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SEVEN CONSECUTIVE PRIMES IN ARITHMETIC PROGRESSION

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ABSTRACT. It is conjectured that there exist arbitrarily long sequences of *consecutive* primes in arithmetic progression. In 1967, the first such sequence of 6 consecutive primes in arithmetic progression was found. Searching for 7 consecutive primes in arithmetic progression is difficult because it is necessary that a prescribed set of at least 1254 numbers between the first and last prime all be composite. This article describes the search theory and methods, and lists the only known example of 7 consecutive primes in arithmetic progression.

1. INTRODUCTION

It is conjectured that the number of primes in arithmetic progression can be as large as you like [2]. A prodigious amount of computer time has been used to search for long strings of primes in arithmetic progression, with the current record being 22 [6]. A related conjecture is the following: there exist arbitrarily long sequences of *consecutive* primes in arithmetic progression [2]. In 1967, Lander and Parkin [4] reported finding the first and smallest sequence of 6 consecutive primes in AP, where the starting prime is 121174811 and the common difference is 30. Since then many other sequences of 6 such primes have been found, as well as other sets of 6 consecutive primes with common differences of 60 and 90 [9].

It is easy to show that 7 primes in AP must have a common difference that is at least 210 (excluding the singular case where the first prime is 7) [7]. But if these are also to be consecutive, this implies that there is a prescribed set of 1254 numbers between the first and last prime that must be composite. This simultaneous requirement for having 7 primes in AP and also a prescribed set of 1254 composite numbers, is the reason for the difficulty in finding 7 consecutive primes in AP. However, by extending a method of Nelson for assuring that a given set of numbers would be composite, such a search became practical [5].

In this article we describe the search theory and methods, and give the only known example of 7 consecutive primes in arithmetic progression.

2. DENSITY OF CONSECUTIVE PRIMES

In order to help estimate the search time for finding 7 consecutive primes in AP we can use the Prime Number Theorem to estimate the density of prescribed sets of primes and prescribed sets of composites as a function of the size of the numbers.

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TABLE 1. Density of consecutive primes in arithmetic progression

number consec primes	number of composites	digits for max density	approx probabilities primes	compos	both
5	116	11	$9*10^{-8}$	$9*10^{-3}$	$9*10^{-10}$
6	145	11	$4*10^{-9}$	$3*10^{-3}$	$1*10^{-11}$
7	1254	78	$1*10^{-16}$	$1*10^{-3}$	$2*10^{-19}$
8	1463	80	$8*10^{-19}$	$3*10^{-4}$	$3*10^{-22}$
9	1672	81	$4*10^{-21}$	$1*10^{-4}$	$5*10^{-25}$
10	1881	82	$2*10^{-23}$	$4*10^{-5}$	$8*10^{-28}$
11	23090	910	$5*10^{-37}$	$1*10^{-5}$	$5*10^{-42}$

Assuming statistical independence, the approximate probability of finding k primes near a number N is,

$$P(k) \approx \left(\frac{1}{\log N}\right)^k, \tag{1}$$

while the minimum number, r, of required composites is,

$$r = (k-1)(d-1), \quad \begin{cases} d = 30 & \text{for } k = 5, 6, \\ d = 210 & \text{for } k = 7, 8, 9, 10, \\ d = 2310 & \text{for } k = 11. \end{cases} \tag{2}$$

Similarly, the approximate probability of finding r composites near N is,

$$C(k) \approx \left(1 - \frac{1}{\log N}\right)^r. \tag{3}$$

Thus, for a random search near N, the approximate probability of finding k consecutive primes in AP is the product of equations (1) and (3),

$$Q(k) \approx \left(\frac{1}{\log N}\right)^k \cdot \left(1 - \frac{1}{\log N}\right)^r. \tag{4}$$

For each k we can calculate the value of N that optimizes the probability, $Q(k)$. Near such an N is where the density of k consecutive primes in AP should be at a maximum. Table 1 summarizes these results.

While simply searching the sequence of primes up to about 10^{10} would be a viable way to look for 6 consecutive primes in AP, the size of the likely candidate values for 7 such primes preclude this approach. However we could search for 7 primes with a common difference of 210 near 78 digit numbers using appropriate sieving methods, and when such a prime set is found, test the intermediate 1254 numbers. About 1 time in 1000 they should all be composite. We estimated that such a search would take about 10 computer-years on a PC 486/66, which was much too long to be practical. The factor of 1000 contributed by the 1254 composites was a major source of difficulty.

3. COMPOSITE SEQUENCES

In 1975 one of the authors solved a problem in the Journal of Recreational Mathematics by developing a method that used a system of simultaneous modular

TABLE 2. Composite-set covering – 1254 numbers

Numbers left vs. number of equations

n	prime	$b(n)$	left
1	2	0	624
2	3	0	414
3	5	0	330
4	7	0	282
5	11	0	255
6	13	7	233
7	17	5	217
8	19	7	204
9	23	6	191
10	29	9	183
11	31	19	175
12	37	9	168
13	41	4	161
14	43	40	154
15	47	22	148
16	53	2	143
17	59	57	137
18	61	2	133
19	67	44	128
20	71	7	124
21	73	33	120
22	79	33	116
23	83	5	113
24	89	7	109
25	97	25	105
26	101	2	102
27	103	20	99
28	107	22	96
29	109	77	93
30	113	11	91
31	127	80	88
32	131	2	86
33	137	5	83
34	139	50	80
35	149	12	77
36	151	136	74
37	157	6	72
38	163	54	70
39	167	143	67
40	173	35	65
41	79	16	62
42	181	80	59
43	191	45	57
44	193	112	55
45	197	4	53
46	199	157	51
47	211	84	49
48	223	69	47
49	227	54	45
50	229	15	43
51	233	86	41
52	239	186	39
53	241	225	37
54	251	123	35
55	257	178	33
56	263	213	31
57	269	7	30
58	271	0	29
59	277	204	27
60	281	21	26
61	283	28	25
62	293	190	23
63	307	56	22
64	311	72	21
65	313	105	20
66	317	304	18
67	331	8	17
68	337	1	16
69	347	2	15
70	349	50	14
71	353	92	13
72	359	5	12
73	367	170	11
74	373	247	10
75	379	267	9
76	383	285	8
77	389	0	7
78	397	27	6
79	401	5	5
80	409	339	4
81	419	352	3
82	421	380	2
83	431	405	1
84	433	427	0

equations to guarantee that a specific sequence of numbers will be composite [5]. This method with modifications was applicable to our 7 prime problem.

Consider the system of modular equations,

$$x \equiv b_j \pmod{p_j}, \qquad p_j = j\text{th prime}. \tag{5}$$

The Chinese Remainder Theorem [8] states that there is always a solution for x satisfying (5), with

$$0 \le x < m, \qquad m = \prod p_j. \tag{6}$$

It is easy to write a computer program to solve for such an x.

For each j, starting at a number determined by x and b_j, every p_jth number is divisible by p_j and therefore composite. For r equations, the set of b's determine a particular pattern of composite numbers. By adding to the number of equations and selecting appropriate b's, more numbers join the composite pattern. In this manner any selection of numbers can be made composite. While assuring that these numbers are composite, by properly restricting b, numbers,

$$q(s) = x + 210 \cdot s + 1, \qquad s = 0, 1, 2, 3, 4, 5, 6 \tag{7}$$

can be kept free of any small factors, p_j, so that these remain candidates for the primes in arithmetic progression.

We start by setting $b_j = 0$ for $p_j = 2, 3, 5, 7$ and then add additional equations until all desired numbers are forced to be composite. When a new equation is added, we determine the new b by trying every possible b and selecting the one that guarantees the most additional composite numbers in our target set while keeping the candidates for primes in AP free of small factors. Usually there are many such "optimum" b's and we arbitrarily chose the smallest one. 7 primes require 1254 composite numbers. Table 2 shows how many numbers are left of the 1254 which are not certain to be composite, as a function of the number of equations with the b's chosen as described. For all 1254 to be composite we needed 84 equations. Note that from equation (6), the size of the solution, x, tends to increase with each new equation.

Other rules were tried which gave different sets of b's, but this did not result in any significant improvement for the number of equations, so that the time estimate for finding 7 primes in AP was hardly affected. It does not seem computationally feasible to determine the absolute optimum set of b's.

4. Search time

From equation (6), let

$$y_N = x + m \cdot N, \qquad N = 0, 1, 2, 3, \ldots . \tag{8}$$

Thus, each y is a potentially good starting point for finding 7 primes in AP since, for the small primes p_j, adding the term, m^*N, has no effect on the divisibility properties of the composite numbers or the 7 possible primes $q(s)$. An array representing the 7 values of $q(s)$ for each y_N can be sieved to eliminate all N for which any $q(s)$ has a factor less than some maximum, pmax. This dramatically reduces the number of q sets that need to be tested for probable primality using Fermat tests.

Our original plan was to use all 84 equations, appropriately sieve N, and test the remaining sets of $q(s)$ until 7 primes were found. However this meant that the size of the primes would be about 185 digits, and the time for a prime test was large enough so that the search time was still quite long. We then investigated the possibility of having fewer equations so that the primes would be smaller, and depend on probability for making the remaining numbers composite. By modifying (1) and (3) to allow for sieving we estimated the search time as a function of the number of equations.

When a random number near N is free of factors up to p, the probability of it being prime becomes approximately,

$$\text{prob} \approx \frac{\log p}{.562 \cdot \log N}. \tag{9}$$

This is based on Merten's theorem [3, p. 351] and a short discussion in Merten's [1], and is valid for p relatively small and N relatively large as is true in this article.

Equations (1) and (3) now become,

$$P(7) \approx \left(\frac{\log(\text{pmax})}{.562 \cdot \log N}\right)^7, \qquad C(7) \approx \left(1 - \frac{\log p_c}{.526 \cdot \log N}\right)^t, \tag{10}$$

where p_c and t depend on the number of equations. The results are shown in Table 3. The test times should be considered approximate because of the use of some simplifying assumptions and because the times are heavily machine dependent, but the relative values are significant. Using 42 instead of 84 equations reduces the expected search time by about a factor of 35.

5. Results

Based on Table 3 we used up to 5 PC's, 486/66 or equivalent, to search for the 7 consecutive probable primes in AP in the appropriate range of equations numbering from 36 to 52. After about a total of 70 computer-days (about 20 days of calendar time) the search was successful for equations = 48. The 7 probable primes were 97 digits long. Their true primality was verified using the APRT-CLE program in UBASIC. The compositeness of all the numbers between these primes was verified using Fermat tests. The solution is shown below.

Number of modular equations $= 48$

Product of primes up to the 48th prime $= m$:

$m = 3670097318273319164650345655501367323398003129553317826194624570 3\_$
9988073311157667212930

Solution for the 48 modular equations $= x$:

$x = 118930613432425504731600916625360539894173228870159415462976014 05\_$
6809082107460202605690

First prime $= Q1 = x + N^* m + 1$, where

$N = 2968677222$

$Q1$ is the 97 digit number,

TABLE 3. Estimated search time for 7 composite primes in AP

number of equations	max prime	comps left	approx digits	average number of prime-sets	average number of composite-sets	test time days
20	71	124	35	3969	5123727.81	14830.2
22	79	116	39	8466	394225.66	2711.9
24	89	109	43	16770	55405.79	832.4
26	101	102	47	31256	11049.37	338.2
28	107	96	51	55366	3119.92	183.5
30	113	91	55	93928	1152.62	124.1
32	131	86	59	153539	489.84	92.4
34	139	80	64	271342	199.64	72.2
36	151	74	68	414781	99.68	58.6
38	163	70	72	618849	60.59	56.2
40	173	65	77	990130	35.04	55.7
42	181	59	81	1411407	21.43	51.0
44	193	55	86	2146577	14.68	56.5
46	199	51	90	2950903	10.78	59.7
48	223	47	95	4308466	7.95	67.8
50	229	43	100	6169606	6.05	77.8
52	239	39	105	8681255	4.73	89.8
54	251	35	109	11278281	3.83	98.0
56	263	31	114	15438012	3.11	114.1
58	271	29	119	20849044	2.76	142.9
60	281	26	124	27810527	2.40	172.2
62	293	23	129	36676290	2.10	207.2
64	311	21	134	47862020	1.92	256.6
66	317	18	139	61853038	1.71	307.0
68	337	16	144	79212711	1.59	377.3
70	349	14	149	100591507	1.48	461.5
72	359	12	154	126736725	1.38	562.1
74	373	10	159	158502934	1.30	681.8
76	383	8	164	196863124	1.22	823.6
78	397	6	170	253162797	1.16	1038.1
80	409	4	175	310115716	1.10	1243.1
82	421	2	180	377715643	1.05	1483.2
84	433	0	185	457572099	1.00	1763.6

Notes:
1. Computer system is PC486/66 with a Cruncher.
2. Approximate Fermat test time = .09·(digits/100) seconds.
3. 7-prime sets are sieved to 1,000,000.
4. Sieving time is included in table.
5. Column 5 is the expected number of prime sets that will have to be tested to find 7 primes in AP.
6. Column 6 is the expected number of composite sets that will have to be tested to find 1254 composites.

$$Q1 = 108953343124705931087578037892295773290803649299313819538521310\_5617421504473089672131417174861 51$$

$$Q2 = Q1 + 210,\ Q3 = Q2 + 210, \ldots, Q7 = Q6 + 210.$$

Finding 8 consecutive primes in AP would be expected to take about 20 times longer than our estimated search time for 7 primes. This is well within the capability of supercomputers or workstation and PC networks. It seems only a matter of time before 8 consecutive primes in AP will be found. Even 9 or 10 such primes seem within the realm of possibility using the above methods.

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