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OPTIMAL INFORMATION FOR APPROXIMATING PERIODIC ANALYTIC FUNCTIONS

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ABSTRACT. Let $S_\beta := \{z \in \mathbb{C} : |\operatorname{Im} z| < \beta\}$ be a strip in the complex plane. For fixed integer $r \geq 0$ let $H^r_{\infty,\beta}$ denote the class of 2π-periodic functions f, which are analytic in S_β and satisfy $|f^{(r)}(z)| \leq 1$ in S_β. Denote by $H^{r,\mathbb{R}}_{\infty,\beta}$ the subset of functions from $H^r_{\infty,\beta}$ that are real-valued on the real axis. Given a function $f \in H^r_{\infty,\beta}$, we try to recover $f(\zeta)$ at a fixed point $\zeta \in \mathbb{R}$ by an algorithm A on the basis of the information

$$If = (a_0(f), a_1(f), \dots, a_{n-1}(f), b_1(f), \dots, b_{n-1}(f)),$$

where $a_j(f)$, $b_j(f)$ are the Fourier coefficients of f. We find the intrinsic error of recovery

$$E(H^r_{\infty,\beta}, I) := \inf_{A\colon \mathbb{C}^{2n-1}\to\mathbb{C}} \sup_{f\in H^r_{\infty,\beta}} |f(\zeta) - A(If)|.$$

Furthermore the $(2n-1)$-dimensional optimal information error, optimal sampling error and n-widths of $H^{r,\mathbb{R}}_{\infty,\beta}$ in C, the space of continuous functions on $[0, 2\pi]$, are determined. The optimal sampling error turns out to be strictly greater than the optimal information error. Finally the same problems are investigated for the class $H_{p,\beta}$, consisting of all 2π-periodic functions, which are analytic in S_β with p-integrable boundary values. In the case $p = 2$ sampling fails to yield optimal information as well in odd as in even dimensions.

INTRODUCTION

Let W be a class of 2π-periodic, real-valued (or complex-valued) functions. Suppose that $W \subset C$, where C is the space of continuous functions on $[0, 2\pi]$ endowed with the supremum norm. Consider the problem of optimal recovery of the linear functional U on W given by $Uf = f(\zeta)$, i.e. point evaluation in ζ, on the basis of the information

$$If = (L_1 f, \dots, L_n f),$$

where $L_1, \dots, L_n$ are continuous linear functionals on W.

By an algorithm we mean any map (not necessarily linear or continuous) A: $Z^n \to Z$, where $Z = \mathbb{R}$ or $\mathbb{C}$ depending on whether W is a set of real-valued or complex-valued functions.

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The algorithm A produces the error

$$E_A(W,I) := \sup_{f\in W} |Uf - A(If)|.$$

The value

$$E(W,I) := \inf_{A\colon Z^n\to Z} E_A(W,I)$$

is called the intrinsic error of the problem. An algorithm A^*, for which

$$E_{A^*}(W,I) = E(W,I)$$

is said to be an optimal algorithm.

The optimal information error for estimating W in C by n linear observations is defined as follows:

$$i_n(W,C) := \inf_{L_1,\dots,L_n} \inf_{A\colon Z^n\to C} \sup_{f\in W} \|f - A(If)\|_C\,. \tag{1}$$

Any continuous linear functionals $L_1^*,\dots,L_n^*$ for which the infimum is attained are called optimal.

If we restrict the class of admissible linear observations to function values, then we have the value

$$s_n(W,C) := \inf_{z_1,\dots,z_n\in[0,2\pi)} \inf_{A\colon Z^n\to C} \sup_{f\in W} \|f - A(f(z_1),\dots,f(z_n))\|_C\,,$$

which is called the optimal sampling error. If the infimum is attained at the points $z_1^*,\dots,z_n^*$, then these points are said to be optimal.

The study of optimal recovery problems has received much attention in the last years. For a detailed survey we refer to the papers of Micchelli and Rivlin [8] and [9] as well as to the book of Traub and Wozniakowski [16]. The values i_n and s_n were considered by Fisher and Micchelli [6] and [7] for the unit balls of Hilbert spaces of nonperiodic functions with simply connected domain of holomorphy.

Let $S_\beta := \{z\in\mathbb{C} : |\operatorname{Im} z| < \beta\}$ be a strip in the complex plane. For fixed integer $r\ge 0$ let $H^r_{\infty,\beta}$ denote the Hardy–Sobolev class of functions f, which are 2π-periodic, analytic in S_β, and satisfy $|f^{(r)}(z)|\le 1$ in S_β. Denote by $H^{r,\mathbb{R}}_{\infty,\beta}$ the subset of functions from $H^r_{\infty,\beta}$ that are real-valued on the real axis. In the case $r=0$ we will omit the upper index r. The Fourier coefficients of f are given by

$$a_k(f) := \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx, \quad k=0,1,\dots,$$
$$b_k(f) := \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx, \quad k=1,2,\dots.$$

In Section 1 we find an optimal algorithm for approximating $f(\zeta)$, $\zeta\in[0,2\pi)$, on the basis of the information

$$If = (a_0(f), a_1(f),\dots,a_{n-1}(f), b_1(f),\dots,b_{n-1}(f)), \tag{2}$$

uniformly for all $f\in H^r_{\infty,\beta}$. We show that the error $E(H^r_{\infty,\beta}, I)$ of an optimal algorithm is given by $\|\Phi^\beta_{n,r}\|_C$, where $\Phi^\beta_{n,r}$ is the r-th indefinite integral of a periodic Blaschke product with $2n$ equidistant, real zeros.

In Section 2 this result is applied to determine the optimal information error $i_{2n-1}(H^{r,\mathbb{R}}_{\infty,\beta}, C)$. We show that the Fourier coefficients are optimal information and that

$$\begin{aligned} i_{2n-1}(H^{r,\mathbb{R}}_{\infty,\beta}, C) &= d_{2n-1}(H^{r,\mathbb{R}}_{\infty,\beta}, C) = d^{2n-1}(H^{r,\mathbb{R}}_{\infty,\beta}, C) \\ &= \delta_{2n-1}(H^{r,\mathbb{R}}_{\infty,\beta}, C) = \|\Phi^{\beta}_{n,r}\|_C\,, \end{aligned}$$

where d_{2n-1}, d^{2n-1} and δ_{2n-1} denote the Kolmogorov, Gel′fand and linear widths, respectively. Osipenko [13] proved that a corresponding equation is valid in the even dimensional case. Thus $i_{2n-1}(H^{r,\mathbb{R}}_{\infty,\beta}, C) = i_{2n}(H^{r,\mathbb{R}}_{\infty,\beta}, C)$ and all three widths of order $2n-1$ and $2n$ coincide and are equal to $\|\Phi^{\beta}_{n,r}\|_C$.

In the case $r = 0$ we find in addition the optimal error $s_{2n-1}(H_{\infty,\beta}, C)$, which coincides with $s_{2n-1}(H^{\mathbb{R}}_{\infty,\beta}, C)$. It turns out that equidistant nodes are optimal. However, $s_{2n-1}(H^{\mathbb{R}}_{\infty,\beta}, C)$ is strictly greater than $i_{2n-1}(H^{\mathbb{R}}_{\infty,\beta}, C)$, i.e. sampling in optimal nodes does not yield optimal information. In particular, we calculate the value

$$\frac{s_{2n-1}(H^{\mathbb{R}}_{\infty,\beta}, C)}{i_{2n-1}(H^{\mathbb{R}}_{\infty,\beta}, C)},$$

which gives a quantitative measure, how much sampling fails to be optimal. This situation is in a sharp contrast to the even dimensional case, where it is known that sampling in equidistant nodes is optimal information (cf. Osipenko [11] and Wilderotter [18]). Moreover, we recall that Fisher and Micchelli [5] proved that for a simply connected domain of holomorphy sampling always yields optimal information.

In Section 3 we consider the problem of optimal recovery and optimal information for the class $H_{p,\beta}$, $1 \le p < \infty$. Here $H_{p,\beta}$ denotes the set of all functions f, which are 2π-periodic, analytic in S_β, and satisfy

$$\sup_{0 \le \eta < \beta} \left(\frac{1}{4\pi} \int_0^{2\pi} \left(|f(t+i\eta)|^p + |f(t-i\eta)|^p \right) dt \right)^{1/p} \le 1.$$

For fixed points $z_1, \dots, z_n \in [0, 2\pi)$ with multiplicities $\nu_1, \dots, \nu_n \in \mathbb{N}$ and $\zeta \in [0, 2\pi)$ we find an optimal algorithm and the intrinsic error for approximating $f(\zeta)$, $f \in H_{p,\beta}$, on the basis of the Hermite information

$$If = \left(f(z_1), \dots, f^{(\nu_1 - 1)}(z_1), \dots, f(z_n), \dots, f^{(\nu_n - 1)}(z_n) \right).$$

We also find the optimal sampling error $s_n(H_{p,\beta}, C)$. It turns out that sampling in equidistant nodes is optimal for all p and all n. Moreover, for $p = 2$ we compare $s_n(H_{2,\beta}, C)$ with the optimal information error $i_n(H_{2,\beta}, C)$. We show that these quantities do not coincide and calculate the ratios

$$\frac{s_{2n-1}(H_{2,\beta}, C)}{i_{2n-1}(H_{2,\beta}, C)}, \qquad \frac{s_{2n}(H_{2,\beta}, C)}{i_{2n}(H_{2,\beta}, C)}.$$

The nonoptimality of sampling in the even dimensional case is quite remarkable. In all examples studied so far for the imbedding of $H_{p,\beta}$ in L_q with $p \ge q$ (see Osipenko [11], Wilderotter [19]) we found that sampling in $2n$ equidistant nodes yields optimal information for i_{2n}. The present paper shows that this fails to be valid for the imbedding of $H_{2,\beta}$ in C.

Throughout the paper we use substantially elliptic function techniques. We emphasize that pretty optimal elliptic function bounds date back already to the classical work of N. I. Achieser [1], which influenced and stimulated the present article.

1. Optimal recovery from Fourier coefficients

This section deals with the optimal recovery of the linear functional $Uf = f(\zeta)$, $\zeta \in [0, 2\pi)$, on $H^r_{\infty,\beta}$, using the information (2). Of central importance for our considerations is the following well known general duality formula due to Smolyak (we use here the complex version of Smolyak's result proved by Osipenko [10]):

$$E(H^r_{\infty,\beta}, I) = \sup_{\substack{f \in H^r_{\infty,\beta} \\ If=0}} |f(\zeta)|. \tag{3}$$

Moreover, the minimal error is achieved by a linear method of the form

$$A^*(If) = c_0 a_0(f) + \sum_{j=1}^{n-1} (c_j a_j(f) + d_j b_j(f)). \tag{4}$$

By an extremal function we mean any function $f_0 \in H^r_{\infty,\beta}$ with $If_0 = 0$ and $|f_0(\zeta)| = E(H^r_{\infty,\beta}, I)$.

Our further strategy will be to determine explicitly an extremal function f_0. For this purpose we need some auxiliary facts about periodic Blaschke products.

In order to introduce periodic Blaschke products, we transfer the analysis from the strip S_β to the annulus $\Omega := \{w \in \mathbb{C} : R < |w| < R^{-1}\}$, where $R = e^{-\beta}$. The universal covering transformation $w = e^{iz}$ maps S_β onto Ω and induces a correspondence $f(z) \to g(w) = f\left(\frac{1}{i} \ln w\right)$ between analytic periodic functions in S_β and analytic functions in Ω.

A Blaschke product B of degree m on Ω is a function of the form

$$B(w) = \exp\left(-\sum_{j=1}^{m} (g(w, \alpha_j) + ih(w, \alpha_j))\right).$$

Here $\alpha_1, \dots, \alpha_m$ are points in Ω, $g(w, \alpha_j)$ is the Green's function for Ω with singularity at α_j and $h(w, \alpha_j)$ is the harmonic conjugate of $g(w, \alpha_j)$. In general B is multiple valued. However, if we choose $m = 2n$ and locate all points $\alpha_1, \dots, \alpha_{2n}$ on the unit circle $\{w \in \mathbb{C} : |w| = 1\}$, it turns out that B is single valued. For a proof of the last fact and further details on Blaschke products we refer to Fisher [4] and Wilderotter [18].

In particular we may choose the $2n$ zeros on the unit circle to be equidistant. Let $\alpha_j^* = \exp\left(i(j-1)\frac{\pi}{n}\right)$ for $j = 1, \dots, 2n$ and

$$B_{2n}(w) = \exp\left(-\sum_{j=1}^{2n} (g(w, \alpha_j^*) + ih(w, \alpha_j^*))\right).$$

Finally we go back again from the annulus to our original setting of the strip and introduce the periodic Blaschke product $\widetilde{B}_{2n}$ on S_β by defining $\widetilde{B}_{2n}(z) := B_{2n}(e^{iz})$.

Blaschke products are closely related to elliptic functions. Throughout the present paper we will use the following terminology (see for example Achieser [2], Bateman [3]): $\operatorname{sn}(z, k)$, $\operatorname{cn}(z, k)$, and $\operatorname{dn}(z, k)$ denote the Jacobi elliptic functions with modulus k (further we will note the dependence of the Jacobi elliptic functions on the modulus only in case the modulus is different from k); the complementary modulus is given by $k' = \sqrt{1-k^2}$ and the complete elliptic integrals of the first kind with moduli k and k' are denoted by K and K', respectively. We always suppose that K and K' satisfy the equation

$$\frac{\pi K'}{2K} = \beta.$$

With this notation $\widetilde{B}_{2n}$ can be written in the form (see Osipenko [11]):

$$\widetilde{B}_{2n}(z) = k^n \prod_{j=1}^{2n} \operatorname{sn}\left(\frac{K}{\pi}z - (j-1)\frac{K}{n}\right).$$

Using the first fundamental transformation of elliptic functions of degree $2n$ one can show that

$$\widetilde{B}_{2n}(z) = -\sqrt{\lambda}\operatorname{sn}\left(\frac{2n\Lambda}{\pi}z, \lambda\right).$$

Here Λ is the complete elliptic integral of the first kind with modulus λ determined by the equation

$$\frac{\Lambda'}{\Lambda} = 2n\frac{K'}{K}.$$

In order to cope with the optimal recovery problem, we introduce the r-th indefinite integral $\Phi_{n,r}^{\beta}$ of $-\widetilde{B}_{2n}$ defined by

$$\Phi_{n,0}^{\beta} := -\widetilde{B}_{2n}, \qquad \Phi_{n,r}^{\beta} := D_r * \Phi_{n,0}^{\beta}, \quad r \geq 1.$$

Here

$$D_r(t) = 2\sum_{k=1}^{\infty} \frac{\cos(kt - \pi r/2)}{k^r}, \quad r = 1, 2, \dots,$$

is the Bernouilli Monospline, while

$$(f * g)(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z-t)g(t)\,dt$$

denotes the convolution of two periodic functions.

Osipenko [13] gave the following explicit representation for $\Phi_{n,r}^{\beta}$ and $\|\Phi_{n,r}^{\beta}\|_C$:

$$\left.\begin{aligned} \Phi_{n,r}^{\beta}(z) &= \frac{\pi}{\sqrt{\lambda}\Lambda n^r}\sum_{k=0}^{\infty}\frac{\sin((2k+1)nz - \pi r/2)}{(2k+1)^r\sinh((2k+1)2n\beta)}, \\ \|\Phi_{n,r}^{\beta}\|_C &= \frac{\pi}{\sqrt{\lambda}\Lambda n^r}\sum_{k=0}^{\infty}\frac{(-1)^{k(r+1)}}{(2k+1)^r\sinh((2k+1)2n\beta)}, \end{aligned}\right. \qquad r = 0, 1, \dots.$$

From this one can read off that $I\Phi_{n,r}^{\beta} = 0$.

We now are ready to formulate our first main result.

Theorem 1. *For all integers $r \geq 0$ and with I defined by (2),*

$$E(H_{\infty,\beta}^r, I) = \|\Phi_{n,r}^{\beta}\|_C.$$

Proof. We can assume without loss of generality that the fixed evaluation point in the problem (3) is equal to $\zeta = 0$. Put

$$\varphi(z) := \begin{cases} \Phi_{n,r}^{\beta}\left(z + \dfrac{\pi}{2n}\right), & r = 2k, \\ \Phi_{n,r}^{\beta}(z), & r = 2k+1. \end{cases}$$

We wish to show that φ is an extremal function of the problem (3). Note that $I\varphi = 0$, $|\varphi(0)| = \|\Phi^{\beta}_{n,r}\|_C$, and φ is an even function. Suppose there exists a function $f_0 \in H^r_{\infty,\beta}$ with $If_0 = 0$ and $|f_0(0)| > |\varphi(0)|$. After scaling f_0 with the factor $\exp(-i \arg f_0(0))$, we may assume $f_0(0)$ to be real and positive. Let us define

$$f_1(z) := \frac{f_0(z) + \overline{f_0(\overline{z})}}{2}, \qquad f_2(z) := \frac{f_1(z) + f_1(-z)}{2}.$$

Then $f_2 \in H^{r,\mathbb{R}}_{\infty,\beta}$, $If_2 = 0$, and $f_2(0) = f_0(0)$. Moreover, f_2 is an even function. Set

$$\rho := \varphi(0)/f_2(0), \qquad F := \varphi - \rho f_2.$$

We claim that the function F has at least $2n+1$ distinct zeros in $[-\pi, \pi)$. Clearly $F(0) = 0$. Moreover, since both φ and f_2 are even functions, F does not change its sign in $\zeta = 0$. On the other side we have $IF = 0$, since $I\varphi = If_2 = 0$. The condition $IF = 0$ means that

$$\int_0^{2\pi} F(x) \cos kx \, dx = 0, \quad k = 0, 1, \ldots, n-1,$$

$$\int_0^{2\pi} F(x) \sin kx \, dx = 0, \quad k = 1, 2, \ldots, n-1.$$

Since the trigonometric polynomials of degree at most $n-1$ are a Tchebycheff system of dimension $2n-1$, it follows from Pinkus [15, Chap. III, Prop. 1.4], that F has at least $2n$ cyclic sign changes. In addition F has a zero in $\zeta = 0$ without sign change. Hence F has altogether at least $2n+1$ zeros in $[-\pi, \pi)$. By Rolle's theorem the same conclusion remains valid for the r-th derivative $F^{(r)} = \varphi^{(r)} - \rho f_2^{(r)}$.

Transferring this result from the strip to the annulus, we see that the function $F^{(r)}\left(\frac{1}{i} \ln w\right)$ is single valued and analytic in Ω and has at least $2n+1$ zeros in Ω. By the definition of $\Phi^{\beta}_{n,r}$ we have

$$\varphi^{(r)}\left(\frac{1}{i} \ln w\right) = \begin{cases} -B_{2n}\left(w \exp\left(i\frac{\pi}{2n}\right)\right), & r = 2k, \\ -B_{2n}(w), & r = 2k+1. \end{cases}$$

The boundary values of the Blaschke product B_{2n} satisfy identically $|B_{2n}(w)| \equiv 1$ on $\partial\Omega$. Consequently we have for $w \in \partial\Omega$

$$\left|\varphi^{(r)}\left(\frac{1}{i} \ln w\right) - F^{(r)}\left(\frac{1}{i} \ln w\right)\right| = \left|\rho f_2^{(r)}\left(\frac{1}{i} \ln w\right)\right| \le |\rho| < 1 = \left|\varphi^{(r)}\left(\frac{1}{i} \ln w\right)\right|.$$

Since B_{2n} has $2n$ zeros in Ω, Rouche's theorem implies that $F^{(r)}\left(\frac{1}{i} \ln w\right)$ has exactly $2n$ zeros in Ω. This is a contradiction and the proof of Theorem 1 is complete. □

2. Optimal information and n-widths of $H^{r,\mathbb{R}}_{\infty,\beta}$

In this section Theorem 1 is applied to determine the optimal information error $i_{2n-1}(H^{r,\mathbb{R}}_{\infty,\beta}, C)$. It turns out that $i_{2n-1}(H^{r,\mathbb{R}}_{\infty,\beta}, C)$ coincides with certain odd dimensional n-widths. Therefore we start by recalling the definition of the various n-widths.

Let V be a subset of a normed linear space X. The Kolmogorov n-widths are defined by

$$d_n(V,X) := \inf_{X_n} \sup_{x \in V} \inf_{y \in X_n} \|x-y\|_X,$$

where X_n runs over all subspaces of X of dimension n or less.

The Gel'fand n-widths are defined by

$$d^n(V,X) := \inf_{X^n} \sup_{x \in X^n \cap V} \|x\|_X,$$

where X^n runs over all subspaces of codimension at most n (here we assume that $0 \in V$).

The linear n-widths are given by

$$\delta_n(V,X) := \inf_{P_n} \sup_{x \in V} \|x - P_n x\|_X,$$

where P_n is any linear operator of X into X of rank at most n.

Much information on n-widths can be found in the book of A. Pinkus [15]. In particular, the following fundamental inequality holds:

$$d_n(V,X)\,,\, d^n(V,X) \le \delta_n(V,X). \tag{5}$$

Analogously to (1) we can define the optimal information error $i_n(V,X)$ for estimating V in X by n linear observations.

Lemma. *Assume that V is a centrally symmetric set and $0 \in V$. Then*

$$d^n(V,X) \le i_n(V,X) \le \delta_n(V,X). \tag{6}$$

Proof. The inequality

$$i_n(V,X) \le \delta_n(V,X)$$

evidently follows from the definition. To prove the lower bound consider any continuous linear functionals $L_1, \ldots, L_n$. For each $\varepsilon > 0$ there exists $x_\varepsilon \in V$ such that $L_1 x_\varepsilon = \cdots = L_n x_\varepsilon = 0$ and

$$\sup_{\substack{x \in V \\ L_1 x = \cdots = L_n x = 0}} \|x\|_X \le \|x_\varepsilon\|_X + \varepsilon.$$

For all algorithms A we have

$$\|x_\varepsilon - A(0,\ldots,0)\|_X + \|-x_\varepsilon - A(0,\ldots,0)\|_X \ge 2\|x_\varepsilon\|_X.$$

Therefore,

$$\sup_{x \in V} \|x - A(L_1 x, \ldots, L_n x)\|_X \ge \|x_\varepsilon\|_X \ge \sup_{\substack{x \in V \\ L_1 x = \cdots = L_n x = 0}} \|x\|_X - \varepsilon \ge d^n(V,X) - \varepsilon.$$

Taking the infimum over A and $L_1 \ldots, L_n$ we obtain

$$i_n(V,X) \ge d^n(V,X). \qquad \square$$

Our result reads now as follows:

Theorem 2. *For all integer $r \geq 0$*

$$i_{2n-1}(H^{r,\mathbb{R}}_{\infty,\beta}, C) = d_{2n-1}(H^{r,\mathbb{R}}_{\infty,\beta}, C) = d^{2n-1}(H^{r,\mathbb{R}}_{\infty,\beta}, C) = \delta_{2n-1}(H^{r,\mathbb{R}}_{\infty,\beta}, C) = \|\Phi^{\beta}_{n,r}\|_C\,.$$

Proof. In view of (5) and (6) to establish upper bounds we may restrict ourselves to $\delta(H^{r,\mathbb{R}}_{\infty,\beta}, C)$. It follows from Theorem 1 that there exists an optimal method (4) such that

$$|f(0) - A^*(If)| \leq \|\Phi^{\beta}_{n,r}\|_C$$

for all $f \in H^{r,\mathbb{R}}_{\infty,\beta}$. Now let η be an arbitrary fixed point in the interval $[0, 2\pi)$ and set $(T_\eta f)(z) := f(z+\eta)$. Since

$$\begin{aligned} a_j(T_\eta f) &= a_j(f)\cos j\eta + b_j(f)\sin j\eta, \\ b_j(T_\eta f) &= -a_j(f)\sin j\eta + b_j(f)\cos j\eta, \end{aligned}$$

we obtain that

$$\Bigg| f(\eta) - c_0 a_0(f) - \sum_{j=1}^{n-1}\Big((c_j\cos j\eta - d_j \sin j\eta)a_j(f) + (c_j \sin j\eta + d_j\cos j\eta)b_j(f)\Big)\Bigg| \leq \|\Phi^{\beta}_{n,r}\|_C.$$

This pointwise estimate holds uniformly in $[0, 2\pi)$. Thus we have

$$\delta_{2n-1}(H^{r,\mathbb{R}}_{\infty,\beta}, C) \leq \|\Phi^{\beta}_{n,r}\|_C\,.$$

As mentioned in the introduction, Osipenko [13] proved that

$$d_{2n}(H^{r,\mathbb{R}}_{\infty,\beta}, C) = d^{2n}(H^{r,\mathbb{R}}_{\infty,\beta}, C) = \delta_{2n}(H^{r,\mathbb{R}}_{\infty,\beta}, C) = \|\Phi^{\beta}_{n,r}\|_C\,. \tag{7}$$

The lower bounds now follow from the monotonicity of the n-widths. □

Combining (7) with Theorem 2, we get in view of (6) that $i_{2n-1}(H^{r,\mathbb{R}}_{\infty,\beta}, C)$ and $i_{2n}(H^{r,\mathbb{R}}_{\infty,\beta}, C)$ as well as all three kinds of widths of order $2n-1$ and $2n$ coincide and are equal to $\|\Phi^{\beta}_{n,r}\|_C$.

The preceding analysis may give the impression that the situation in odd and even dimensions is identical. This is definitely not true. Although the different values of the widths are all the same, the properties of optimal information are substantially different in odd and even dimensions. In the sequel we will restrict ourselves to the case $r = 0$. Our course of proof showed that the Fourier coefficients $(a_0(f), a_1(f), \dots, a_{n-1}(f), b_1(f), \dots, b_{n-1}(f))$ are optimal information for $i_{2n-1}(H^{\mathbb{R}}_{\infty,\beta}, C)$ and consequently also for $i_{2n}(H^{\mathbb{R}}_{\infty,\beta}, C)$. However, Osipenko [11] and Wilderotter [18] proved that in the even dimensional case sampling in $2n$ equidistant nodes yields optimal information as well, that is $s_{2n}(H^{\mathbb{R}}_{\infty,\beta}, C) = i_{2n-1}(H^{\mathbb{R}}_{\infty,\beta}, C)$. We now try to find the optimal sampling error $s_{2n-1}(H^{\mathbb{R}}_{\infty,\beta}, C)$.

For this purpose we consider in a first step fixed sampling points $z_1, \dots, z_{2n-1} \in [0, 2\pi)$. From the results of Ovchincev [14] and Wilderotter [17] it follows that

$$\inf_{A\colon \mathbb{R}^{2n-1}\to C}\ \sup_{f \in H^{\mathbb{R}}_{\infty,\beta}} \|f - A(f(z_1), \dots, f(z_{2n-1}))\|_C = k^n \left\| \prod_{j=1}^{2n-1} \operatorname{sn}\left(\frac{K}{\pi}(\cdot - z_j)\right)\right\|_C.$$

In a second step we minimize the right-hand side of the last equation over all possible choices of sampling points. Osipenko [11] showed in a different context that

$$\inf_{z_1,\dots,z_n\in[0,2\pi)} k^{n/2}\left\|\prod_{j=1}^{n} \operatorname{sn}\left(\frac{K}{\pi}(\cdot - z_j)\right)\right\|_C = \sqrt{\lambda_n}, \tag{8}$$

where

$$\lambda_n = 4e^{-\beta n}\left(\frac{\sum_{m=0}^{\infty} e^{-2\beta n m(m+1)}}{1+2\sum_{m=1}^{\infty} e^{-2\beta n m^2}}\right)^2 = 4e^{-\beta n} + O(e^{-3\beta n}) \tag{9}$$

(λ_n can also be defined as a solution of the equation $\Lambda'/\Lambda = nK'/K$). Moreover, equidistant nodes are the unique nodes (up to a shift), for which the infimum in (8) is attained. Thus

$$s_{2n-1}(H^{\mathbb{R}}_{\infty,\beta}, C) = \sqrt{k\lambda_{2n-1}}.$$

On the other side we have

$$i_{2n-1}(H^{\mathbb{R}}_{\infty,\beta}, C) = i_{2n}(H^{\mathbb{R}}_{\infty,\beta}, C) = s_{2n}(H^{\mathbb{R}}_{\infty,\beta}, C) = \|\Phi_{n,0}\|_C = \sqrt{\lambda_{2n}}.$$

Set $z_j^* := (j-1)\dfrac{2\pi}{2n-1}$, $j = 1, \dots, 2n-1$,

$$b_1(z) := \sqrt{k}\,\operatorname{sn}\left(\frac{K}{\pi}(z - z_{n+1}^*)\right), \qquad b_2(z) := k^{n-1/2}\prod_{j=1}^{2n-1} \operatorname{sn}\left(\frac{K}{\pi}(z - z_j^*)\right).$$

Using the first fundamental transformation of elliptic functions of degree $2n-1$ it can be shown that

$$b_2(z) = \sqrt{\lambda_{2n-1}}\,\operatorname{sn}\left(\frac{(2n-1)\Lambda_{2n-1}}{\pi}z, \lambda_{2n-1}\right),$$

where Λ_{2n-1} is the complete elliptic integral of the first kind with modulus λ_{2n-1}. It is easy to check that

$$\|b_1\|_C = -b_1\left(\frac{\pi}{2n-1}\right) = \sqrt{k}, \qquad \|b_2\|_C = b_2\left(\frac{\pi}{2n-1}\right) = \sqrt{\lambda_{2n-1}}.$$

Consequently

$$\|b_1 b_2\|_C = \sqrt{k\lambda_{2n-1}}.$$

Since equidistant nodes are unique optimal nodes in the extremal problem (8) we obtain that $\sqrt{\lambda_{2n}} < \sqrt{k\lambda_{2n-1}}$. Thus

$$s_{2n-1}(H^{\mathbb{R}}_{\infty,\beta}, C) > i_{2n-1}(H^{\mathbb{R}}_{\infty,\beta}, C),$$

i.e. sampling does not yield optimal information in odd dimensions.

More precisely we may calculate the following ratio, which gives a quantitative measure, how much sampling fails to be optimal:

$$\frac{s_{2n-1}(H^{\mathbb{R}}_{\infty,\beta}, C)}{i_{2n-1}(H^{\mathbb{R}}_{\infty,\beta}, C)} = \frac{\sqrt{k\lambda_{2n-1}}}{\sqrt{\lambda_{2n}}} = \sqrt{k}e^{\beta/2} + O(e^{-4\beta n}).$$

For $n = 1$ from (9) it follows that

$$k = 4e^{-\beta} \left(\frac{\sum_{m=0}^{\infty} e^{-2\beta m(m+1)}}{1 + 2\sum_{m=1}^{\infty} e^{-2\beta m^2}} \right)^2 .$$

Using this equality it is easy to show that

$$e^{-\beta/2} < \sqrt{k} < 2e^{-\beta/2}.$$

Thus $1 < \sqrt{k} e^{\beta/2} < 2$ for all $\beta \in (0, +\infty)$.

3. Optimal sampling and information in $H_{p,\beta}$

Denote by $\mathcal{H}_{p,\beta}$, $1 \leq p \leq \infty$, the space of all 2π-periodic functions f, which are analytic in S_β and satisfy

$$\|f\|_{\mathcal{H}_{p,\beta}} := \sup_{0 \leq \eta < \beta} \left(\frac{1}{4\pi} \int_0^{2\pi} (|f(t+i\eta)|^p + |f(t-i\eta)|^p)\, dt \right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$
$$\|f\|_{\mathcal{H}_{\infty,\beta}} := \sup_{z \in S_\beta} |f(z)| < \infty.$$

Let $H_{p,\beta}$ be the closed unit ball of $\mathcal{H}_{p,\beta}$. Given an evaluation point $\zeta \in [0, 2\pi)$ consider the problem of optimal recovery of $f(\zeta)$, $f \in H_{p,\beta}$, on the basis of the Hermite information

$$If = (f(z_1), \ldots, f^{(\nu_1 - 1)}(z_1), \ldots, f(z_n), \ldots, f^{(\nu_n - 1)}(z_n)), \qquad N := \sum_{j=1}^{n} \nu_j,$$

where $z_1, \ldots, z_n \in [0, 2\pi)$. The case $p = \infty$ was obtained by Ovchincev [14] and Wilderotter [17]. The solution of this recovery problem for $1 \leq p < \infty$ reads as follows.

Theorem 3. *Set*

$$W(z) = k^{N/2} \prod_{j=1}^{n} \mathrm{sn}^{\nu_j} \left(\frac{K}{\pi}(z - z_j) \right).$$

Then

$$E(H_{p,\beta}, I) = \begin{cases} \left(\dfrac{2K}{\pi} \right)^{1/p} |W(\zeta)|, & N \text{ even}, \\ \left(\dfrac{2K}{\pi} \right)^{1/p} \sqrt{k} |W(\zeta)|, & N \text{ odd}. \end{cases}$$

An optimal method of recovery is given by

$$f(\zeta) \approx \sum_{j=1}^{n} \sum_{\nu=0}^{\nu_j - 1} c_{j\nu}(\zeta, p) f^{(\nu)}(z_j), \tag{10}$$

where

$$c_{j\nu}(\zeta,p) = \frac{K}{\pi}\frac{W(\zeta)}{\nu!(\nu_j-\nu-1)!}$$
$$\times \lim_{z\to z_j}\left(\frac{(z-z_j)^{\nu_j}\gamma_N(z)\,\mathrm{dn}^{\frac{p-2}{p}}\left(\frac{K}{\pi}(\zeta-z)\right)}{W(z)\,\mathrm{sn}\left(\frac{K}{\pi}(\zeta-z)\right)}\right)^{(\nu_j-\nu-1)},$$

$$\gamma_N(z) = \begin{cases} \mathrm{cn}\left(\frac{K}{\pi}(\zeta-z)\right), & N \textit{ even},\\ \mathrm{dn}\left(\frac{K}{\pi}(\zeta-z)\right), & N \textit{ odd}.\end{cases}$$

Proof. The function

$$b(z) = \sqrt{k}\,\mathrm{sn}\,\frac{K}{\pi}z$$

is analytic in S_β. Moreover, $b(z+2\pi) = -b(z)$ and $|b(x+i\beta)| \equiv 1$ for all $x \in \mathbb{R}$. Thus $\overline{W(z)} = W^{-1}(z)$ for $z \in \partial S_\beta$.

Suppose N is an even number. Consider the function

$$g(z) = W(z)\,\mathrm{dn}^{2/p}\left(\frac{K}{\pi}(\zeta-z)\right).$$

Since $\mathrm{dn}\,\frac{K}{\pi}z$ is 2π-periodic and does not vanish in the strip S_β, $g \in \mathcal{H}_{\infty,\beta}$. Set

$$\alpha := \frac{2K}{\pi}g(\zeta).$$

For $f \in \mathcal{H}_{p,\beta}$ consider the integral

$$Jf := \frac{\alpha}{4\pi}\int_{\Gamma_0}\overline{g(z)}|g(z)|^{p-2}f(z)\,dz,$$

where $\Gamma_0 := [-i\beta, 2\pi - i\beta] \cup [i\beta, 2\pi + i\beta]$. Using the properties of elliptic functions, we have

$$\overline{\mathrm{dn}\left(\frac{K}{\pi}(x\pm i\beta)\right)} = \pm i\frac{\mathrm{cn}\left(\frac{K}{\pi}(x\pm i\beta)\right)}{\mathrm{sn}\left(\frac{K}{\pi}(x\pm i\beta)\right)}.$$

The element of integration in Jf is 2π-periodic. So we can rewrite Jf in the following form

$$Jf = \frac{KW(\zeta)}{\pi}\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{\mathrm{cn}\left(\frac{K}{\pi}(\zeta-z)\right)\mathrm{dn}^{\frac{p-2}{p}}\left(\frac{K}{\pi}(\zeta-z)\right)}{W(z)\,\mathrm{sn}\left(\frac{K}{\pi}(\zeta-z)\right)}f(z)\,dz,$$

where Γ_ε is the boundary of the rectangle $-\varepsilon \le \mathrm{Re}\,z \le 2\pi - \varepsilon$, $|\mathrm{Im}\,z| \le \beta$, and ε such that $\zeta, z_1, \ldots, z_n$ lie inside this rectangle. By the residue theorem

$$Jf = f(\zeta) - \sum_{j=1}^{n}\sum_{\nu=0}^{\nu_j-1} c_{j\nu}(\zeta,p)f^{(\nu)}(z_j). \tag{11}$$

For $f(z) = g(z)$ this equality gives

$$\|g\|_{\mathcal{H}_{p,\beta}} = \left(\frac{\pi}{2K}\right)^{1/p}.$$

If $f \in H_{p,\beta}$, then by Hölder's inequality we obtain

$$|Jf| \le |\alpha| \, \|g\|_{\mathcal{H}_{p,\beta}}^{p/q} \|f\|_{\mathcal{H}_{p,\beta}} \le \left(\frac{2K}{\pi}\right)^{1/p} |g(\zeta)|, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

In view of (11) we have

$$E(H_{p,\beta}, I) \le \left(\frac{2K}{\pi}\right)^{1/p} |g(\zeta)|.$$

On the other hand, $g_0 := g/\|g\|_{\mathcal{H}_{p,\beta}} \in H_{p,\beta}$ and $Ig_0 = 0$. Consequently,

$$E(H_{p,\beta}, I) = \sup_{\substack{f \in H_{p,\beta} \\ If=0}} |f(\zeta)| \ge |g_0(\zeta)| = \left(\frac{2K}{\pi}\right)^{1/p} |g(\zeta)|.$$

Hence

$$E(H_{p,\beta}, I) = \left(\frac{2K}{\pi}\right)^{1/p} |g(\zeta)|$$

and (10) is an optimal method of recovery.

For odd N the same scheme of proof is applied to Jf with

$$g(z) = \sqrt{k}\,\mathrm{sn}\left(\frac{K}{\pi}(z - \zeta + \pi)\right) W(z)\,\mathrm{dn}^{2/p}\left(\frac{K}{\pi}(\zeta - z)\right)$$

(here we use that $\mathrm{sn}(u + K) = \mathrm{cn}\,u/\,\mathrm{dn}\,u$). $\square$

Taking into account the equality (8), we have

Corollary. *For all $1 \le p < \infty$ and $n \in \mathbb{N}$,*

$$s_n(H_{p,\beta}, C) = \begin{cases} \left(\dfrac{2K}{\pi}\right)^{1/p} \sqrt{\lambda_n}, & n \text{ even}, \\ \left(\dfrac{2K}{\pi}\right)^{1/p} \sqrt{k\lambda_n}, & n \text{ odd}, \end{cases}$$

where λ_n is defined by (9). Moreover, equidistant nodes are optimal.

Finally we compare in the case $p = 2$ the optimal sampling error $s_n(H_{2,\beta}, C)$ with the optimal information error $i_n(H_{2,\beta}, C)$. Osipenko [12] proved that

$$\delta_{2n-1}(H_{2,\beta}, C) = d^{2n-1}(H_{2,\beta}, C) = \left(2\sum_{j=n}^{\infty} \frac{1}{\cosh 2j\beta}\right)^{1/2}$$

$$= \frac{2}{\sqrt{1-e^{-2\beta}}} e^{-\beta n} + O(e^{-5\beta n}),$$

$$\delta_{2n}(H_{2,\beta}, C) = d^{2n}(H_{2,\beta}, C) = \left(\frac{1}{\cosh 2n\beta} + 2\sum_{j=n+1}^{\infty} \frac{1}{\cosh 2j\beta}\right)^{1/2}$$

$$= \sqrt{2\frac{1+e^{-2\beta}}{1-e^{-2\beta}}}\, e^{-\beta n} + O(e^{-5\beta n}).$$

In view of (6) the same equalities hold for $i_n(H_{2,\beta}, C)$. Thus we obtain

$$\frac{s_{2n-1}(H_{2,\beta}, C)}{i_{2n-1}(H_{2,\beta}, C)} = 2\sqrt{\frac{kK}{\pi} \sinh\beta} + O(e^{-4\beta n}),$$

$$\frac{s_{2n}(H_{2,\beta}, C)}{i_{2n}(H_{2,\beta}, C)} = 2\sqrt{\frac{K}{\pi} \tanh\beta} + O(e^{-4\beta n}).$$

The last result is very interesting, inasmuch as it is the first example known so far of a periodic Hardy space imbedding, for which sampling in equidistant nodes does not yield optimal information in even dimensions.

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