CONVERGENCE OF NON-STATIONARY PARALLEL MULTISPLITTING METHODS FOR HERMITIAN POSITIVE DEFINITE MATRICES

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ABSTRACT. Non-stationary multisplitting algorithms for the solution of linear systems are studied. Convergence of these algorithms is analyzed when the coefficient matrix of the linear system is hermitian positive definite. Asynchronous versions of these algorithms are considered and their convergence investigated.

1. INTRODUCTION

In this paper, we study the parallel solution of linear systems of the form

$$(1.1) \qquad\qquad Ax = b,$$

where $A \in \mathbb{C}^{n \times n}$ is a nonsingular matrix and $x, b \in \mathbb{C}^n$.

In order to get an iterative method to solve system (1.1) on a parallel computer, O'Leary and White [23] introduced the multisplitting technique. Later, this technique was studied by many authors; see e.g., Frommer and Mayer [11], [12], Neumann and Plemmons [22], or White [28], [29], [30].

The multisplitting method (see [23]) consists of having a collection of splittings

$$(1.2) \qquad A = F_j - G_j, \quad 1 \le j \le r, \quad \det(F_j) \ne 0,$$

and diagonal nonnegative weighting matrices E_j, $1 \le j \le r$, which add to identity, and the following algorithm is performed.

Algorithm 1. (MULTISPLITTING).
Given the initial vector $x^{(0)}$.

$$\text{For } l = 0, 1, 2, \ldots, \text{ until convergence.}$$
$$\text{For } j = 1 \text{ to } r$$
$$F_j y_j = G_j x^{(l)} + b$$
$$x^{(l+1)} = \sum_{j=1}^{r} E_j y_j \ .$$

As it can be appreciated, Algorithm 1 corresponds to the following iteration

$$(1.3) \qquad x^{(l+1)} = \sum_{j=1}^{r} E_j P_j x^{(l)}, \quad l = 0, 1, 2, \dots,$$

where the operators $P_j : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $1 \le j \le r$, are defined as

$$(1.4) \qquad P_j x = F_j^{-1} G_j x + F_j^{-1} b.$$

Thus, iteration (1.3) can be rewritten as

$$x^{(l+1)} = T x^{(l)} + \sum_{j=1}^{r} E_j F_j^{-1} b, \quad l = 0, 1, 2, \dots,$$

where $T = \sum_{j=1}^{r} E_j F_j^{-1} G_j$ is the iteration matrix.

Conditions on the splittings (1.2) and on the weighting matrices which ensure the convergence of Algorithm 1 in some important cases were given by O'Leary and White [23]. In particular they showed that convergence (i.e., $\rho(T) < 1$, where $\rho(T)$ denotes the spectral radius of T) when A is a symmetric positive definite matrix and the splittings (1.2) are P–regular (see definitions in Section 2). In an efficient implementation of a multisplitting method on a multiprocessor system, each operator P_j, defined in (1.4), represents the task assigned to one of the r processors to obtain its local approximation. Each local approximation is updated exactly once using the same previous iterate $x^{(l)}$. However, it is possible to update that approximation more than once using different iterates computed earlier. In this case we call this method a *non-stationary* method. The main idea of these methods is that at the lth iteration each processor j solves the system defined by its operator P_j, $q(l, j)$ times, updating each time the right-hand side by using the new calculated vector, i.e.

Algorithm 2. (NON–STATIONARY MULTISPLITTING).
   Given the initial vector $x^{(0)}$

$$\text{For } l = 0, 1, 2 \dots, \text{ until convergence}$$
$$\text{In processor } j, \ j = 1 \text{ to } r$$
$$y_j^{(0)} = x^{(l)}$$
$$\text{For } k = 1 \text{ to } q(l, j)$$

$$(1.5) \qquad F_j y_j^{(k)} = G_j y_j^{(k-1)} + b$$

$$(1.6) \qquad x^{(l+1)} = \sum_{j=1}^{r} E_j y_j^{(q(l,j))}.$$

Bru, Elsner and Neumann [3] showed the convergence of this algorithm when $A^{-1} \ge 0$ and the splittings (1.2) are weak regular ($F_j^{-1} \ge 0$ and $F_j^{-1} G_j \ge 0$, $1 \le j \le r$). They used the term *chaotic* for these non-stationary methods; however we have chosen the non-stationary term since in the classical literature chaotic is synonymous with asynchronous, e.g., as in [8].

In Algorithm 2 a relaxation parameter $\omega \in \mathbb{R}$, $\omega \neq 0$, can be introduced by replacing the computation of $x^{(l+1)}$ in (1.6) with the equation

$$(1.7) \qquad x^{(l+1)} = \omega \sum_{j=1}^{r} E_j y_j^{(q(l,j))} + (1 - \omega)x^{(l)}.$$

Clearly, with $\omega = 1$, equation (1.6) is recovered. In the case of $\omega \neq 1$ we have a Relaxed Non-stationary Multisplitting Algorithm (Algorithm 3). The convergence of algorithms 2 and 3 when A is an H–matrix was studied by Mas, Migallón, Penadés and Szyld [20]. Furthermore, in [20] the authors report computational results of those algorithms on a multiprocessor system that show a better behavior of the non-stationary models than the stationary ones (Algorithm 1). For a background on parallel non-stationary models see also [5], [6], [7], [13], [14] or [15].

In this paper we concentrate our study on the case where A is a hermitian positive definite matrix. In Section 3 we study the convergence of Algorithm 2, together with its relaxed version, Algorithm 3. We also study their extension to asynchronous algorithms where the solution of the systems (1.5) proceed in each processor without waiting for the completion of the computation of the iterates in the other processors; see Section 4. Previously, in Section 2 we present some notation, definitions and preliminary results which we refer to later.

2. NOTATION AND PRELIMINARIES

For any matrix $A \in \mathbb{C}^{n \times n}$, $|A|$ denotes the matrix whose entries are the modulus of the corresponding entries of A; the matrices A^T and A^H denote respectively the transpose and the conjugate transpose of A. Similarly the transpose and the conjugate transpose of a vector $x \in \mathbb{C}^n$ are denoted by x^T and x^H, respectively. A matrix $A \in \mathbb{C}^{n \times n}$ is symmetric if $A = A^T$, and hermitian if $A = A^H$. Obviously a real symmetric matrix is a special case of a hermitian matrix.

Recall that a complex, not necessarily hermitian matrix A, is positive definite if the real part of $x^H A x$ is positive, for all complex $x \neq 0$. When A is hermitian, this is equivalent to requiring that $x^H A x > 0$, for all complex $x \neq 0$. In addition, a general matrix A is positive definite if and only if the hermitian matrix $A + A^H$ is positive definite. Let $A \in \mathbb{C}^{n \times n}$ be a hermitian positive definite matrix, then $\langle x, y \rangle = (x^H A y)^{1/2}$ defines an inner product on $\mathbb{C}^n$. Hence, $\|x\|_A = (x^H A x)^{1/2}$ is a vector norm on $\mathbb{C}^n$. The matrix norm induced by that vector norm will also be denoted by $\|\cdot\|_A$. In addition, $\|\cdot\|_\infty$ denotes the infinite matrix norm. Let $A \in \mathbb{C}^{n \times n}$, the splitting $A = M - N$ is called P–regular if the matrix $M^H + N$ is positive definite; see e.g., [2], [24], [25], for an extensive bibliography on hermitian matrices and positive definite matrices.

The following theorem gives convergence conditions for iterative methods based on a single splitting $A = M - N$, when A is a hermitian matrix. The proof can be found, e.g, in [2].

Theorem 2.1. *Let $A = M - N$ be a P–regular splitting of a hermitian matrix A. Then $\rho(M^{-1}N) < 1$ if and only if A is positive definite.*

A similar result for Algorithm 1 was obtained by O'Leary and White [23], assuming that the weighting matrices E_j, $1 \le j \le r$, of Algorithm 1 have the form $E_j = \alpha_j I$. Although that result was given for symmetric matrices, it can be extended without difficulty to hermitian matrices.

Theorem 2.2. *Let $A = F_j - G_j$, $1 \le j \le r$, be P–regular splittings of a hermitian positive definite matrix, and $E_j = \alpha_j I$, $1 \le j \le r$, with $\alpha_j > 0$ and $\sum_{j=1}^{r} \alpha_j = 1$. Then Algorithm 1 is convergent.*

We point out that if the weighting matrices E_j are not of the form $E_j = \alpha_j I$, Algorithm 1 may not converge when A is hermitian positive definite, even though the splittings of A are P–regular. Here we report an example, different to that used in [23], that illustrates this situation. Later, we will use this example for the non-stationary methods.

Example 2.3. Consider

$$A = \begin{bmatrix} 0.75 & 0 \\ 0 & 0.75 \end{bmatrix} = F_1 - G_1 = F_2 - G_2,$$

where

$$F_1 = \begin{bmatrix} 0.3934 & -2.0660 \\ 2.0660 & 7.6244 \end{bmatrix}, \quad G_1 = \begin{bmatrix} -0.3566 & -2.0660 \\ 2.0660 & 6.8744 \end{bmatrix},$$

and

$$F_2 = \begin{bmatrix} 7.6244 & 2.0660 \\ -2.0660 & 0.3934 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 6.8744 & 2.0660 \\ -2.0660 & -0.3566 \end{bmatrix}.$$

Setting $E_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, the iteration matrix $T = E_1 F_1^{-1} G_1 + E_2 F_2^{-1} G_2 = \begin{bmatrix} 0.9594 & 0.2132 \\ 0.2132 & 0.9594 \end{bmatrix}$, has spectral radius equal to 1.1726, and so Algorithm 1 is not convergent. However, Algorithm 1 converges setting other weighting matrices that do not satisfy Theorem 2.2. For example, if the above matrices E_1 and E_2 are interchanged, the spectral radius of the resulting iteration matrix is 0.4264.

The following lemmas will be very useful in our convergence analysis.

Lemma 2.4. *Given a nonsingular matrix A and a matrix T such that $(I - T)^{-1}$ exists, there is a unique pair of matrices P, Q such that P is nonsingular, $T = P^{-1}Q$ and $A = P - Q$. The matrices are $P = A(I - T)^{-1}$ and $Q = P - A$.*

Proof. See Lemma 8 of Lanzkron, Rose and Szyld [18]. □

Lemma 2.5. *Let $T^{(l)}$, $l = 0, 1, 2, \ldots$, be a sequence of square complex matrices. If there exists a matrix norm $\| \cdot \|$ such that $\|T^{(l)}\| \le \theta < 1$, $l = 0, 1, 2, \ldots$, then $\lim_{l \to \infty} T^{(l)} T^{(l-1)} \cdots T^{(0)} = 0$.*

Proof. See Lemma 2 of Bru and Fuster [4]. □

Lemma 2.6. *Let A be a hermitian positive definite matrix. If the splittings $A = F - G = P - Q$ are P–regular, then the matrix $T = P^{-1}QF^{-1}G$ has spectral radius less that one. Moreover, the unique splitting $A = B -_i C$ induced by the iteration matrix T, such that $T = B^{-1}C$, is also P-regular.*

Proof. See Theorem 4.6 of Benzi and Szyld [1]. This theorem was given for symmetric matrices. The hermitian case is analogous. □

3. Convergence of non-stationary methods

Given an initial vector $\dot{x}^{(0)}$, the Non-stationary Multisplitting Algorithm 2 produces the sequence of vectors

$$(3.1) \qquad x^{(l+1)} = \sum_{j=1}^{r} E_j P_j^{q(l,j)} x^{(l)}, \quad l = 0, 1, 2, \dots ,$$

where the operators P_j, $1 \leq j \leq r$, are defined in (1.4). We rewrite (3.1) as

$$(3.2) \qquad x^{(l+1)} = T^{(l)} x^{(l)} + c^{(l)}, \quad l = 0, 1, 2, \dots ,$$

where $T^{(l)}$ are the iteration matrices

$$(3.3) \qquad T^{(l)} = \sum_{j=1}^{r} E_j \left(F_j^{-1} G_j \right)^{q(l,j)}, \quad l = 0, 1, 2 \dots ,$$

and

$$c^{(l)} = \sum_{j=1}^{r} E_j \left(\sum_{i=0}^{q(l,j)-1} (F_j^{-1} G_j)^i \right) F_j^{-1} b, \quad l = 0, 1, 2 \dots .$$

The exact solution of the linear system (1.1) is a fixed point of the operators P_j, $1 \leq j \leq r$. Then, using error analysis, it is easy to see that the sequence of vectors generated by iteration (3.1) (or equivalently, iteration (3.2)) converges to the solution of the linear system (1.1) if and only if $\lim_{l \to \infty} T^{(l)} T^{(l-1)} \cdots T^{(0)} = 0$.

The following lemma is very useful when analyzing the iteration matrices (3.3).

Lemma 3.1. *Let A be a hermitian positive definite matrix. Assume the splitting $A = F - G$ is P–regular. Given $s \geq 1$, there exists a unique splitting $A = M - N$ such that $(F^{-1}G)^s = M^{-1}N$. Moreover, the splitting is P–regular.*

Proof. The proof of the existence and uniqueness follows from Theorem 2.1 and Lemma 2.4. To show the P–regularity we proceed by induction. For $s = 1$ the result follows from the uniqueness. Suppose that the result is true for $s - 1$, that is, we know that the unique splitting $A = P - Q$, such that $(F^{-1}G)^{s-1} = P^{-1}Q$, is P–regular. Now, using Lemma 2.6 with the P–regular splittings $A = F - G = P - Q$ the proof is completed. $\qquad\square$

Theorem 3.2. *Let A be a hermitian positive definite matrix. Let $A = F_j - G_j$, $1 \leq j \leq r$, be P–regular splittings and $E_j = \alpha_j I$, $1 \leq j \leq r$, with $\alpha_j > 0$ and $\sum_{j=1}^{r} \alpha_j = 1$. Then, there exists a unique splitting $A = R^{(l)} - S^{(l)}$, induced by each iteration matrix $T^{(l)}$, $l = 0, 1, 2, \dots$, defined in (3.3), such that $T^{(l)} = (R^{(l)})^{-1} S^{(l)}$. Moreover, that splitting is P–regular.*

Proof. Since $A = F_j - G_j$, $1 \leq j \leq r$, are P–regular splittings of a hermitian positive definite matrix then, from Lemma 3.1, for each j, l there exists a unique P–regular splitting $A = M_j^{(l)} - N_j^{(l)}$ such that $(F_j^{-1} G_j)^{q(l,j)} = (M_j^{(l)})^{-1} N_j^{(l)}$. Then, the iteration matrices (3.3) can be written as

$$T^{(l)} = \sum_{j=1}^{r} E_j \left(F_j^{-1} G_j \right)^{q(l,j)} = \sum_{j=1}^{r} E_j (M_j^{(l)})^{-1} N_j^{(l)}, \quad l = 0, 1, 2, \dots .$$

Now, for each fixed l, $T^{(l)}$ is the iteration matrix corresponding to Algorithm 1 by setting the splittings $A = M_j^{(l)} - N_j^{(l)}$, $1 \leq j \leq r$. Since these splittings are P–regular, Theorem 2.2 ensures that $\rho(T^{(l)}) < 1$. On the other hand, since each $M_j^{(l)}$ is positive definite, the matrix $\sum_{j=1}^{r} E_j (M_j^{(l)})^{-1}$ is also positive definite, and therefore it is nonsingular. Thus, each matrix $T^{(l)}$, $l = 0, 1, 2, \ldots$, is also the iteration matrix of an iterative method based on the single splitting

$$(3.4) \qquad\qquad\qquad A = R^{(l)} - S^{(l)},$$

where $R^{(l)} = \left(\sum_{j=1}^{r} E_j (M_j^{(l)})^{-1} \right)^{-1}$, and $S^{(l)} = R^{(l)} T^{(l)}$. Since $\rho(T^{(l)}) < 1$, from Lemma 2.4 these splittings are unique. Moreover, from Theorem 3.2 of [21], it follows that the splittings (3.4) are P–regular.   □

From the above theorem follows the convergence of Algorithm 2 when the number of times $q(l, j)$ that the jth processor works is fixed in each iteration l. That is, if $q(l, j) = q(j)$, $l = 0, 1, 2, \ldots$, $1 \leq j \leq r$, then there is a unique iteration matrix $T = T^{(l)}$, $l = 0, 1, 2, \ldots$; thus, from Theorem 3.2, $\rho(T) < 1$. However, since the product of convergent matrices may not tend to zero (see e.g., Johnson and Bru [17], or Robert, Charnay and Musy [27]), then other tools are needed to show the convergence of Algorithm 2 for any sequence of integers $q(l, j) \geq 1$. So, we have the following result.

Theorem 3.3. *Let A be a hermitian positive definite matrix. Let $A = F_j - G_j$, $1 \leq j \leq r$, be P–regular splittings and $E_j = \alpha_j I$, $1 \leq j \leq r$, with $\alpha_j > 0$ and $\sum_{j=1}^{r} \alpha_j = 1$. Assume that $q(l, j) \geq 1$, $1 \leq j \leq r$, $l = 0, 1, 2, \ldots$. Then, the Non-stationary Multisplitting Algorithm 2 converges to the solution of the linear system (1.1), for any initial vector $x^{(0)}$.*

Proof. Since $A = F_j - G_j$, $1 \leq j \leq r$, are P–regular splittings of a hermitian matrix, the matrix $A - (F_j^{-1} G_j)^H A (F_j^{-1} G_j) = (F_j^{-1} A)^H (F_j^H + G_j)(F_j^{-1} A)$ is hermitian positive definite. Then, letting the vector norm $\| \cdot \|_A$, we have

$$\| F_j^{-1} G_j x \|_A^2 = x^H (F_j^{-1} G_j)^H A (F_j^{-1} G_j) x < x^H A x = \| x \|_A^2, \quad \text{for all } x \neq 0.$$

Hence $\| F_j^{-1} G_j \|_A < 1$, $1 \leq j \leq r$, and there exists a real constant $0 \leq \theta < 1$ such that $\| F_j^{-1} G_j \|_A \leq \theta < 1$, $1 \leq j \leq r$. Hence, for all $l = 0, 1, 2, \ldots$, we have

$$\| T^{(l)} \|_A = \| \sum_{j=1}^{r} E_j \left(F_j^{-1} G_j \right)^{q(l,j)} \|_A \leq \sum_{j=1}^{r} \alpha_j \| \left(F_j^{-1} G_j \right)^{q(l,j)} \|_A \leq \theta < 1.$$

From Lemma 2.5, this implies that $\lim_{l \to \infty} T^{(l)} T^{(l-1)} \cdots T^{(0)} = 0$, and then the proof is completed.   □

Theorem 3.3 generalizes Theorem 1.b of [23] (Theorem 2.2 of Section 2) for the non-stationary case, and it gives an alternative proof of the convergence result by O'Leary and White [23].

Note that in Theorem 3.3 we have restricted the weighting matrices in the same way as in Theorem 2.2. If we consider the splittings of Example 2.3 and we compute the iteration matrices of Algorithm 2, setting $q(l,1) = q(l,2) = 2$ or 3, $l = 0, 1, 2, \ldots$, we obtain $\rho(T^{(l)}) > 1$, $l = 0, 1, 2, \ldots$, in the first choice of weighting matrices. However, if we set $q(l,1) = q(l,2) = 4$, $l = 0, 1, 2, \ldots$, we get $\rho(T^{(l)}) < 1$, $l = 0, 1, 2, \ldots$, in both choices of weighting matrices. Consequently it motivates the convergence study of Algorithm 2 without the additional hypothesis $E_j = \alpha_j I$, $0 < \alpha_j \leq 1$. The following theorem shows the convergence of Algorithm 2 when each processor performs enough local iterations (1.5).

Theorem 3.4. *Let A be a hermitian positive definite matrix. Let $A = F_j - G_j$, $1 \leq j \leq r$, be P–regular splittings. Assume further that $\lim_{l \to \infty} q(l,j) = \infty$, $1 \leq j \leq r$. Then the Non-stationary Multisplitting Algorithm 2 converges to the solution of the linear system (1.1), for any initial vector $x^{(0)}$.*

Proof. Since $A = F_j - G_j$, $1 \leq j \leq r$, are P–regular splittings of a hermitian positive definite matrix, by Theorem 2.1, $\rho(F_j^{-1}G_j) < 1$, $1 \leq j \leq r$. Then, $\lim_{i \to \infty} (F_j^{-1} G_j)^i = 0$, $1 \leq j \leq r$. Thus, given an $\epsilon > 0$, there exists an integer i_0 such that for any matrix norm $\| \cdot \|$, $\|(F_j^{-1}G_j)^i\| \leq \epsilon$, for all $i \geq i_0$. Particularly, we can choose the infinite norm. Since $\lim_{l \to \infty} q(l,j) = \infty$, there exists an l_0 such that

$$\|(F_j^{-1}G_j)^{q(l,j)}\|_\infty \leq \epsilon, \quad \forall l \geq l_0, \quad 1 \leq j \leq r.$$

Hence, for $l \geq l_0$,

$$\|T^{(l)}\|_\infty = \|\sum_{j=1}^{r} E_j (F_j^{-1}G_j)^{q(l,j)}\|_\infty \leq \max_{1 \leq j \leq r} \|(F_j^{-1}G_j)^{q(l,j)}\|_\infty \leq \epsilon.$$

Setting $\epsilon < 1$ the convergence of Algorithm 2 follows from Lemma 2.5. $\square$

It is well known when the Jacobi, Gauss-Seidel and SOR splittings (and their block splitting versions) of a hermitian positive definite matrix are P–regular; see e.g., [2] and [25]. Another class of P–regular splittings of a hermitian positive definite matrix A is the unique splitting induced by the iteration matrix of an alternating iterative method of the form

$$x^{(l+1/2)} = M^{-1}Nx^{(l)} + M^{-1}b, \quad x^{(l+1)} = P^{-1}Qx^{(l+1/2)} + P^{-1}b, \quad l = 0, 1, 2, \ldots,$$

where $A = M - N = P - Q$ are P–regular splittings; see [1]. So, the SSOR splitting is an important example of this class. Besides SSOR, one can consider, for example, the alternating iterations based on a splitting of the form $A = A_1 + A_2$, where A_1, A_2 are positive definite and $A_2 = A_1^T$; choosing $M = \beta I + A_1$, $N = \beta I - A_2$, $P = \beta I + A_2$ and $Q = \beta I - A_1$, with $\beta > 0$, the splittings $A = M - N = P - Q$ are P–regular; see e.g., [19]. Furthermore, there are other ways to construct r P–regular splittings of a hermitian positive definite matrix; for example, see [23], let $A = \sum_{j=1}^{r} A_j$ be a hermitian positive definite matrix, and let D_j, $1 \leq j \leq r$, be diagonal matrices such that $F_j = A_j + D_j$, $1 \leq j \leq r$, are nonsingular. If the matrices $2(A_j + D_j) - A$, $1 \leq j \leq r$, are positive definite, then the splittings $A = F_j - G_j$, $1 \leq j \leq r$, are P–regular, and so we can apply the above convergence results.

On the other hand, the proof of Theorem 3.4 shows that $\|(F_j^{-1}G_j)\|_\infty < 1$, $1 \le j \le r$, is a sufficient condition for the convergence of Algorithm 2 for any matrix A, without additional hypotheses on the weighting matrices. So we have the following result.

Corollary 3.5. *Let $A = F_j - G_j$, $1 \le j \le r$, be splittings of the matrix $A \in \mathbb{C}^{n \times n}$ such that $\|(F_j^{-1}G_j)\|_\infty < 1$, $1 \le j \le r$. If $q(l,j) \ge 1$, $1 \le j \le r$, $l = 0, 1, 2, \ldots$, then the Non-stationary Multisplitting Algorithm 2 converges to the solution of the linear system (1.1), for any initial vector $x^{(0)}$.*

Proof. Since $\|(F_j^{-1}G_j)\|_\infty < 1$, $1 \le j \le r$, there exists a real constant $0 \le \theta < 1$ such that $\|(F_j^{-1}G_j)\|_\infty \le \theta < 1$, $1 \le j \le r$. By computing the infinite norm of $T^{(l)}$, defined in (3.3), we obtain for $l = 0, 1, 2, \ldots$, $\|T^{(l)}\|_\infty \le \max_{1 \le j \le r} \|F_j^{-1}G_j\|_\infty^{q(l,j)} \le \theta < 1$. $\qquad\square$

We note that Corollary 3.5 can be proved in the same way if the infinite norm is replaced with any matrix norm such that if for arbitrary matrices T, T_j and weighting matrices E_j, $1 \le j \le r$, such that $T = \sum_{j=1}^{r} E_j T_j$, then $\|T\| \le \max_{1 \le j \le r} \|T_j\|$; see e.g., [4]. In particular one can use any weighted max-norm associated with a positive vector; see e.g., [15], [16] or [26] for descriptions and applications of these norms.

To finish this section we analyze the convergence of the Relaxed Non-stationary Multisplitting Algorithm 3. If in addition, we assume $0 < \omega < \frac{2}{1 + \max_{1 \le j \le r} \|F_j^{-1}G_j\|_A}$ in Theorem 3.3 and $0 < \omega < \frac{2}{1 + \max_{1 \le j \le r} \|F_j^{-1}G_j\|_\infty}$ in Corollary 3.5 the same results hold for Algorithm 3. Since its iteration matrices can be written as $\omega T^{(l)} + (1-\omega)I$, with $T^{(l)}$ defined in (3.3), the proofs follow in a similar way. Furthermore, if in Theorem 3.4 we assume $0 < \omega < \frac{2}{1+\epsilon}$, then the convergence of Algorithm 3 also holds. So, the convergence of Algorithm 3 is shown under the same hypotheses as in the above results, setting ω in an interval $(0, \omega_0)$, with $\omega_0 > 1$.

4. ASYNCHRONOUS ITERATIONS

The motivation of defining an asynchronous non-stationary method is to obtain a parallel implementation of the non-stationary iterative methods on a multiprocessor system such that the communication and synchronization between the cooperating processes are reduced. To illustrate that, consider Algorithm 2; this algorithm is synchronous in the sense that step (1.6) is performed only after all processors have computed their iterate vectors $y_j^{(q(l,j))}$, $1 \le j \le r$. Alternatively, we can consider a parallel implementation of a non-stationary multisplitting method in which each part of $x^{(l+1)}$, i.e., $E_j y_j^{(q(l,j))}$, can be updated without waiting for the other parts of $x^{(l+1)}$ to be updated. In order to construct an asynchronous version of Algorithm 2, we consider a different scheme where all processors are always working without waiting for information from the other processors. In a formal way, let $\{j_l\}_{l=0}^\infty$, $1 \le j_l \le r$, be a sequence of integers that indicates the processor which updates the approximation to the solution at the lth iteration. Let $r_l - 1$ be the number of times that processors other than the j_lth processor update the approximation of the solution during the time interval in which the j_lth processor's

calculations are performed. This implies that r_l is the smallest positive integer such that $j_l = j_{l+r_l}$. Assume that there exists a positive integer K such that $0 \leq r_l - 1 < K$, that is, in carrying out the evaluation of the lth iterate, a process cannot make use of any value of the components of the jth iterate if $j < l - K$. With this notation we consider the following asynchronous scheme

$$(4.1) \qquad x^{(l+r_l)} = (I - E_{j_l})x^{(l+r_l-1)} + E_{j_l}P_{j_l}^{q(l,j_l)}x^{(l)}, \quad l = 0, 1, 2, \ldots,$$

where the operators P_j, $1 \leq j \leq r$, are defined in (1.4) and $x^{(-K)} = x^{(-K+1)} = \cdots = x^{(0)}$. As it can be appreciated, scheme (4.1) corresponds to the following algorithm.

Algorithm 4. (ASYNCHRONOUS NON-STATIONARY MULTISPLITTING).
  Given the initial vectors $x^{(-K)} = x^{(-K+1)} = \cdots = x^{(0)}$.

$$\text{In processor } j_l, \ l = 0, 1, 2, \ldots, \text{ until convergence.}$$

$$y_{j_l}^{(0)} = x^{(l)}$$
$$\text{For } k = 1 \text{ to } q(l, j_l)$$
$$F_{j_l}y_{j_l}^{(k)} = G_{j_l}y_{j_l}^{(k-1)} + b$$
$$(4.2) \qquad x^{(l+r_l)} = (I - E_{j_l})x^{(l+r_l-1)} + E_{j_l}y_{j_l}^{(q(l,j_l))}.$$

In the same way as in the synchronous case, a relaxation parameter $\omega \in \mathbb{R}$, $\omega \neq 0$, can be introduced in Algorithm 4, by replacing the vector $y_{j_l}^{(q(l,j_l))}$ in (4.2) with the equation $\omega y_{j_l}^{(q(l,j_l))} + (1 - \omega)x^{(l)}$. When $\omega \neq 1$ we have a Relaxed Asynchronous Non-stationary Multisplitting Algorithm (Algorithm 5).

The following theorem shows the convergence of Algorithm 4 under similar hypotheses as those for the synchronous Algorithm 2 in Theorem 3.3.

Theorem 4.1. *Let A be a hermitian positive definite matrix. Let $A = F_j - G_j$, $1 \leq j \leq r$, be P-regular splittings and $E_j = \alpha_j I$, $1 \leq j \leq r$, with $0 < \alpha_j \leq 1$. Assume that there exists a positive integer K such that $0 \leq r_l - 1 < K$. If $q(l, j) \geq 1$, $1 \leq j \leq r$, $l = 0, 1, 2, \ldots$, then the Asynchronous Non-stationary Multisplitting Algorithm 4 converges to the solution of the linear system (1.1), for any initial vector $x^{(0)}$.*

Proof. In order to analyze the convergence of the asynchronous iteration (4.1) (or Algorithm 4), we will construct a procedure in $\mathbb{C}^{nK}$, with K satisfying $0 \leq r_l - 1 < K$. To this purpose, let us use the following notation. Let ξ be the unique solution of the linear system (1.1) and let $e^{(l)} = x^{(l)} - \xi$ be the error vector in the lth iteration of the asynchronous scheme (4.1). Define

$$(4.3) \qquad \overline{e}_l = \left((e^{(l)})^T, (e^{(l-1)})^T, \ldots, (e^{(l-K+1)})^T \right)^T \in \mathbb{C}^{nK}.$$

Since $0 \leq r_l - 1 < K$, it follows that $l + r_l - K \leq l \leq l + r_l - 1$, $l = 0, 1, 2, \ldots$, and then we can write

$$(4.4) \qquad e^{(l)} = S_l \overline{e}_{l+r_l-1},$$

where S_l, $l = 0, 1, 2, \ldots$, is an $n \times nK$ matrix with an $n \times n$ identity block in the r_l position and the remaining $K - 1$ blocks are zero. From (4.1) and (4.4), and knowing that $E_j = \alpha_j I$, $1 \leq j \leq r$, it follows that

$$\overline{e}_{l+r_l} = B_{l+r_l}\overline{e}_{l+r_l-1},$$

where $B_{l+r_l} \in \mathbb{C}^{nK \times nK}$ is defined as

$$
(4.5) \qquad B_{l+r_l} =
\begin{bmatrix}
(1-\alpha_{j_l})I & O & \dots & O & O \\
I & O & \dots & O & O \\
\vdots & \vdots & & \vdots & \vdots \\
O & O & \dots & I & O
\end{bmatrix}
+
\begin{bmatrix}
\alpha_{j_l}(F_{j_l}^{-1}G_{j_l})^{q(l,j_l)}S_l \\
O \\
\vdots \\
O
\end{bmatrix}.
$$

Then $\bar{e}_{l+K} = B_{l+K}B_{l+K-1}\cdots B_{l+1}\bar{e}_l$.

Let μ be an arbitrary nonzero nK-dimensional vector partitioned as in (4.3), that is, $\mu = (\mu^i)_{i=1}^K \in \mathbb{C}^{nK}$, with $\mu^i \in \mathbb{C}^n$. We define a vector norm on $\mathbb{C}^{nK}$ by

$$
\|\mu\| = \max_{1 \leq i \leq K} \|\mu^i\|_A,
$$

and its induced matrix norm on $\mathbb{C}^{nK \times nK}$. We show that there exists a real constant $0 \leq \gamma < 1$, such that $\|B_{l+K}B_{l+K-1}\cdots B_{l+1}\| \leq \gamma < 1$, $l = 0, 1, 2, \dots$, and therefore $\lim\limits_{l\to\infty} \bar{e}_l = 0$.

Since $A = F_j - G_j$, $1 \leq j \leq r$, are P–regular splittings, reasoning as in Theorem 3.3 it follows that there exists a real constant $0 \leq \theta < 1$ such that $\|F_j^{-1}G_j\|_A \leq \theta$, and then $\|(F_j^{-1}G_j)^s\|_A \leq \theta < 1$, for all $s \geq 1$, $1 \leq j \leq r$. Let

$$
(4.6)
$$

$$
v_\nu = B_{l+\nu}B_{l+\nu-1}\cdots B_{l+1}\mu :=
\begin{bmatrix}
v_\nu^1 \\
v_\nu^2 \\
\vdots \\
v_\nu^K
\end{bmatrix},
\qquad v_\nu^i \in \mathbb{C}^n,\ i = 1, 2, \dots, K,\ \nu \geq 1.
$$

From (4.4) and (4.5), $v_1^1 = (1-\alpha_{j_l})\mu_1 + \alpha_{j_l}(F_{j_l}^{-1}G_{j_l})^{q(l,j_l)}\mu_i$, $1 \leq i \leq K$. Then

$$
\|v_1^1\|_A \leq (1-\alpha_{j_l})\|\mu_1\|_A + \alpha_{j_l}\|(F_{j_l}^{-1}G_{j_l})^{q(l,j_l)}\|_A\|\mu_i\|_A
$$
$$
(4.7) \qquad\qquad \leq [(1-\alpha_{j_l}) + \alpha_{j_l}\theta]\,\|\mu\| \leq \gamma\|\mu\|,
$$

where $\gamma = \max\limits_{1 \leq j \leq r} [(1-\alpha_j) + \alpha_j\theta] < 1$. Then, from (4.5)

$$
(4.8) \qquad\qquad \|v_1^i\|_A = \|\mu^{i-1}\|_A \leq \|\mu\|, \quad i = 2, 3, \dots, K.
$$

Now, from expression (4.6), $v_\nu^{1+i} = v_{\nu-1}^i$, $i = 1, 2, \dots, K-1$. Then, using the bounds (4.7) and (4.8) it follows that

$$
\|v_\nu^i\|_A \leq \begin{cases} \gamma\|\mu\|, & i = 1, 2, \dots, \nu, \\ \|\mu\|, & i = \nu+1, \dots, K. \end{cases}
$$

Hence, $\|v_K\| \leq \gamma\|\mu\|$, for all nonzero vectors $\mu \in \mathbb{C}^{nK}$. Therefore, we have that $\|B_{l+K}B_{l+K-1}\cdots B_{l+1}\| \leq \gamma < 1$, $l = 0, 1, 2, \dots$. Thus, the proof is complete. $\square$

From now on, we assume, as is customary in the description of asynchronous algorithms (see e.g., [3] and [9]), that the sequence of integers $\{j_l\}_{l=0}^\infty$, $1 \leq j_l \leq r$, is a regulated sequence. This means that there exists a positive integer K, such that each of the integers $1, 2, \dots, r$, appears at least once in every K consecutive elements of the sequence. That actually implies $0 \leq r_l - 1 < K$. We point out that in Theorem 4.1 we have only needed the latter bounds; this is due to the fact that the weighting matrices are of the form $E_j = \alpha_j I$, $1 \leq j \leq r$. Another similar way of describing asynchronous iterations has been considered by Frommer [10] and other authors, e.g., Bru, Migallón, Penadés and Szyld [7] and the authors cited therein.

Now, we study the convergence of the asynchronous scheme (4.1) (or Algorithm 4) under hypotheses similar to those of Theorem 3.4 and Corollary 3.5.

Theorem 4.2. *Let A be a hermitian positive definite matrix. Let $A = F_j - G_j$, $1 \leq j \leq r$, be P-regular splittings. Assume further that $\lim_{l \to \infty} q(l, j) = \infty$, $1 \leq j \leq r$. Given a regulated sequence of integers $\{j_l\}_{l=0}^{\infty}$, $1 \leq j_l \leq r$, the Asynchronous Non-stationary Multisplitting Algorithm 4 converges to the solution of the linear system (1.1), for any initial vector $x^{(0)}$.*

Proof. Reasoning as in Theorem 3.4 there exists an l_0 and a real constant θ, such that $\|(F_j^{-1}G_j)^{q(l,j)}\|_\infty \leq \theta < 1$, for all $l \geq l_0$. Thus, setting $z = (1, 1, \ldots, 1)^T$,

$$|(F_j^{-1}G_j)^{q(l,j)}|z \leq \theta z, \quad l \geq l_0.$$

By using error analysis as in Theorem 4.1 and reasoning in a similar way as in the proof of [20, Theorem 3.2], which in turn is based on [3, Theorem 2.2], it follows that

$$|B_{l+2K-1}B_{l+2K-2}\cdots B_{l+1}|z \leq \theta z, \quad l \geq l_0,$$

where the matrices B_l are obtained by replacing $(1 - \alpha_{j_l})I$ in (4.5) with $I - E_{j_l}$. Then,

$$\|B_{l+2K-1}B_{l+2K-2}\cdots B_{l+1}\|_\infty \leq \theta < 1, \quad l \geq l_0.$$

Thus, the proof is completed. $\square$

Corollary 4.3. *Let $A = F_j - G_j$, $1 \leq j \leq r$, be splittings of the matrix $A \in \mathbb{C}^{n \times n}$ such that $\|(F_j^{-1}G_j)\|_\infty < 1$, $1 \leq j \leq r$. Assume $q(l, j) \geq 1$, $1 \leq j \leq r$, $l = 0, 1, 2, \ldots$. Given a regulated sequence of integers $\{j_l\}_{l=0}^{\infty}$, $1 \leq j_l \leq r$, the Asynchronous Non-stationary Multisplitting Algorithm 4 converges to the solution of the linear system (1.1), for any initial vector $x^{(0)}$.*

Proof. The proof follows in a similar way as in the proof of Theorem 4.2. $\square$

Analogously, setting ω as at the end of Section 3, the convergence of Algorithm 5 is done under the same hypotheses as in the above results.

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