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ON THE KHINTCHINE CONSTANT

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ABSTRACT. We present rapidly converging series for the Khintchine constant and for general "Khintchine means" of continued fractions. We show that each of these constants can be cast in terms of an efficient free-parameter series, each series involving values of the Riemann zeta function, rationals, and logarithms of rationals. We provide an alternative, polylogarithm series for the Khintchine constant and indicate means to accelerate such series. We discuss properties of some explicit continued fractions, constructing specific fractions that have limiting geometric mean equal to the Khintchine constant. We report numerical evaluations of such special numbers and of various Khintchine means. In particular, we used an optimized series and a collection of fast algorithms to evaluate the Khintchine constant to more than 7000 decimal places.

1. INTRODUCTION

The Khintchine constant arises in the measure theory of continued fractions. Every positive irrational number can be written uniquely as a simple continued fraction $[a_0; a_1, a_2, \ldots, a_n, \ldots]$, i.e., with a_0 a nonnegative integer and all other a_i positive integers. The *Gauss-Kuz′min distribution* [13] predicts that the density of occurrence of some chosen positive integer k in the fraction of a random real number is given by

$$\mathrm{Prob}(a_n = k) = -\log_2 \left[1 - \frac{1}{(k+1)^2}\right].$$

In his celebrated text, Khintchine [13] uses the Gauss-Kuz′min distribution to show that for almost all positive irrationals the limiting geometric mean of the positive elements a_i of the relevant continued fraction exists and equals

$$K_0 := \prod_{k=1}^{\infty} \left[1 + \frac{1}{k(k+2)}\right]^{\log_2 k} = \prod_{k=1}^{\infty} k^{\log_2\left[1+\frac{1}{k(k+2)}\right]}.$$

The fundamental constant K_0 is the *Khintchine constant.* Ever since Khintchine's elegant discovery there has been a keen interest in the numerical evaluation of K_0 [14, 20, 26, 27, 24]. It is known that this constant can be cast in terms of various converging series, the following example of which having been used by Shanks and

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Wrench to provide the first high-precision numerical values for K_0:

$$\log(K_0)\log(2) = \sum_{s=1}^{\infty} \frac{\zeta(2s)-1}{s}\left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2s-1}\right). \tag{1}$$

This series can be rendered even more computationally efficient via the introduction of a free integer parameter. We used a carefully optimized free-parameter series to resolve K_0 to 7350 decimal places ($K_0 = 2.68545200106\ldots$ see §5).

The Khintchine constant can be thought of as a member of a certain class of constants we shall call *Khintchine means* K_p, for real numbers $p < 1$. The *Hölder mean* of order p of the continued fraction elements, namely $\lim_k[(a_1^p + a_2^p + \cdots + a_k^p)/k]^{1/p}$, also exists with probability one and again with probability one equals the constant:

$$K_p := \left\{\sum_{k=1}^{\infty} -k^p \log_2\left[1 - \frac{1}{(k+1)^2}\right]\right\}^{1/p}.$$

(See the final section of Khintchine's book [13] for a proof for $p < \frac{1}{2}$, or more modern references on ergodic theory for a proof for $p < 1$ [19].) We may interpret K_0 as the limiting instance of the K_p definition as $p \to 0$. We shall show in Theorem 6 below that for any negative integer p the Khintchine mean of order p satisfies an identity

$$(K_p)^p \log(2) = \sum_{s=2}^{\infty}(\zeta(s-p)-1)Q_{sp}, \tag{2}$$

where each coefficient Q_{sp} is rational. Again, there is a free-parameter generalization, which we employed to resolve the *harmonic mean* K_{-1} also to over 7000 decimal places ($K_{-1} = 1.74540566240\ldots$ see §5). It is of interest that, evidently, only K_0 can be written as a series involving exclusively even zeta arguments. The computational implications of this unique property of K_0 are discussed in §5. We should mention that aside from the series (1) for K_0 there are other previously known formulae for Khintchine means, some of which involve derivatives of the zeta function [24].

In the next section we establish the series forms (1) and (2) for K_0 and the general Khintchine means K_p, respectively. Actually, (1) and (2) can be thought of as degenerate cases of free-parameter forms in which an integer parameter can be optimized for numerical efficiency. Then in §3 we present polylogarithm series and a certain zeta-like function whose evaluations can be used to accelerate the polylogarithm series. In §4 we discuss explicit continued fractions with a view toward determining whether Hölder means exist and coincide with Khintchine means. In particular, some numbers known to have geometric mean (zeroth Hölder mean) equal to K_0 are presented. Finally, in §5 we discuss numerical details relevant to very-high-precision evaluation of Khintchine means.

2. Fundamental identities

This section is devoted to presenting the basic identities. We begin with a list of preliminary, largely elementary, results needed in the paper. The first lemma amounts to a set of observations due to Wrench and Shanks [27].

Lemma 1. (a) *We have*

$$-\log(1-x)\log(1+x) = \sum_{k=1}^{\infty} \frac{A_k}{k} x^{2k},$$

where $A_s := \sum_{m=1}^{2s-1} (-1)^{m-1}/m$.

(b) *Further,*

$$\sum_{k=2}^{N} \log(1-\frac{1}{k})\log(1+\frac{1}{k}) - \sum_{k=2}^{N} \log(k-1)\log(1-\frac{1}{k^2}) = -\log(N)\log(1+\frac{1}{N}).$$

(c) *Thus,*

$$\sum_{k=2}^{\infty} \log(1-\frac{1}{k})\log(1+\frac{1}{k}) = -\log(K_0)\log(2).$$

Proof. Part (a) is most easily seen by differentiating both sides. The left–hand side becomes $f(x) - f(-x)$ where $f(x) := \log(1+x)/(1-x)$. Using the standard relationship

$$\sum_{k=1}^{\infty} \frac{a_k}{1-x} x^k = \sum_{k=1}^{\infty} \{\sum_{j=1}^{k} a_j\} x^k$$

produces (a).

Part (b) is easily established inductively after expanding the left–hand side.

Part (c) follows on taking limits and noting that

$$-\sum_{k=2}^{N} \log(k-1)\log(1-\frac{1}{k^2}) = \log(K_0)\log(2),$$

as follows from the definition of K_0. □

We shall find it convenient to use the *Hurwitz zeta function* defined by

$$\zeta(s, N) := \sum_{n=1}^{\infty} \frac{1}{(n+N)^s},$$

so that $\zeta(s) = \zeta(s, 0)$ and so that for N a nonnegative integer

$$\zeta(s, N) = \zeta(s) - \sum_{n=1}^{N} \frac{1}{n^s}.$$

With this notation we have:

Lemma 2. (a) *For N a positive integer,*

$$\sum_{n=2}^{\infty} \zeta(n, N) = \frac{1}{N}.$$

(b) *For N a positive integer,*

$$\sum_{n=1}^{\infty} \frac{\zeta(2n, N)}{n} = \log(\frac{N+1}{N}).$$

(c)

$$\int_0^1 \frac{\log(1-t^2)}{t(1+t)} dt = -\log^2(2).$$

Proof. The proofs of the first two identities are similar and rely on expanding the zeta terms, rearranging the order of summation and re-evaluating. In both cases, the result telescopes to the desired conclusion.

Part (c) is less immediate. Actually, the indefinite integral is evaluable with the aid of the theory of the dilogarithm [15]. The integral

$$\int_0^t \frac{\log(1-x^2)}{x(1+x)} dx$$

equals the log terms

$$-\frac{\log^2(1+t)}{2} - \log^2(2) + \log(2)\log(1-t) - \log(1+t)\log(1-t) + \log(t)\log(1-t)$$

plus the dilog terms

$$\operatorname{dilog}(t) - \operatorname{dilog}(1+t) - \operatorname{dilog}(\frac{1+t}{2}),$$

and the limit as $t \to 1$ yields the desired result $-\log^2(2)$. □

We are now in a position to establish a general Shanks-Wrench identity [20] for K_0.

Theorem 3. *For any positive integer* N,

$$\log(K_0)\log(2) = \sum_{s=1}^{\infty} \zeta(2s, N)\frac{A_s}{s} - \sum_{k=2}^{N} \log(1-\frac{1}{k})\log(1+\frac{1}{k}), \tag{3}$$

where $A_s := \sum_{m=1}^{2s-1}(-1)^{m-1}/m$.

Remark. The integer N is a free parameter that can be optimized in actual computations to significantly reduce the number of zeta evaluations required. Variation of this parameter also provides a kind of error check, for whatever the choice of positive integer N, one expects the same result for the left-hand side. Note that in the case $N = 1$ the second summation is empty, and we recover precisely the K_0 identity (1) of §1.

Proof. Let $f(N)$ denote the right-hand side of (1). Then

$$f(N-1) - f(N) = \sum_{s=1}^{\infty} \frac{A_s}{s} N^{-2s} + \log(1-\frac{1}{N})\log(1+\frac{1}{N})$$

which equals zero by Lemma 1(a). Thus, since $\zeta(2s, N) \to 0$ sufficiently rapidly,

$$\begin{aligned} f(1) = f(N) &= \lim_{N\to\infty} f(N) \\ &= -\sum_{k=2}^{\infty} \log(1-\frac{1}{k})\log(1+\frac{1}{k}). \end{aligned}$$

By Lemma 1(c), this sum agrees with $\log(K_0)\log(2)$. □

As a companion relation to the identity of Theorem 3, we can establish an elegant integral representation for the left-hand side. There is a powerful generalization of Lemma 2(b) in the form of a generating function based on Euler's product for $\sin(\pi t)/(\pi t)$ (see [23, p. 249]). For real t in [0,1) define $g(t)$ by

$$g(t) := \sum_{s=1}^{\infty} \frac{\zeta(2s)-1}{s} t^{2s} = -\log\left(\frac{\sin(\pi t)}{\pi t}\right) + \log(1-t^2), \tag{4}$$

and define also the limiting case $g(1) := \log 2$. We only need observe now that, on the basis of Theorem 3 with parameter $N = 1$,

$$\log(K_0)\log(2) = \int_0^1 \frac{\log(2) + g(t)/t}{1+t}\,dt,$$

and with the help of the previous dilogarithm integral evaluation we thus arrive at an integral representation. (Reference [20] contains an equivalent integral identity.)

Corollary 4. *The following integral representation holds for K_0:*

$$\int_0^1 \frac{\log[\sin(\pi t)/(\pi t)]}{t(1+t)}\,dt = -\log(K_0)\log(2).$$

It is amusing to observe that Lemma 1(c) may also be turned into an analogous integral form:

$$\log(K_0)\log(2) = \int_1^\infty \frac{\log(\lfloor t\rfloor)}{t(1+t)}\,dt = \int_0^1 \frac{\log(\lfloor 1/t\rfloor)}{1+t}\,dt.$$

This was observed from a very different starting point by Robert Corless [11] but follows immediately on breaking the first integral up at integer points.

We now derive new, corresponding identities for the higher–order Khintchine means. They are in some sense simpler, since one logarithmic term is replaced by a negative integral power. There is an observation that leads directly to a zeta function expansion for these general Khintchine means. Note that a sum of terms $k^p \log(1-(k+1)^{-2})$ can be expressed, via expansion of the logarithm, in terms of sums of the form (note p is assumed to be a negative integer):

$$\sum_{n=2}^\infty \frac{1}{n^{2s-p}(1-1/n)^{-p}}.$$

Upon expansion of the term

$$1/(1-1/n)^{-p}$$

in powers of $1/n$, we obtain an identity for the pth power of K_p as a series of zeta functions. The result, after the same manner of free-parameter manipulation we used for K_0, is a new series that can be thought of as a companion identity to the Shanks-Wrench expansion of Theorem 3.

Theorem 5. *For negative integer p and positive integer N we have*

$$K_p^p \log(2) = \sum_{n=1}^\infty \frac{\sum_{j=0}^\infty \binom{j-p-1}{-p-1}\zeta(2n+j-p, N)}{n} - \sum_{k=2}^N \log(1-\frac{1}{k^2})(k-1)^p.$$

Remark. Note that for $N = 1$ the final sum is empty, the coefficient of any given $\zeta(s)$ is an easily computed rational, and we immediately establish a general series with rational coefficients, (2) of §1.

Corollary 6. *The constant K_{-1} satisfies, for any integer $N > 0$,*

$$\frac{\log(2)}{K_{-1}} = \sum_{n=1}^\infty \frac{\frac{1}{N} - \sum_{k=2}^{2n}\zeta(k,N)}{n} - \sum_{k=2}^N \frac{\log(1-k^{-2})}{k-1}.$$

Proof. It suffices to show that for every positive integer n

$$\sum_{j=0}^{\infty} \zeta(2n+j+1, N) + \sum_{k=2}^{2n} \zeta(k, N) = \frac{1}{N}.$$

This follows immediately from Lemma 2(a). □

3. Polylogarithm series

There exist some interesting identities for the Khintchine constant in terms of polylogarithm evaluations. One particularly interesting polylogarithm identity is obtained by resolving the integral representation of Corollary 4 in polylogarithm terms [28]. One may employ the Euler product for $\sin z/z$ to write the integral as a sum of logarithmic integrals, each in turn expressible in terms of polylogarithms. This procedure leads to the series

$$\log(K_0)\log(2) = \log^2(2) + \mathrm{Li}_2(-\frac{1}{2}) + \frac{1}{2}\sum_{n=2}^{\infty}(-1)^n \mathrm{Li}_2(\frac{4}{n^2}),$$

where, $\mathrm{Li}_m(z)$ is the *polylogarithm* function: $\mathrm{Li}_m(z) := \sum_{k=1}^{\infty} z^k k^{-m}$.

A somewhat different application of polylogarithms is to use the classical Abel identity [15]

$$\log(1-x)\log(1-y) = \mathrm{Li}_2(\frac{x}{1-y}) + \mathrm{Li}_2(\frac{y}{1-x}) - \mathrm{Li}_2(x) - \mathrm{Li}_2(y) - \mathrm{Li}_2(\frac{xy}{(1-x)(1-y)})$$

together with Lemma 1(c), setting $x := 1/n, y := -1/n$ to obtain

$$\log(K_0)\log(2) = \frac{\pi^2}{6} - \frac{1}{2}\log^2(2) + \sum_{n=2}^{\infty} \mathrm{Li}_2(\frac{-1}{n^2-1}).$$

An interesting line of analysis starting from this last polylogarithm series is to "peel off" parts of the Li_2 function, casting the corrections in closed form. Such a procedure gives polylogarithm-based analogues of Theorem 3. For example, one can replace the last Li_2 summand above with a more rapidly decaying term

$$\mathrm{Li}_2(\frac{-1}{n^2-1}) - \frac{-1}{n^2-1} - \frac{1}{4}\frac{1}{(n^2-1)^2}$$

and add back a correction

$$-\Omega(1) + \frac{1}{4}\Omega(2) = \frac{\pi^2}{48} - \frac{59}{64},$$

where Ω is the zeta-like function

$$\Omega(m) := \sum_{n=2}^{\infty} \frac{1}{(n^2-1)^m}.$$

Careful Eulerian partial fraction decomposition (as detailed in [7]) can be used to produce a closed-form evaluation of $\Omega(m)$ for any positive integer m. In this way one may accelerate the convergence of relevant polylogarithm sums.

4. Explicit continued fractions

It is remarkable that, even though a random fraction's limiting geometric mean exists and furthermore equals the Khintchine constant with probability one, not a single explicit real number (e.g., a real number cast in terms of fundamental constants) has been demonstrated to have elements whose geometric mean equals K_0. Likewise, for any negative integer p, not a single explicit real number has been shown to have elements whose Hölder mean equals K_p. In any event, it is worthwhile to mention some classical continued fractions with respect to this theoretical impasse.

The continued fraction for e is

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, \dots].$$

The elements are eventually comprised of a meshing of two arithmetic progressions, one of which has zero common difference while the other has difference two and diverges. Thus the meshing has diverging geometric mean. Thus, e does not possess geometric mean K_0. The harmonic mean for e does exist, but equals $3/2$, which is not K_{-1}. It turns out that any fraction with elements lying in a single arithmetic progression can be evaluated in terms of special functions. Explicitly, for any positive integers a, d we have [2, eq. 9.1.73]

$$[a; a+d, a+2d, a+3d, ...] = \frac{I_{a/d-1}(\frac{2}{d})}{I_{a/d}(\frac{2}{d})},$$

where I_ν is the *modified Bessel function* of order ν. These arithmetic progression fractions are certainly interesting, and not beyond deep analysis. It was known, for example, to C. L. Siegel that these fractions are transcendental [22]. But each such fraction has diverging geometric mean and indeed diverging Hölder means. Note that the means are monotone nondecreasing in p, and so a fraction with $\liminf$ of its elements infinite has infinite means.

Another example of interest is π, whose continued fraction expansion is

$$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, \dots].$$

The continued fraction elements do not appear to follow any pattern and are widely suspected to be in some sense random. Based on the first 17,001,303 continued fraction elements, the geometric mean (of the fraction elements yielding the same precision) is 2.686393 and the harmonic mean is 1.745882 [12]. These values are reasonably close to K_0 and K_{-1}, but of course no conclusion can be drawn beyond this.

It is a well-known theorem of Lagrange that the elements of a simple continued fraction form an eventually periodic sequence if and only if the fraction is an irrational quadratic surd. All Hölder means for $p = 0, -1, -2, ...$ then exist, and are completely determined by one period of elements. Hence, each Hölder mean of a quadratic surd is an algebraic number. Clearly, for any algebraic number $c = a^{1/b}$ formed from integers a, b, one can write down a quadratic surd having geometric mean c. Along these lines, it is not hard to show that if there exists an integer $m > 2$ such that

$$\frac{\log(K_0/m)}{\log(2/m)}$$

is rational, then there exists a quadratic surd with geometric mean K_0. Thus the issue of transcendence for K_0 and related numbers is an interesting one, and one we return to in the next section.

Even though no explicit real number is known to have elements whose geometric mean is K_0, it is still possible to fabricate explicit lists of *elements* whose geometric mean does equal K_0. If one were in possession of some representation of K_0 to arbitrary accuracy, one could of course construct a fraction having geometric mean K_0 by appending a "2" (respectively, "3") to the element list whenever the current geometric mean were above (below) K_0. There seems to be no way to determine *a priori* the value of, say, the nth element. Thus, such a constructed fraction is not explicit.

But it is possible to give an explicit list of elements having the desired property. One successful construction has been given by [1], as follows. Consider the naturally ordered rationals in (0,1); that is, consider

$$1/2, 1/3, 2/3, 1/4, 2/4, 3/4, 1/5, 2/5, 3/5, 4/5, \dots ,$$

where for each successive denominator $d = 2, 3, 4, \dots$ we employ in increasing order all numerators between 1 and $d-1$ inclusive. Now consider the (finite) set of fraction elements for each rational in the list. (We also demand the caveat that no such terminating fraction is allowed to end with element 1, so for example 2/3 is the fraction $[1, 2]$ rather than $[1, 1, 1]$.) If we concatenate the elements from all the terminating fractions, the infinite chain of elements has limiting geometric mean equal to K_0. The resulting sequence of elements starts out:

$$A = [2; 3, 1, 2, 4, 2, 1, 3, 5, 2, 2, 1, 1, 2, 1, 4, 6, 3, 2, 1, 2, 1, 5, 7, 3, 2, \dots].$$

The geometric mean of the first 15,000 elements of A is 2.35821 ... , which appears low but note that as the denominator d increases during construction of the elements, larger and larger elements (such as d itself) appear.

But one may construct elements whose geometric mean converges much more rapidly to K_0. One such construction is based on a deterministic stochastic sampling of the Gauss-Kuz′min density, and proceeds as follows. First, for non-negative integer n define the *van der Corput discrepancy sequence* [16] to be a set of the base-2 numbers

$$d(n) = 0.b_0 b_1 b_2 \dots ,$$

where the b_i are the binary bits of n, with b_0 being least significant. As n runs through positive integers, the sequence of $d(n)$ is confined to (0,1) and has appealing pseudorandom properties. The construction of the number we shall call Z_2 then starts with $a_0 := 0$, and loops as follows:

For $n = 1$ to ∞, set $a_n := \lfloor 1/(2^{d(n)} - 1) \rfloor$.

The continued fraction elements a_n thus determined start out

$$Z_2 = [0; 2, 5, 1, 11, 1, 3, 1, 22, 2, 4, 1, 7, 1, 2, 1, 45, 2, 4, 1, 8, 1, 3, 1, 14, 1, \dots].$$

On the idea that the discrepancy sequence is in a certain sense equidistributed, we are moved to posit:

Conjecture. *The geometric mean of the number Z_2 is in fact the Khintchine constant K_0. Furthermore, every limiting p-th Hölder mean of Z_2 for $p = -1, -2, \dots$ is the respective Khintchine mean K_p.*

With regard to the above conjecture, S. Plouffe [18] has reported a computation of the geometric and harmonic means through 5206016 continued fraction elements of Z_2. His results are 2.6854823207 and 1.7454074435, respectively, which are remarkably close to the expected theoretical values. The authors have also been informed by T. Wieting that he has an unpublished proof of the basic conjecture, i.e., that the limiting Hölder means of Z_2 exist for $p = 0, -1, -2, \ldots$ and furthermore equal the corresponding Khintchine means [25]. In the wake of such positive results, one can ask in addition for, say, the rate of convergence to K_0. Again on the basis of the distribution of discrepancy values we think it reasonable to conjecture that the geometric mean G_n of the first n elements of Z_2 satisfies

$$|G_n - K_0| < c/n^{1/2}$$

for some absolute constant c.

Yet a third, and novel construction runs as follows. First, for the correct Gauss-Kuz'min density of 1's, namely $p_1 = \log_2(4/3)$, generate a list of 0's and $1's$ by assigning element values $a_n = k(\lceil np_1 \rceil - \lceil (n-1)p_1 \rceil)$ for $n > 0$ and $k = 1$. Now replace the remaining 0's in the sequence with 2's, by replacing p_1 with $p_2/(1-p_1)$ and incrementing k to 2, in which case the index n on a_n refers, naturally, to the nth zero of the preceding list. Continuation of this construction for $k = 3, 4, \cdots$ gives a real number:

$$R = [0; 1, 2, 1, 3, 1, 4, 2, 1, 5, 6, 2, 1, 3, 1, 7, 1, 8, 2, 1, 4, 1, 3, 2, 1, 9, 1, \ldots].$$

Through 100,000 elements the geometric and harmonic means for R work out to be 2.6753 and 1.7454, respectively. Although we have not done so, it should be possible to prove for example that the limiting geometric mean of these elements is indeed K_0.

5. Computation of Khintchine means

As intimated in our introduction, calculation of K_0 has occupied the attention of various researchers. For example, Gosper recently computed K_0 to 2217 digits [12]. Our 7350-digit value is in complete agreement with Gosper's 2217 digits. There is also the interesting problem of computing fraction elements from decimal representations of certain real numbers, a task that one may wish to do in, say, statistical experiments involving Khintchine means. We mention that in [21] an interesting algorithm is presented for computation of fraction elements without recourse to decimal input. Instead, differential properties of an appropriate function are used. For example, Shiu resolved 10,000 elements of the fraction for e^{π} using properties of the function $f(t) = \sin(\log(t))$.

By exploiting various modern algorithms to be described presently, the present authors have explicitly computedcdots K_0 and K_{-1} to more than 7350 decimal digit accuracy. These computations were performed with the aid of the MPFUN multiprecision software [4, 5], which was found to be significantly faster for our purposes than other available multiprecision facilities. One utilizes this software by writing ordinary Fortran-90 code, with multiprecision variables declared to be of type `mp_integer`, `mp_real` or `mp_complex`. In the computations described below, the level of precision was sufficiently high that the "advanced" routines of the Fortran-90 MPFUN library were employed. These routines employ special algorithms, including fast Fourier transform (FFT) multiplication, which are efficient for extra-high levels of precision.

The constant K_0 was computed using the formula given above in Theorem 3, with the free integer parameter $N = 100$, and with $N = 120$ as a check. The implementation of this formula was straightforward except for the computation of the Riemann zeta function. To obtain 7350-digit accuracy in the final result, 2048 terms of the indicated series were evaluated, which requires $\{\zeta(2k),\ 0 \le k \le 2048\}$ to be computed. One approach to compute these zeta function values is to apply formulas due to P. Borwein [8]. These formulas are very efficient for computing one or a few zeta function values, but when many values are required as in this case, another approach was found to be more efficient. This method is based on an observation that has previously been used in numerical approaches to Fermat's "Last Theorem" [9, 10], namely

$$\begin{aligned}\coth(\pi x) &= \frac{-2}{\pi x}\sum_{k=0}^{\infty}\zeta(2k)(-1)^k x^{2k} \\ &= \cosh(\pi x)/\sinh(\pi x) \\ &= \frac{1}{\pi x}\cdot\frac{1+(\pi x)^2/2!+(\pi x)^4/4!+(\pi x)^6/6!+\cdots}{1+(\pi x)^2/3!+(\pi x)^4/5!+(\pi x)^6/7!+\cdots}.\end{aligned}$$

Let $N(x)$ and $D(x)$ be the numerator and denominator polynomials obtained by truncating these two series to n terms. Then the approximate reciprocal $Q(x)$ of $D(x)$ can be obtained by applying the Newton iteration

$$Q_{k+1}(x) := Q_k(x) + [1 - D(x)Q_k(x)]Q_k(x).$$

Once $Q(x)$ has been computed to sufficient accuracy, the quotient polynomial is simply the product $N(x)Q(x)$. The required values $\zeta(2k)$ can then be obtained from the coefficients of this polynomial.

Computation time for the Newton iteration procedure can be reduced by starting with a modest polynomial length and precision level, iterating to convergence, doubling each, etc., until the final length and precision targets are achieved. Computation time can be further economized by performing the two polynomial multiplications indicated in the above formula using a FFT-based convolution scheme. In our implementation, FFTs were actually performed at two levels of this computation: (i) to multiply pairs of polynomials, where the data elements to be transformed are the multiprecision polynomial coefficients, and (ii) to multiply pairs of multiprecision numbers, where the data elements to be transformed are integers representing successive sections of the binary representations of the two multiprecision numbers.

The constant K_{-1} was computed by applying the formula in Corollary 6. Again, the challenge here is to precompute values of the Riemann zeta function for integer values. But in this case both odd and even values are required. The odd values can be economically computed by applying the following two formulas, the first given by Ramanujan, but simplified slightly and known earlier; the second derived by differentiating a companion identity of Ramanujan [6, ch. 14]:

$$
\begin{aligned}
\zeta(4N+3) = &-2\sum_{k=1}^{\infty}\frac{1}{k^{4N+3}(\exp(2k\pi)-1)}\\
&-\pi(2\pi)^{4N+2}\sum_{k=0}^{2N+2}(-1)^k\frac{B_{2k}B_{4N+4-2k}}{(2k)!(4N+4-2k)!},\\
\zeta(4N+1) = &-\frac{1}{N}\sum_{k=1}^{\infty}\frac{(2\pi k+2N)\exp(2\pi k)-2N}{k^{4N+1}(\exp(2k\pi)-1)^2}\\
&-\frac{1}{2N}\pi(2\pi)^{4N}\sum_{k=1}^{2N+1}(-1)^k\frac{B_{2k}B_{4N+2-2k}}{(2k-1)!(4N+2-2k)!}.
\end{aligned}
$$

Here B_{2k} is as always the $2k$th Bernoulli number.

Alternatively, the formulas can be written in terms of the even zetas as

$$
\begin{aligned}
\zeta(4N+3) = &-2\sum_{k=1}^{\infty}\frac{1}{k^{4N+3}(\exp(2k\pi)-1)}\\
&+\frac{1}{\pi}\{\frac{(4N+7)}{2}\zeta(4N+4)-\sum_{k=1}^{N}2\zeta(4k)\zeta(4N+4-4k)\},\\
\zeta(4N+1) = &-\frac{1}{N}\sum_{k=1}^{\infty}\frac{(2\pi k+2N)\exp(2\pi k)-2N}{k^{4N+1}(\exp(2k\pi)-1)^2}\\
&+\frac{1}{2N\pi}\{\sum_{k=1}^{2N}(-1)^k 2k\zeta(2k)\zeta(4N+2-2k)+(2N+1)\zeta(4N+2)\}.
\end{aligned}
$$

These two formulas are not very economical for computing a single odd value or just a few odd values of $\zeta(k)$ — again, the formulas in [8] are more efficient for such purposes. But these Ramanujan formulas are quite efficient when a large number of odd zetas are required. Note that the infinite series in the two formulas can be inexpensively evaluated for many N simultaneously, since the expensive parts of these expressions do not involve N. Further, the evaluation of the infinite series can be cut off once terms for a given N are smaller than the "epsilon" of the numeric precision level being used. Happily, convergence here is fairly rapid for large N.

At first glance, the latter summations in these two formulas may appear quite expensive to evaluate. But note that each is merely the polynomial product of two vectors consisting principally of even zeta values. Thus, both sets of summation results can be computed using multiprecision FFT-based convolutions.

Computation of K_0 to 7350 digit precision required 2.5 hours on an IBM RS6000/590 workstation, and computation of K_{-1} also to 7350 digits required some 12 hours. Excerpts of the resulting decimal expansions for each are included in the appendix. The complete expansions are available from the authors.

One intriguing question that was raised decades ago [27] is whether the continued fraction elements of K_0 themselves enjoy a limiting geometric mean K_0. We can of course ask more generally whether, for the fraction elements of any Khintchine mean K_p, the limiting Hölder mean of order q is in fact K_q. During the task of computing from a given decimal representation a Hölder mean of some order, the issue of where to terminate the list of continued fraction elements is an interesting

TABLE 1. Continued fraction statistics

Constant	Cont. Frac. Elements	Geometric mean	Harmonic mean
K_0	7182	2.660716	1.745541
K_{-1}	7052	2.722471	1.746871
A	15000	2.358251	1.745395
R	100000	2.6753	1.7454
Z_2	5206016	2.685482	1.7454074
π	17001303	2.686393	1.745882

one. We employed a simple criterion: if x is known numerically, to D decimals to the right of the decimal point, generate continued fraction elements for x until a convergent p/q has $2q^2 > 10^D$. The motivation for choosing this simple criterion is the theorem that at least one of any two successive convergents must satisfy

$$|\frac{p}{q} - x| < \frac{1}{2q^2}$$

and conversely, any reduced ratio p/q satisfying this inequality must be a convergent of x [13].

Our results are shown in Table 1, together with results for the constants K_{-1}, A, R, Z_2 (which were defined above), and π. To give statistical perspective to our results for K_0 and K_{-1}, we computed the geometric and harmonic means of the first 7000 fraction elements for each of 100 pseudorandom multiprecision numbers of the same precision, namely 7350 decimal digits. The average and standard deviation of their geometric means were 2.683740 and 0.030124, respectively. The same statistics for their harmonic means were 1.745309 and 0.011148, respectively. Note that these two averages are in good agreement with the theoretical values K_0 and K_{-1}. In any event, it appears that the geometric and harmonic means for the first 7182 elements of our 7350-digit K_0 are within reasonable statistical limits of the expected theoretical values.

A question implicitly asked in the previous section is whether K_0 or K_{-1} is algebraic. This question can be numerically explored by means of integer relation algorithms. A vector of real numbers $(x_1, x_2, \ldots, x_n)$ is said to possess an *integer relation* if there exist integers a_k such that $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$. It can easily be seen that a real number α is algebraic of degree $n-1$ if and only if the vector $(1, \alpha, \alpha^2, \ldots, \alpha^{n-1})$ possesses an integer relation. Even if α is not algebraic, integer relation algorithms produce bounds that allow one to exclude relations within a region.

We employed the "PSLQ" algorithm developed by Ferguson and one of the authors, a simplified version of which is given in [3]. This algorithm, when applied to power vectors generated from our computed values of K_0 or K_{-1}, found no relations for either. On the contrary, we obtained the following result: if K_0 satisfies a polynomial of the form

$$0 = a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + \cdots + a_{50}\alpha^{50}$$

in the variable α, then the magnitude of some integer coefficient a_k exceeds 10^{70}. The same was found to be true for K_{-1}.

In a second experiment, we explored the possibility that K_0 or K_{-1} is given by a multiplicative formula involving powers of primes and some well-known mathematical constants. To that end, let p_k denote the kth prime. We established, using PSLQ, that neither K_0 nor K_{-1} satisfies a relation of the form

$$
\begin{aligned}
0 = a_0 \log \alpha + \sum_{k=1}^{15} a_k \log p_k \\
+ a_{16} \log \pi + a_{17} \log e + a_{18} \log \gamma + a_{19} \log \zeta(3) + a_{20} \log \log 2
\end{aligned}
$$

with integer coefficients a_k of absolute value 10^{20} or less. By exponentiating this expression, it follows that neither K_0 nor K_{-1} satisfies a corresponding multiplicative formula with exponents of absolute value 10^{20} or less.

There are many other tests that might be applied. For example, further work might be to rule out the possibility that $\log K_0$, $(\log K_0)(\log 2)$, or one of many other forms involving K_0 be an algebraic number of low degree.

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Appendix

The Khintchine Constant K_0 to 7,350 Digits

```
2.
68545200106530644530971483548179569382038229399446
29530511523455572188595371520028011411749318476979
95153465905288090082897677716410963051792533483259
66838185231542133211949962603932852204481940961806
86641664289308477880620360737053501033672633577289
04990427070272345170262523702354581068631850103237
46558037750264425248528694682341899491573066189872
07994137235500057935736698933950879021244642075289
74145914769301844905060179349938522547040420337798
56398310157090222339100002207725096513324604444391
.....
36909874406573435125594396103980583983755664559601
```

The Khintchine Harmonic Mean K_{-1} to 7,350 Digits

```
1.
74540566240734686349459630968366106729493661877798
42565950137735160785752208734256520578864567832424
20977343982577985596531102601834294460206578713176
15026238960612981165718728271638949622593992929776
06160830078357479801549029312671643067241248453710
96077711207484391474195803753220015690822609477078
44894635568203493582068440202422591615018316479048
29229656977731436622109918063888425816505999997697
61391683577259217628635718712601565066754443340174
00283376465305136584406098398017126202832041200630
```

. . . . .
7855312824966647368030403476149746733070847943628O

Khintchine Means K_p to 50 Digits for Various Negative p

p	K_p
-2	1.45034032849563040605298307668069788140829997960590 4...
-3	1.31350707868798576671733944707278682815812986148479 2...
-4	1.23696180942373005262622724445342256742024113154893 7...
-5	1.18900392646551315406236373277140339738609251263967 1...
-6	1.15655237442151442315260599874341004684021307071876 1...
-7	1.13332336395086579491028969490886836359909828241179 7...
-8	1.11596440897871669061915641934534969576949118223040 0...
-9	1.10254313667072801383609340252256835102222128414931 8...
-10	1.09187704120961267827611097947763825649327265142965 6...

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