x) = \frac{(-1)^{k+1}x^\beta}{\psi^{(k)}(x) - (\log(x))^{(k)}}$$

is star-shaped on $(0,\infty)$ if and only if $\beta \ge -k$.

Proof. Let g_β be star-shaped on $(0,\infty)$. We assume (for a contradiction) that $\beta < -k$. We consider two cases. If $k = 0$, then inequality

$$g_\beta(0;ax) \le ag_\beta(0;x) \qquad (x > 0; 0 < a < 1)$$

and Theorem 1 imply that

$$0 < \log(x) - \psi(x) \le \frac{a^{-\beta}}{x}(ax)[\log(ax) - \psi(ax)]. \tag{5.2}$$

If we let a tend to 0, then we conclude from $\beta < 0$ that the product on the right-hand side of (5.2) tends to 0. Let $k \ge 1$; from

$$\begin{aligned}(-1)^{k+1}\psi^{(k)}(x) &= k!\sum_{i=0}^{\infty}\frac{1}{(x+i)^{k+1}} > k!\int_0^\infty \frac{dt}{(x+t)^{k+1}} \\ &= \frac{(k-1)!}{x^k} = (-1)^{k+1}(\log(x))^{(k)}\end{aligned} \tag{5.3}$$

and

$$g_\beta(k;ax) \le ag_\beta(k;x)$$

we obtain

$$\begin{aligned}0 &< x^{-\beta}\left[(-1)^{k+1}\psi^{(k)}(x) - \frac{(k-1)!}{x^k}\right] \\ &\le \frac{1}{x}[(-1)^{k+1}(ax)^{1-\beta}\psi^{(k)}(ax) - (k-1)!(ax)^{1-\beta-k}] \\ &= \frac{1}{x}[(-1)^{k+1}(ax)^{1-\beta}\psi^{(k)}(ax+1) + k!(ax)^{-\beta-k} \\ &\qquad - (k-1)!(ax)^{1-\beta-k}].\end{aligned} \tag{5.4}$$

Since $\beta < -k$, we conclude that each term on the right-hand side of (5.4) tends to 0 if a tends to 0. Hence, if g_β is star-shaped on $(0, \infty)$, then $\beta \geq -k$.

Next, we assume that $\beta \geq -k$; to prove

$$g_\beta(k; ax) \leq a g_\beta(k; x) \tag{5.5}$$

for $x > 0$ and $a \in (0, 1)$, we reconsider two cases.

Case 1: $k = 0$. Then inequality (5.5) is equivalent to

$$\log(x) - \psi(x) \leq a^{1-\beta}[\log(ax) - \psi(ax)] = F(a), \quad \text{say.}$$

It suffices to show that F is decreasing on $(0, 1]$. We obtain

$$a^\beta F'(a) = (1 - \beta)[\log(ax) - \psi(ax)] + 1 - (ax)\psi'(ax).$$

If we set

$$G(z) = (1 - \beta)[\log(z) - \psi(z)] + 1 - z\psi'(z) \qquad (z > 0),$$

then we conclude from (5.3) (with $k = 1$) and the right-hand side inequality of (4.1) (with $n = 2$) that

$$G'(z) = \beta(\psi'(z) - 1/z) + 1/z - (z\psi(z))'' > 0.$$

From (2.2) and (4.3) we get

$$G(z) < \lim_{z \to \infty} G(z) = 0,$$

which implies $F'(a) < 0$ for all $a \in (0, 1]$.

Case 2: $k \geq 1$. Then inequality (5.5) can be written as

$$H(1) \leq H(a), \tag{5.6}$$

where

$$H(a) = a^{1-\beta}[(-1)^{k+1}\psi^{(k)}(ax) - (k - 1)!/(ax)^k].$$

Differentiation yields

$$\begin{aligned} a^\beta H'(a) = {} & (1 - \beta)[(-1)^{k+1}\psi^{(k)}(ax) - (k - 1)!/(ax)^k] \\ & + (-1)^{k+1} ax\psi^{(k+1)}(ax) + k!/(ax)^k. \end{aligned} \tag{5.7}$$

We replace ax by z and denote the right-hand side of (5.7) by $J(z)$. Then we obtain

$$\begin{aligned} J'(z) = {} & (1 - \beta)[(-1)^{k+1}\psi^{(k+1)}(z) + k!/z^{k+1}] \\ & + (-1)^{k+1}\psi^{(k+1)}(z) + (-1)^{k+1} z\psi^{(k+2)}(z) - k!k/z^{k+1}. \end{aligned} \tag{5.8}$$

From the second inequality of (4.1) we obtain

$$\begin{aligned} k!/z^{k+1} & > (-1)^k (z\psi(z))^{(k+2)} \\ & = (-1)^k [z\psi^{(k+2)}(z) + (k + 2)\psi^{(k+1)}(z)]. \end{aligned} \tag{5.9}$$

Using (5.3), (5.8), and (5.9) we get

$$J'(z) > (\beta + k)[(-1)^k \psi^{(k+1)}(z) - k!/z^{k+1}] \geq 0.$$

Thus, J is strictly increasing on $(0, \infty)$. From (4.3) we conclude that $\lim_{z\to\infty} z^k J(z) = 0$, which implies that $J(z) \leq 0$ for all $z > 0$. Therefore, H in decreasing on $(0, 1]$ which leads to inequality (5.6). This completes the proof of Theorem 5. □

Theorem 6. *Let $k \geq 0$ be an integer and let β be a real number. The function*

$$x \mapsto g_\beta(k;x) = \frac{(-1)^{k+1}x^\beta}{\psi^{(k)}(x) - (\log(x))^{(k)}}$$

is super-additive on $(0,\infty)$ if and only if $\beta \geq -k$.

Proof. If $\beta \geq -k$, then we conclude from Theorem 5 that g_β is star-shaped, which implies that g_β is super-additive. Next, we suppose that

$$g_\beta(k;x) + g_\beta(k;y) \leq g_\beta(k;x+y) \tag{5.10}$$

holds for all $x, y > 0$. We set in (5.10) $x = y$ and obtain after simple manipulations

$$2^{-\beta} \leq \frac{x(\log(x) - \psi(x))}{2x(\log(2x) - \psi(2x))} \quad \text{if } k = 0,$$

and

$$2^{-\beta-k} \leq \frac{(-1)^k x^{k+1}\psi^{(k)}(x+1) + (k-1)!x - k!}{(-1)^k(2x)^{k+1}\psi^{(k)}(2x+1) + (k-1)!(2x) - k!} \quad \text{if } k \geq 1.$$

If we let x tend to 0, then we obtain $\beta \geq -k$. □

Remark. In 1989, S. Y. Trimble et al. [47] introduced an interesting subclass of the completely monotonic functions. A function g is called strongly completely monotonic on $(0,\infty)$ if

$$x \mapsto (-1)^n x^{n+1} g^{(n)}(x)$$

is nonnegative and decreasing on $(0,\infty)$ for $n = 0, 1, 2, \ldots$. The authors showed that these functions have a close connection to star-shaped functions. Indeed, one of their results states: If g is strongly completely monotonic on $(0,\infty)$ and $g \not\equiv 0$, then $1/g$ is star-shaped.

6

In the past many articles were published providing different inequalities for the ratio $\Gamma(x+1)/\Gamma(x+s)$, where $x > 0$ and $s \in (0,1)$; see, e.g., [2], [13], [18], [25], [26], [29]–[31], [45], [50]. In this section we present upper and lower bounds for the difference $\psi(x+1) - \psi(x+s)$. In 1972, Y. L. Luke [33] considered the special case $s = 1/2$. He pointed out that this difference can be represented in terms of Gauss' hypergeometric series

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$, namely,

$$\psi(x+1) - \psi(x+1/2) = \frac{1}{x+1/2}\,{}_2F_1(1, 2x+1; 2x+2; -1),$$

and used well-known Padé-approximation for ${}_2F_1$ to obtain rational bounds for $\psi(x+1) - \psi(x+1/2)$. By using a different approach we get the following sharp inequalities for $\psi(x+1) - \psi(x+s)$.

Theorem 7. *Let $n \geq 0$ be an integer and let $x > 0$ and $s \in (0,1)$ be real numbers. Then we have*

$$A_n(s;x) < \psi(x+1) - \psi(x+s) < A_n(s;x) + \delta_n(s;x), \tag{6.1}$$

where

$$A_n(s;x) = (1-s)\left[\frac{1}{x+s+n} + \sum_{i=0}^{n-1} \frac{1}{(x+i+1)(x+i+s)}\right]$$

and

$$\delta_n(s;x) = \frac{1}{x+n+s} \log \frac{(x+n)^{(x+n)(1-s)}(x+n+1)^{(x+n+1)s}}{(x+n+s)^{x+n+s}}.$$

Proof. From Theorem 4 we conclude that the function $h(x) = x\psi(x)$ is strictly convex on $(0,\infty)$. If we set in Jensen's inequality

$$h(su + (1-s)v) < sh(u) + (1-s)h(v) \qquad (u, v > 0; u \neq v; 0 < s < 1),$$

$u = x+1$ and $v = x$, and make use of the identity $\psi(x+1) - \psi(x) = 1/x$, then we get

$$\frac{1-s}{x+s} < \psi(x+1) - \psi(x+s). \tag{6.2}$$

Next, we replace in (6.2) x by $x+1$ and obtain the following sharpening of (6.2):

$$\frac{1-s}{x+s+1} + \frac{1-s}{(x+1)(x+s)} < \psi(x+1) - \psi(x+s).$$

Repeating this process n times we get

$$\frac{1-s}{x+s+n} + (1-s)\sum_{i=0}^{n-1} \frac{1}{(x+i+1)(x+i+s)} < \psi(x+1) - \psi(x+s),$$

that is, the left-hand inequality of (6.1). Using the same method of proof with $\tilde{h}(x) = x(\log(x) - \psi(x))$ instead of h, we obtain the second inequality of (6.1). We omit the details. □

Remark. A simple calculation shows that $\lim_{n\to\infty} \delta_n(s;x) = 0$.

7

In 1964, H. Minc and L. Sathre [37] proved that the inequalities

$$0 < \log\Gamma(x) - \left(x - \frac{1}{2}\right)\log(x) + x - \frac{1}{2}\log(2\pi) < \frac{1}{x} \tag{7.1}$$

are valid for $x > 1$. Since the function $\log\Gamma(x)$ is asymptotically equal to the (divergent) series

$$\left(x - \frac{1}{2}\right)\log(x) - x + \frac{1}{2}\log(2\pi) + \sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)x^{2i-1}},$$

where B_i $(i = 0, 1, 2, \dots)$ are Bernoulli numbers, defined by

$$\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} B_i \frac{t^i}{i!}$$

(see [16, p. 823]), it is natural to ask whether it is possible to determine the sign of

$$S_k(x) = \log\Gamma(x) - \left(x - \frac{1}{2}\right)\log(x) + x - \frac{1}{2}\log(2\pi) - \sum_{i=1}^{k} \frac{B_{2i}}{2i(2i-1)x^{2i-1}} \qquad (0 \le k \in \mathbb{Z}).$$

As by-products of the next theorem we obtain sgn $S_k(x) = (-1)^k$ for $x > 0$ and $k \ge 0$, and we get that (7.1) (with the upper bound $1/(12x)$) holds for all $x > 0$. Further refinements of (7.1) can be found in [22].

Muldoon [39] investigated $S_0(x)$ and proved that this function is completely monotonic on $(0, \infty)$. This result can be extended:

Theorem 8. *Let $n \ge 0$ be an integer. The functions*

$$F_n(x) = \log\Gamma(x) - \left(x - \frac{1}{2}\right)\log(x) + x - \frac{1}{2}\log(2\pi) - \sum_{i=1}^{2n} \frac{B_{2i}}{2i(2i-1)x^{2i-1}}$$

and

$$G_n(x) = -\log\Gamma(x) + \left(x - \frac{1}{2}\right)\log(x) - x + \frac{1}{2}\log(2\pi) + \sum_{i=1}^{2n+1} \frac{B_{2i}}{2i(2i-1)x^{2i-1}}$$

are strictly completely monotonic on $(0, \infty)$.

Proof. We only establish that F_n is strictly completely monotonic; the proof for G_n is similar. In [16, pp. 823–824] the following representations for F_n and F_n' are given:

$$F_n(x) = \frac{B_{4n+2}}{(4n+1)(4n+2)} \frac{\theta}{x^{4n+1}} \qquad (0 < \theta < 1)$$

and

$$F_n'(x) = -\frac{B_{4n+2}}{4n+2} \frac{\tilde{\theta}}{x^{4n+2}} \qquad (0 < \tilde{\theta} < 1).$$

Since $B_{4n+2} > 0$ (see [4, p. 267]), we obtain $F_n(x) > 0$ and $F_n'(x) < 0$ for $x > 0$. Let $k \ge 1$; differentiation yields

$$\begin{aligned} \frac{1}{k!}(-1)^{k+1}F_n^{(k+1)}(x) = &\sum_{i=0}^{\infty} \frac{1}{(x+i)^{k+1}} - \frac{1}{kx^k} - \frac{1}{2x^{k+1}} \\ &+ \frac{(-1)^{k+1}}{k!} \sum_{i=1}^{2n} \left[\frac{B_{2i}}{2i} \prod_{j=0}^{k-1}(-2i-j)\right] \frac{1}{x^{2i+k}}. \end{aligned} \tag{7.2}$$

To find a lower bound for this sum we make use of Euler's summation formula [1, p. 806]:

$$\begin{aligned}\sum_{i=0}^{p} f(a+i) &= \int_a^b f(t)\,dt + \frac{1}{2}(f(a)+f(b)) \\ &+ \sum_{i=1}^{m} \frac{B_{2i}}{(2i)!}(f^{(2i-1)}(b) - f^{(2i-1)}(a)) \\ &+ \frac{B_{2m+2}}{(2m+2)!}\sum_{i=0}^{p-1} f^{(2m+2)}(a+i+\theta)\end{aligned} \tag{7.3}$$

where $b = a + p$ and $\theta \in (0,1)$. We set $f(x) = 1/x^{k+1}$, $a = x$, and $m = 2n$ in (7.3) and let p tend to ∞. Then we obtain

$$\begin{aligned}\sum_{i=0}^{\infty} \frac{1}{(x+i)^{k+1}} &= \frac{1}{kx^k} + \frac{1}{2x^{k+1}} - \sum_{i=1}^{2n}\left[\frac{B_{2i}}{(2i)!}\prod_{j=0}^{2i-2}(-k-1-j)\right]\frac{1}{x^{2i+k}} \\ &+ \frac{B_{4n+2}}{(4n+2)!}\left[\prod_{j=0}^{4n+1}(-k-1-j)\right]\sum_{i=0}^{\infty}\frac{1}{(x+\theta+i)^{4n+k+3}}.\end{aligned} \tag{7.4}$$

Using $B_{4n+2} > 0$ and $\prod_{j=0}^{4n+1}(-k-1-j) > 0$ we get from (7.4):

$$\sum_{i=0}^{\infty} \frac{1}{(x+i)^{k+1}} > \frac{1}{kx^k} + \frac{1}{2x^{k+1}} - \sum_{i=1}^{2n}\left[\frac{B_{2i}}{(2i)!}\prod_{j=0}^{2i-2}(-k-1-j)\right]\frac{1}{x^{2i+k}}, \tag{7.5}$$

so that (7.2) and (7.5) imply

$$\begin{aligned}&\frac{1}{k!}(-1)^{k+1}F_n^{(k+1)}(x) \\ &> \sum_{i=1}^{2n}\left[\frac{(-1)^{k+1}}{k!}\frac{1}{2i}\prod_{j=0}^{k-1}(-2i-j) - \frac{1}{(2i)!}\prod_{j=0}^{2i-2}(-k-1-j)\right]\frac{B_{2i}}{x^{2i+k}} = 0,\end{aligned}$$

since the term in square brackets is equal to 0. Thus, F_n is strictly completely monotonic on $(0,\infty)$. □

Using the inequalities $(-1)^{k+1}F_n^{(k+1)}(x) > 0$ and $(-1)^{k+1}G_n^{(k+1)}(x) > 0$ for $k \ge 1$, we obtain the following rational bounds for $(-1)^{k+1}\psi^{(k)}(x)$.

Theorem 9. *Let $k \ge 1$ and $n \ge 0$ be integers. Then we have for all real $x > 0$:*

$$S_k(2n;x) < (-1)^{k+1}\psi^{(k)}(x) < S_k(2n+1;x),$$

where

$$S_k(p;x) = \frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} + \sum_{i=1}^{p}\left[B_{2i}\prod_{j=1}^{k-1}(2i+j)\right]\frac{1}{x^{2i+k}}.$$

Remark. Related inequalities for the special case $k = 1$ are given in [22].

8

In 1986, J. Bustoz and M.E.H. Ismail [9] proved that the function

$$p(x;a,b)=\frac{\Gamma(x)\Gamma(x+a+b)}{\Gamma(x+a)\Gamma(x+b)}\qquad (a,b>0)$$

is completely monotonic on $(0,\infty)$. This generalizes a proposition of K. B. Stolarsky [46], who established that p is decreasing in x. The next theorem provides an extension of these results.

Theorem 10. *Let a_i and b_i $(i=1,\dots,n)$ be real numbers such that $0\le a_1\le\dots\le a_n$, $0\le b_1\le\dots\le b_n$, and $\sum_{i=1}^k a_i\le\sum_{i=1}^k b_i$ for $k=1,\dots,n$. Then,*

$$x\mapsto\prod_{i=1}^n\frac{\Gamma(x+a_i)}{\Gamma(x+b_i)}$$

is completely monotonic on $(0,\infty)$.

In order to prove Theorem 10 we need the following two lemmas.

Lemma 1. *If h' is completely monotonic on $(0,\infty)$, then $\exp(-h)$ is also completely monotonic on $(0,\infty)$.*

An extension of Lemma 1 can be found in [5] and [15].

Lemma 2. *Let a_i and b_i $(i=1,\dots,n)$ be real numbers such that $a_1\le\dots\le a_n$, $b_1\le\dots\le b_n$, and $\sum_{i=1}^k a_i\le\sum_{i=1}^k b_i$ for $k=1,\dots,n$. If the function f is decreasing and convex on $\mathbb{R}$, then*

$$\sum_{i=1}^n f(b_i)\le\sum_{i=1}^n f(a_i).$$

A proof of Lemma 2 is given in [36, p. 10].

Proof of Theorem 10. Let

$$h(x)=\sum_{i=1}^n(\log\Gamma(x+b_i)-\log\Gamma(x+a_i)).$$

Then we have for $k\ge0$:

$$(h'(x))^{(k)}=\sum_{i=1}^n(\psi^{(k)}(x+b_i)-\psi^{(k)}(x+a_i)).$$

Using the integral representations

$$\psi(z)=-\gamma+\int_0^\infty\frac{e^{-t}-e^{-tz}}{1-e^{-t}}dt\qquad (z>0)$$

and

$$\psi^{(m)}(z)=(-1)^{m+1}\int_0^\infty\frac{e^{-tz}t^m}{1-e^{-t}}\,dt\qquad (z>0;m=1,2,\dots)$$

(see [16, p. 802], [34, p. 16]), we obtain for $k\ge0$:

$$(8.1)\qquad (-1)^k(h'(x))^{(k)}=\int_0^\infty\frac{e^{-tx}t^k}{1-e^{-t}}\sum_{i=1}^n(e^{-ta_i}-e^{-tb_i})\,dt.$$

Since the function $z \mapsto e^{-tz}$ $(t \geq 0)$ is decreasing and convex on $\mathbb{R}$, we conclude from Lemma 2 that $\sum_{i=1}^{n}(e^{-ta_i} - e^{-tb_i}) \geq 0$, so that (8.1) implies

$$(-1)^k (h'(x))^{(k)} \geq 0 \quad \text{for } x > 0 \text{ and } k \geq 0.$$

Hence, h' is completely monotonic on $(0, \infty)$. Applying Lemma 1 we obtain that

$$\exp(-h(x)) = \prod_{i=1}^{n} \frac{\Gamma(x+a_i)}{\Gamma(x+b_i)}$$

is also completely monotonic on $(0, \infty)$.

Remark. Since

$$\lim_{x\to\infty} \frac{\Gamma(x+a)}{\Gamma(x+b)} x^{b-a} = 1,$$

we conclude from Theorem 10 that the inequality

$$\prod_{i=1}^{n} \frac{\Gamma(x+a_i)}{\Gamma(x+b_i)} \geq 1 \qquad (x > 0)$$

holds for all real numbers a_i and b_i $(i = 1, \ldots, n)$ which satisfy $0 \leq a_1 \leq \cdots \leq a_n$, $0 \leq b_1 \leq \cdots \leq b_n$, $\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i$ for $k = 1, \ldots, n-1$, and $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$. This generalizes an inequality given in [9].

In a recently published paper L. Maligranda et al. [35] established that the function

$$x \mapsto \Gamma(x)^{n-1}\Gamma\left(x + \sum_{i=1}^{n} a_i\right) / \prod_{i=1}^{n} \Gamma(x+a_i)$$

$(a_i > 0; i = 1, \ldots, n)$ is decreasing on $(0, \infty)$. From Theorem 10 we conclude that this function is not only decreasing, but even completely monotonic on $(0, \infty)$. The following theorem presents a slight extension of this result.

Theorem 11. *Let α be a real number and let a_i $(i = 1, \ldots, n; n \geq 2)$ be positive real numbers. The function*

$$x \mapsto \Gamma(x)^{\alpha}\Gamma\left(x + \sum_{i=1}^{n} a_i\right) / \prod_{i=1}^{n} \Gamma(x+a_i)$$

is strictly completely monotonic on $(0, \infty)$ if and only if $\alpha = n-1$.

Proof. Let

$$p_\alpha(x) = \Gamma(x)^{\alpha}\Gamma(x+b) / \prod_{i=1}^{n} \Gamma(x+a_i)$$

with $b = \sum_{i=1}^{n} a_i$. Slight modifications of the proof of Theorem 10 show that p_{n-1} is strictly completely monotonic on $(0, \infty)$. We assume now that p_α is strictly completely monotonic on $(0, \infty)$. Then, p_α is decreasing, so that we obtain for $x > 0$:

$$\frac{\partial}{\partial x} \log p_\alpha(x) = \alpha\psi(x) + \psi(x+b) - \sum_{i=1}^{n} \psi(x+a_i) \leq 0.$$

This implies for all sufficiently large x :

$$\alpha \leq \sum_{i=1}^{n} \frac{\psi(x+a_i)}{\psi(x)} - \frac{\psi(x+b)}{\psi(x)}. \tag{8.2}$$

Since p_α is completely monotonic on $(0,\infty)$, we obtain

$$\begin{aligned} 0 &\leq (p_\alpha(x))^{-2}\left[p_\alpha(x)\frac{\partial^2 p_\alpha(x)}{\partial x^2} - \left(\frac{\partial p_\alpha(x)}{\partial x}\right)^2\right] \\ &= \alpha\psi'(x) + \psi'(x+b) - \sum_{i=1}^{n}\psi'(x+a_i); \end{aligned}$$

see [17]. Hence, we have for $x > 0$:

$$\sum_{i=1}^{n} \frac{\psi'(x+a_i)}{\psi'(x)} - \frac{\psi'(x+b)}{\psi'(x)} \leq \alpha. \tag{8.3}$$

Since

$$\lim_{x\to\infty} \psi(x+A)/\psi(x) = \lim_{x\to\infty} \psi'(x+A)/\psi'(x) = 1 \qquad (A>0),$$

we conclude from (8.2) and (8.3) that $\alpha = n-1$. □

We conclude with an application to probability theory. A probability measure $d\mu$ is infinitely divisible if for every natural number n there exists a probability measure $d\mu_n$ such that

$$d\mu = d\mu_n * d\mu_n * \cdots * d\mu_n \quad (n \text{ times}),$$

where $*$ denotes convolution.

A proof for the following proposition, which provides a connection between infinitely divisible probability measures and completely monotonic functions, can found in [15, p. 450].

Proposition. *A probability measure $d\mu$ supported on a subset of $[0,\infty)$ is infinitely divisible if and only if*

$$\int_0^\infty e^{-xt} d\mu(t) = \exp(-h(x)) \qquad (x>0),$$

where h has a completely monotonic derivative on $(0,\infty)$ and $h(0)=0$.

Using the Proposition and the results of this section, we obtain

Theorem 12. *Let $\varepsilon > 0$ be a real number, and let a_i and b_i $(i = 1,\ldots,n)$ be real numbers such that $0 \leq a_1 \leq \cdots \leq a_n$, $0 \leq b_1 \leq \cdots \leq b_n$, and $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$ for $k = 1,\ldots,n$. The function*

$$x \mapsto \prod_{i=1}^{n} \frac{\Gamma(x+\varepsilon+a_i)\Gamma(\varepsilon+b_i)}{\Gamma(x+\varepsilon+b_i)\Gamma(\varepsilon+a_i)}$$

is Laplace transform of an infinitely divisible probability measure.

Related results are given in [2], [9], [24].

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