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THE d_2-TRANSFORMATION FOR INFINITE DOUBLE SERIES AND THE D_2-TRANSFORMATION FOR INFINITE DOUBLE INTEGRALS

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ABSTRACT. New transformations for accelerating the convergence of infinite double series and infinite double integrals are presented. These transformations are generalizations of the univariate d- and D-transformations. The D_2-transformation for infinite double integrals is efficient if the integrand satisfies a p.d.e. of a certain type. Similarly, the d_2-transformation for double series works well for series whose terms satisfy a difference equation of a certain type. In both cases, the application of the transformation does not require an explicit knowledge of the differential or the difference equation. Asymptotic expansions for the remainders in the infinite double integrals and series are derived, and nonlinear transformations based upon these expansions are presented. Finally, numerical examples which demonstrate the efficiency of these transformations are given.

1. INTRODUCTION

We discuss the problem of accelerating the convergence of infinite double integrals and infinite double series. The methods that are presented in this paper are generalizations of the D- and d-transformations for univariate infinite integrals and series, which were developed in [11]. In the following we review some useful definitions that were used in the course of development of the transformations for the univariate case:

Definition 1.1. A function $p(x)$ is said to belong to the set $A^{(\gamma)}$, if, as $x \to \infty$, it has a Poincaré-type asymptotic expansion of the form

$$p(x) \sim x^{\gamma} \sum_{i=0}^{\infty} \alpha_i / x^i \ . \tag{1.1}$$

Definition 1.2. $B^{(m)}$ is defined as the set of functions f which are integrable on $(0, \infty)$ and which satisfy a linear mth-order differential equation of the form

$$f(x) = \sum_{k=1}^{m} p_k(x) f^{(k)}(x) \ , \tag{1.2}$$

where $p_k \in A^{(k)},\ k = 1, \ldots, m$.

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Definition 1.3. $\tilde{B}^{(m)}$ is defined as the set of infinite sequences $\{a_n\}$ whose elements satisfy a linear mth-order difference equation of the form

$$a_n = \sum_{k=1}^{m} p_k(n)\Delta^k a_n \ , \tag{1.3}$$

where $\Delta a_n = a_{n+1} - a_n$ and p_k, considered as functions of a continuous variable, are in $A^{(k)}$, $k = 1, 2, \ldots, m$.

In [11] it is shown that under certain conditions, infinite integrals whose integrands belong to $B^{(m)}$, have an asymptotic expansion that can be used to obtain an approximation to these integrals, namely—the D-transformation. Similarly, the d-transformation is derived by applying the discrete analogue of the technique to series with terms in $\tilde{B}^{(m)}$. The important observation in [11] is the fact that the sets $B^{(m)}$ and $\tilde{B}^{(m)}$ include a very large variety of functions and sequences which appear in applied mathematics. As shown there, the d-transformation is actually *a family* of transformations, which includes the ϵ-algorithm [18] and the u-transformation [7]. The range of applications of the D- and d-transformations includes very complicated integrals and series which cannot be handled by other methods.

In the two-dimensional case, we are interested in accelerating the convergence of infinite double integrals on $\mathbb{R}_+^2$

$$I = \int_0^\infty \int_0^\infty f(s,t)\, dt\, ds \ , \tag{1.4}$$

and infinite double series

$$S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} \ . \tag{1.5}$$

As in the one-dimensional case, we aim at integrals and series which belong to certain classes which cover a large variety of important cases. The first step is generalizing Definitions 1.1-1.3 to the two-dimensional case in the following manner:

Definition 1.4. $A^{(\gamma,\delta)}$ is defined as the set of functions $g(x,y)$ which have Poincaré type asymptotic expansions in inverse powers of x and y, as $\min\{x,y\} \to \infty$, of the form

$$g(x,y) \sim x^\gamma y^\delta \left(\alpha_{0,0} + \frac{\alpha_{1,0}}{x} + \frac{\alpha_{0,1}}{y} + \frac{\alpha_{2,0}}{x^2} + \frac{\alpha_{1,1}}{xy} + \frac{\alpha_{0,2}}{y^2} + \cdots\right) , \tag{1.6}$$

such that the partial derivatives of g, of any order, have asymptotic expansions which can be obtained by differentiating that in (1.6) formally term by term. The asymptotic expansion is in the sense

$$g(x,y) = x^\gamma y^\delta \Big(\sum_{i+j\le k} \frac{\alpha_{i,j}}{x^i y^j} + O((min\{x,y\})^{-k-1}) \Big) \ , \quad \text{as } min\{x,y\} \to \infty \ . \tag{1.7}$$

Definition 1.5. $B^{(m,n)}$ is defined as the set of bivariate functions that are integrable on $(0,\infty) \times (0,\infty)$, and which satisfy a linear partial differential equation of the form

$$f(x,y) = \sum_{k=0}^{m} \sum_{l=0}^{n} p_{k,l}(x,y) \partial_x^k \partial_y^l f(x,y) \ , \tag{1.8}$$

where $p_{k,l} \in A^{(k,l)}$ and $p_{0,0} \equiv 0$.

Definition 1.6. $\tilde{B}^{(m,n)}$ is defined as the set of double sequences whose terms satisfy a linear difference equation of the form

$$a_{q,r} = \sum_{k=0}^{m} \sum_{l=0}^{n} p_{k,l}(q,r) \Delta_1^k \Delta_2^l a_{q,r} \;, \tag{1.9}$$

where $\Delta_1 a_{i,j} = a_{i+1,j} - a_{i,j}$, $\Delta_2 a_{i,j} = a_{i,j+1} - a_{i,j}$ and $p_{k,l}$, considered as continuous bivariate functions, are in $A^{(k,l)}$, with $p_{0,0} \equiv 0$.

The derivation of the transformations in the bivariate case follows the idea and the technique in [11]. Using the above definitions, we derive asymptotic expansions for the remainders in double integrals and series whose associated integrands and series terms belong to $B^{(m,n)}$ and $\tilde{B}^{(m,n)}$, respectively. The derivation of the asymptotic expansions is based on repetitive integration or summation by parts and appropriate assumptions regarding the decay of the integrand/series terms at infinity. As we shall show, for infinite integrals the asymptotic expansion is given by

$$\begin{aligned} &\int_x^\infty \int_y^\infty f(s,t)\, dt\, ds \\ &\quad \sim \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} x^{k+1} y^{l+1} \partial_x^k \partial_y^l f(x,y) \sum_{i=0}^{\infty} \sum_{j=0}^{i} \frac{\beta_{i-j,j}^{k,l}}{x^{i-j} y^j} \\ &\quad + \sum_{k=0}^{m-1} x^{k+1} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \int_y^\infty \frac{\gamma_{i-j,j}^k}{x^{i-j} t^j} \partial_x^k f(x,t)\, dt \\ &\quad + \sum_{l=0}^{n-1} y^{l+1} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \int_x^\infty \frac{\delta_{j,i-j}^l}{s^j y^{i-j}} \partial_y^l f(s,y)\, ds \;, \end{aligned} \tag{1.10}$$

and for infinite series the asymptotic expansion is given by

$$\begin{aligned} \sum_{q=Q}^{\infty} \sum_{r=R}^{\infty} a_{q,r} &\sim \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} Q^{k+1} R^{l+1} \Delta_1^k \Delta_2^l a_{Q,R} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \frac{\beta_{i-j,j}^{k,l}}{Q^{i-j} R^j} \\ &\quad + \sum_{k=0}^{m-1} Q^{k+1} \sum_{r=R}^{\infty} \Delta_1^k a_{Q,r} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \frac{\gamma_{i-j,j}^k}{Q^{i-j} r^j} \\ &\quad + \sum_{l=0}^{n-1} R^{l+1} \sum_{q=Q}^{\infty} \Delta_2^l a_{q,R} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \frac{\delta_{j,i-j}^l}{q^j R^{i-j}} \;. \end{aligned} \tag{1.11}$$

We use these expansions to formulate linear systems of equations in which one unknown is the approximation to the underlying integral or sum, and the other unknowns are the coefficients of the truncated asymptotic expansion. We argue that all one needs to know in order to apply the transformations, is the *order* of the partial differential equation satisfied by the integrand, or alternatively, in the case of series, the order of the difference equation satisfied by the terms of the series. The derivation of the asymptotic expansions is quite tedious, but the resulting linear systems are easy to implement. See also [5].

Another point that should be mentioned is that in the bivariate transformations certain infinite univariate integrals or series have to be approximated. In general, we assume that this can be efficiently done, e.g., by the D-transformation for 1D integrals, and by d-transformation or Padé approximants for 1D series.

An outline of the rest of this paper follows: in Section 2 we review the univariate d- and D- transformations. In Section 3 we derive the D_2-transformation for infinite double integrals. In Section 4 we present the d_2-transformation for infinite double series. Sections 3 and 4 also include a brief review of some other techniques for evaluation of infinite 2D integrals and series. Finally, in Section 5 we bring several numerical examples in which various aspects of the two new transformations are discussed.

2. The D- and d-transformations

We now review the main results and definitions for the d- and D-transformations. We start with a theorem which is the key to the construction of the latter (see [11] for the proof).

Theorem 2.1. *Let f be integrable on $[0,\infty)$ and satisfy a linear mth-order differential equation of the form (1.2) with $p_k \in A^{(i_k)}$, $i_k \le k$, $k = 1,\dots,m$. If $lim_{x\to\infty}[p_k^{(i-1)}(x)][f^{(k-i)}(x)] = 0$ for $i \le k \le m$, $1 \le i \le m$, and if for any integer $l \ge -1$ we have*

$$\sum_{k=1}^{m} l(l-1)\cdots(l-k+1)\cdot \lim_{x\to\infty}[p_k(x)/x^k] \neq 1 \ , \tag{2.1}$$

then, as $x\to\infty$, $\int_x^\infty f(t)\,dt$ has an asymptotic expansion of the form

$$\int_x^\infty f(t)\,dt \sim \sum_{k=0}^{m-1} f^{(k)}(x)x^{j_k}(\beta_{k,0} + \frac{\beta_{k,1}}{x} + \frac{\beta_{k,2}}{x^2} + \dots) \ , \tag{2.2}$$

where $j_k \le max(i_{k+1}, i_{k+2}-1, \dots, i_m - m + k + 1)$, $k = 0,\dots,m-1$.

The D-transformation is defined by truncating the asymptotic expansion (2.2) and forming the following linear system of $N = 1 + \sum_{k=0}^{m-1} n_k$ equations:

(2.3)
$$D^{(m)}_{n_0,n_1,\dots,n_{m-1}} = \int_0^{x_r} f(t)\,dt + \sum_{k=0}^{m-1} f^{(k)}(x_r){x_r}^{j_k}\sum_{i=0}^{n_k-1}\frac{\overline{\beta}_{k,i}}{{x_r}^i} \ , \qquad r = 1,\dots,N \ .$$

The unknowns in this system are $D^{(m)}_{n_0,n_1,\dots,n_{m-1}}$, which represents the approximation to $\int_0^\infty f(t)\,dt$, and the set of coefficients $\{\overline{\beta}_{k,i}\}$. An appropriate choice of the points $\{x_r\}_{r=1}^N$ (which yields fast convergence) is discussed by Levin and Sidi in [11], and by Sidi in [13], [16], [17].

The d-transformation for accelerating the convergence of infinite series is merely a discrete analogue of the D-transformation [11] :

Theorem 2.2. *Let a_n, $n = 1,2,\dots$, satisfy the linear mth-order difference equation of the form (1.3), with $p_k \in A^{(i_k)}$, $i_k \le k$, $k = 1,\dots,m$. If*

$$\lim_{n\to\infty}[\Delta^{i-1}p_k(n)][\Delta^{k-i}a_n] = 0 \tag{2.4}$$

for $i \le k \le m$, $1 \le i \le m$, *and* $\sum_{k=1}^{m} l(l-1)\cdots(l-k+1)\cdot lim_{n\to\infty}[p_k(n)/n^k] \neq 1$ *for* $l = -1, 1, 2, 3, \ldots$, *then for* $N \to \infty$

$$\sum_{n=N}^{\infty} a_n \sim \sum_{k=0}^{m-1} N^{j_k}(\Delta^k a_N)\beta_k(N) \ , \tag{2.5}$$

where $\beta_k \in A^{(0)}$ *and* $j_k \le max(i_{k+1}, i_{k+2}-1, \cdots, i_m - m + k + 1)$, $0 \le k \le m-1$.

Theorem 2.2 serves as the basis for the definition of the d-transformation. We demand that the approximation $d^{(m)}_{n_0,n_1,\ldots,n_{m-1}}$ to $\sum_{n=1}^{\infty} a_n$ satisfy the $N = 1 + \sum_{k=0}^{m-1} n_k$ equations

(2.6)

$$d^{(m)}_{n_0,n_1,\ldots,n_{m-1}} = \sum_{n=1}^{N_l} a_n + \sum_{k=0}^{m-1} [\Delta^k a_{N_l+1}](N_l+1)^{j_k} \sum_{i=0}^{n_k-1} \frac{\beta_{k,i}}{(N_l+1)^i} \ , \quad l = 1, \cdots, N \ .$$

From (2.6) it can be observed that setting $m = 1$ and $j_0 = 0$ yields Levin's t-transformation, and $m = j_0 = 1$ yields Levin's u-transformation (see [6],[7]). The full derivation of the d- and D-transformations, including special cases, plus examples which demonstrate their efficiency can be found in [11]. The convergence analysis of this method has been presented in a series of papers by Sidi [13], [14], [15]. In [12] it is shown that when applied to power series, the d-transformation is a rational approximant with some properties that are similar to those of Padé approximants.

3. The D_2-transformation for infinite 2D integrals

We consider approximations to infinite double integrals, $I = \int_0^\infty \int_0^\infty f(s,t)\, dt\, ds$. In analogy to the 1D case, we assume here that $f \in B^{(m,n)}$. We start by reviewing the 2D analogue of the confluent ϵ-algorithm, which was developed by the second author [10] for a special subset of $B^{(m,n)}$ integrands. The assumption that stands in the basis of this transformation is that the integrand, $f(x,y)$, satisfies a linear partial differential equation with constant coefficients of the form:

$$f(x,y) = \sum_{(k,l)\in T} \alpha_{k,l}\partial_x^k \partial_y^l f(x,y) \ , \tag{3.1}$$

where T is a finite set of pairs, $T \subseteq \Omega^+ \backslash \{(0,0)\}$ where $\Omega^+ = \{(i,j)|\ i,j \ge 0\}$. Let R be a subset of Ω^+, with the same number of elements as T, and let $A_{x,y}$ be defined as

$$A_{x,y} = \{(s,t)|\ s \le x \text{ or } t \le y, \quad s,t \ge 0\} \ . \tag{3.2}$$

Then, for $x, y \ge 0$, the 2D analogue of the confluent ϵ-algorithm is defined as follows:

$$\begin{aligned} \epsilon_T^{(2)}(x,y) = \int\int_{A_{x,y}} f(s,t)\, dt\, ds + \sum_{(k,l)\in T, k\cdot l\neq 0} \alpha_{k,l}\partial_x^{k-1}\partial_y^{l-1} f(x,y) \\ - \sum_{(k,0)\in T} \alpha_{k,0} \int_y^\infty \partial_x^{k-1} f(x,t)\, dt - \sum_{(0,l)\in T} \alpha_{0,l} \int_x^\infty \partial_y^{l-1} f(s,y)\, ds \ , \end{aligned} \tag{3.3}$$

where the coefficients $\alpha_{k,l}$ are determined by the system of linear equations:

$$\partial_x^i \partial_y^j f(x,y) = \sum_{(k,l)\in T} \alpha_{k,l} \partial_x^{k+i} \partial_y^{l+j} f(x,y) \ , \qquad (i,j) \in R \ . \tag{3.4}$$

The following theorem, presented in [10], describes the class of exactness of the algorithm:

Theorem 3.1. *Let f satisfy* (3.1) *for $s \geq x$ and $t \geq y$, and suppose that*

$$\lim_{t\to\infty} \partial_x^k \partial_y^{l-1} f(s,t) = 0 \ , \qquad (k,l) \in T, \ l \neq 0 \ , \tag{3.5}$$

and

$$\lim_{s\to\infty} \partial_x^{k-1} \partial_y^{l} f(s,t) = 0, \qquad (k,l) \in T, \ k \neq 0 \ . \tag{3.6}$$

Then, if the system (3.4) *has a unique solution, $\epsilon_T^{(2)}(x,y)$ integrates f exactly in* $[0,\infty) \times [0,\infty)$.

The above algorithm handles very well integrals whose associated integrands satisfy p.d.e.'s with constant coefficients. The D_2-transformation presented below is meant to be appropriate for the case of variable coefficients. The theoretical basis for the definition of this transformation is the generalization of Theorem 2.1 to the two-dimensional case. In the following we refer to functions $f \in B^{(m,n)}$, and we present a series of propositions leading to the derivation of the asymptotic expansion (1.10). Throughout the derivation we shall make certain assumptions with regard to the decay of f and its derivatives. These assumptions are analogous to the ones given in Theorem 2.1.

Assumption 3.2. Given (1.8), the following holds:

$$\lim_{x\to\infty} \partial_x^{i-1} p_{k,l}(x,y) \partial_x^{k-i} \partial_y^l f(x,y) = 0,$$

for $k = i, \ldots, m$, $i = 1, \ldots, m$, $l = 0, \ldots, n$, and for any $y > 0$.

We now state the following result, concerning one-dimensional integration of f:

Proposition 3.3. *Let f satisfy* (1.8) *and assume that the conditions stated in Assumption 3.2 hold, then*

(3.7)

$$\int_x^\infty f(s,y)\, ds = \sum_{l=0}^{n} \{ \sum_{k=0}^{m-1} \bar{a}_{0,k,l}(x,y) \partial_x^k \partial_y^l f(x,y) + \int_x^\infty \bar{b}_{0,l}(s,y) \partial_y^l f(s,y)\, ds \} \ ,$$

where

(3.8)

$$\bar{a}_{0,k,l}(x,y) = \sum_{j=k+1}^{m} (-1)^{j+k} \partial_x^{j-k-1} p_{j,l}(x,y) \ , \quad k = 0, \ldots, m-1 \ , \quad l = 0, \ldots, n \ ,$$

and

$$\bar{b}_{0,l}(x,y) = \sum_{j=0}^{m} (-1)^j \partial_x^j p_{j,l}(x,y) \ , \qquad l = 0, \ldots, n \ . \tag{3.9}$$

Proof. Intergating (1.8) with respect to x we have

$$\int_x^\infty f(s,y)\,ds = \sum_{l=0}^{n}\{\sum_{k=0}^{m}\int_x^\infty p_{k,l}(s,y)\partial_x^k\partial_y^l f(s,y)\,ds\}\ . \tag{3.10}$$

In a way similar to that used in the derivation of the D-transformation, we now perform a sequence of integration by parts: by Assumption 3.2 with $i=1$, we obtain

$$\begin{aligned}\int_x^\infty f(s,y)\,ds = \sum_{l=0}^{n}\{&-\sum_{k=1}^{m} p_{k,l}(x,y)\partial_x^{k-1}\partial_y^l f(x,y)\\ &+\int_x^\infty p_{0,l}(s,y)\partial_y^l f(s,y)\,ds - \sum_{k=1}^{m}\int_x^\infty \partial_x p_{k,l}(s,y)\partial_x^{k-1}\partial_y^l f(s,y)\,ds\}\ .\end{aligned} \tag{3.11}$$

Integrating by parts the last term of the right hand side of (3.11) and using Assumption 3.2 with $i=2$ leads to

$$\begin{aligned}\int_x^\infty f(s,y)\,ds = \sum_{l=0}^{n}\{&-\sum_{k=1}^{m} p_{k,l}(x,y)\partial_x^{k-1}\partial_y^l f(x,y)\\ &+\int_x^\infty [p_{0,l}(s,y)-\partial_x p_{1,l}(s,y)]\partial_y^l f(s,y)\,ds\\ &+\sum_{k=2}^{m}\int_x^\infty \partial_{xx} p_{k,l}(s,y)\partial_x^{k-2}\partial_y^l f(s,y)\,ds\}\ .\end{aligned} \tag{3.12}$$

At this point it is clear that assuming the decay conditions stated in Assumption 3.2, we can carry on repeated integration by parts, until all the x-derivatives of f in the last term of the right hand side of (3.12) disappear, and the result (3.7) is obtained. $\square$

We now turn to integrate (3.7) in the y-direction:

$$\begin{aligned}\int_x^\infty\int_y^\infty f(s,t)\,dt\,ds = &\sum_{l=0}^{n}\sum_{k=0}^{m-1}\int_y^\infty \bar{a}_{0,k,l}(x,t)\partial_x^k\partial_y^l f(x,t)\,dt\\ &+\sum_{l=0}^{n}\int_y^\infty\int_x^\infty \bar{b}_{0,l}(s,t)\partial_y^l f(s,t)\,ds\,dt\ .\end{aligned} \tag{3.13}$$

Performing a sequence of successive integration by parts, we assume the following decay conditions :

Assumption 3.4. Given (3.8) and (3.9), the following decay conditions hold for $x>0$: $lim_{y\to\infty}\partial_y^{i-1}\bar{b}_{0,l}(x,y)\partial_y^{l-i}f(x,y)=0$, for $i=1,\dots,n,\ \ l=i,i+1,\dots,n$, and $lim_{y\to\infty}\partial_y^{i-1}\bar{a}_{0,k,l}(x,y)\partial_x^k\partial_y^{l-i}f(x,y)=0$, for $k=0,\dots,m-1,\ i=1,\dots,n$, $l=i,i+1,\dots,n$.

Provided Assumption 3.4 holds, we now present formulas for each of the terms on the right hand side of (3.13).

Proposition 3.5.

$$\sum_{l=0}^{n}\sum_{k=0}^{m-1}\int_y^\infty \overline{a}_{0,k,l}(x,t)\partial_x^k\partial_y^l f(x,t)\,dt \tag{3.14}$$
$$= \sum_{k=0}^{m-1}\sum_{l=0}^{n-1} a^*_{0,k,l}(x,y)\partial_x^k\partial_y^l f(x,y) + \sum_{k=0}^{m-1}\int_y^\infty b^*_{0,k}(x,t)\partial_x^k f(x,t)\,dt\ ,$$

where

$$a^*_{0,k,l}(x,y) = \sum_{i=k+1}^{m}\sum_{j=l+1}^{n}(-1)^{i+j+k+l}\partial_x^{i-k-1}\partial_y^{j-l-1}p_{i,j}(x,y)\ , \tag{3.15}$$
$$k=0,\dots,m-1\ ,\qquad l=0,\dots,n-1\ ,$$

and

$$b^*_{0,k}(x,y) = \sum_{i=k+1}^{m}\sum_{j=0}^{n}(-1)^{i+j+k}\partial_x^{i-k-1}\partial_y^j p_{i,j}(x,y)\ ,\qquad k=0,\dots,m-1. \tag{3.16}$$

Proof. The left hand side of (3.14) is similar in essence to the one on the right hand side of (3.10) handled in Proposition 3.3. Therefore, interchanging order of summation, assuming the conditions in Assumption 3.4 concerning $\overline{a}_{0,k,l}$ and performing successive integration by parts in a way analogous to the one performed in Proposition 3.3, we obtain (3.14) with

$$a^*_{0,k,l}(x,y) = \sum_{j=l+1}^{n}(-1)^{j+l}\partial_y^{j-l-1}\overline{a}_{0,k,j}(x,y)\ , \tag{3.17}$$
$$k=0,\dots,m-1\ ,\quad l=0,\dots,n-1\ ,$$

and

$$b^*_{0,k}(x,y) = \sum_{j=0}^{n}(-1)^j\partial_y^j\overline{a}_{0,k,j}(x,y),\qquad k=0,\dots,m-1\ . \tag{3.18}$$

In terms of the $p_{k,l}$'s, the above two functions are given by (3.15) and (3.16). □

Proposition 3.6.

$$\sum_{l=0}^{n}\int_y^\infty\int_x^\infty \overline{b}_{0,l}(s,t)\partial_y^l f(s,t)\,ds\,dt \tag{3.19}$$
$$= \sum_{l=0}^{n-1}\int_x^\infty c^*_{0,l}(s,y)\partial_x^l f(s,y)\,ds + \int_y^\infty\int_x^\infty \overline{d}_0(s,t)f(s,t)\,dt\,ds,$$

where

$$c^*_{0,l}(x,y) = \sum_{i=0}^{m}\sum_{j=l+1}^{n}(-1)^{i+j+l}\partial_x^i\partial_y^{j-l-1}p_{i,j}(x,y)\ ,\qquad l=0,\dots,n-1\ , \tag{3.20}$$

and

$$\overline{d}_0(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\partial_x^i\partial_y^j p_{i,j}(x,y)\ . \tag{3.21}$$

Proof. Interchanging the order of integration, using the conditions in Assumption 3.4 concerning $\bar{b}_{0,l}$ and repeating the process of integration by parts, we obtain (3.19), with

$$c^*_{0,l}(x,y) = \sum_{j=l+1}^{n} (-1)^{j+l} \partial_y^{j-l-1} \bar{b}_{0,j}(x,y) \ , \qquad l = 0, \ldots, n-1 \ , \tag{3.22}$$

and

$$\bar{d}_0(x,y) = \sum_{j=0}^{n} (-1)^j \partial_y^j \bar{b}_{0,j}(x,y) \ . \tag{3.23}$$

Using (3.9) to express the above in terms of the $p_{k,l}$'s yields (3.20) and (3.21). □

Let us examine the effect of Propositions 3.3, 3.5 and 3.6. Using (3.13), (3.14) and (3.19) yields

$$\begin{aligned} \int_x^\infty \int_y^\infty f(s,t)\, dt\, ds &= \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} a^*_{0,k,l}(x,y) \partial_x^k \partial_y^l f(x,y) \\ &+ \sum_{k=0}^{m-1} \int_y^\infty b^*_{0,k}(x,t) \partial_x^k f(x,t)\, dt + \sum_{l=0}^{n-1} \int_x^\infty c^*_{0,l}(s,y) \partial_y^l f(s,y)\, ds \\ &+ \int_x^\infty \int_y^\infty \bar{d}_0(s,t) f(s,t)\, dt\, ds \ . \end{aligned} \tag{3.24}$$

From (3.15), (3.16), (3.20) and (3.21) the following conclusions can be made:

$$a^*_{0,k,l} \in A^{(k+1,l+1)} \ ; \quad b^*_{0,k} \in A^{(k+1,0)} \ ; \quad c^*_{0,l} \in A^{(0,l+1)} \ ; \quad \bar{d}_0 \in A^{(0,0)} \ , \tag{3.25}$$

for $0 \leq k \leq m-1$ and $0 \leq l \leq n-1$. Since $\bar{d}_0 \in A^{(0,0)}$, i.e.,

$$\bar{d}_0 = \bar{\alpha}_{0,0} + \frac{\bar{\alpha}_{1,0}}{x} + \frac{\bar{\alpha}_{0,1}}{y} + \frac{\bar{\alpha}_{2,0}}{x^2} + \frac{\bar{\alpha}_{1,1}}{xy} + \frac{\bar{\alpha}_{0,2}}{y^2} + \cdots \ , \tag{3.26}$$

the dominant factor in the last term of the right hand side of (3.24) is

$$\bar{\alpha}_{0,0} \int_x^\infty \int_y^\infty f(s,t)\, dt\, ds.$$

Provided $\bar{\alpha}_{0,0} \neq 1$, we may move this factor to the left hand side and divide by $1 - \bar{\alpha}_{0,0}$ to obtain a new expression for $\int_x^\infty \int_y^\infty f(s,t)\, dt\, ds$. In the following we are going to repeat this process several times, and we shall need to generalize the assumption regarding $\bar{\alpha}_{0,0}$ in the following manner (which will be clarified in the proof of Lemma 3.8):

Assumption 3.7. For integer values $\mu, \nu \geq 0$

$$\alpha^{\mu,\nu} = \sum_{i=0}^{m} \sum_{j=0}^{n} [\prod_{q=1}^{i} \prod_{r=1}^{j} (\mu - q)(\nu - r)]\ \bar{p}_{i,j} \neq 1 \ , \tag{3.27}$$

where $\bar{p}_{i,j}$ is the leading term in the asymptotic expansion of $p_{i,j}$.

Remark. We refer to $\prod_{q=1}^{0}(\ldots)$ as being identically equal to 1.

We now present the generalization of the process started above:

Lemma 3.8. *Let $d(x,y) \in A^{(-\mu,-\nu)}$ where μ,ν are nonnegative integer values, and assume that for $0 \le i \le m,\ 0 \le j \le n$, $\partial_x^i \partial_y^j d$ exist. Under Assumptions 3.2, 3.4, 3.7,*

$$\begin{aligned}(3.28)\qquad \int_x^\infty \int_y^\infty d(s,t) f(s,t)\, dt\, ds &= \sum_{k=0}^{m-1}\sum_{l=0}^{n-1} a_{k,l}(x,y)\partial_x^k \partial_y^l f(x,y) \\ &+ \sum_{k=0}^{m-1} \int_y^\infty b_k(x,t)\partial_x^k f(x,t)\, dt \ + \sum_{l=0}^{n-1} \int_x^\infty c_l(s,y)\partial_y^l f(s,y)\, ds \\ &+ \int_x^\infty \int_y^\infty d^*(s,t) f(s,t)\, dt\, ds \ ,\end{aligned}$$

where

$$(3.29)\qquad a_{k,l} \in A^{(k-\mu+1,l-\nu+1)};\ b_k \in A^{(k-\mu+1,-\nu)};\ c_l \in A^{(-\mu,l-\nu+1)}; \\ d^* \in A^{(-\mu-1,-\nu)} \cup A^{(-\mu,-\nu-1)}$$

with $k = 0,\dots,m-1$ and $l = 0,\dots,n-1$.

Proof. The procedure presented in Propositions 3.3, 3.5 and 3.6 starts with representing f by (1.8). Here we start by representing $d \cdot f$ by a similar representation with the functions $p_{i,j}$ replaced by $d \cdot p_{i,j}$. Repeating the same steps yields an equation of the type (3.28), with the exception that the last term on the right hand side is now $\int_x^\infty \int_y^\infty \overline{d}(s,t) f(s,t)\, ds\, dt$, with $\overline{d} \in A^{(-\mu,-\nu)}$. In the process of successive integration by parts, the required decay conditions are guaranteed to be satisfied since $-\mu, -\nu \le 0$ and Assumptions 3.2 and 3.4 hold. Since d and $\overline{d}$ both belong to $A^{(-\mu,-\nu)}$, we can write

$$(3.30)\qquad \overline{d}(x,y) = \alpha \cdot d(x,y) + d^*(x,y)\ ,$$

with $d^* \in A^{(-\mu-1,-\nu)} \cup A^{(-\mu,-\nu-1)}$. α can be computed as follows: since the $p_{i,j}$ are replaced by $d \cdot p_{i,j}$, we have by Proposition 3.6 that α is the coefficient of the leading term in the asymptotic expansion of

$$(3.31)\qquad \overline{d}_0(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} (-1)^{i+j} \partial_x^i \partial_y^j [d \cdot p_{i,j}]\ .$$

By writing the formal asymptotic expansion of the form (1.6) for $d \cdot p_{i,j} \in A^{(-\mu+i,-\nu+j)}$, differentiating i times in the x-direction and j times in the y-direction for each required value of i and j, the leading term in $\overline{d}_0$ is $\alpha x^\mu y^\nu$ with

$$(3.32)\qquad \alpha = \sum_{i=0}^{m}\sum_{j=0}^{n} (-1)^{i+j} [\prod_{q=1}^{i}\prod_{r=1}^{j} (-\mu+q)(-\nu+r)]\ \overline{p}_{i,j}\ .$$

Note that $\alpha \equiv \alpha^{\mu,\nu}$ [see (3.27)]. We can now rewrite $\int_x^\infty \int_y^\infty \overline{d}(s,t) f(s,t)\, ds\, dt$ replacing $\overline{d}(x,y)$ by $\alpha \cdot d(x,y) + d^*(x,y)$. Transferring $\int_x^\infty \int_y^\infty \alpha\, d(s,t) f(s,t)\, ds\, dt$ to the left hand side and dividing by $1-\alpha$, which is non-zero by Assumption 3.7, we obtain (3.28). □

We now use Lemma 3.8 recursively. First we obtain

$$
\begin{aligned}
(3.33)\qquad \int_x^\infty \int_y^\infty f(s,t)\,dt\,ds &= \sum_{k=0}^{m-1}\sum_{l=0}^{n-1} a_{1,k,l}(x,y)\partial_x^k\partial_y^l f(x,y) \\
&+ \sum_{k=0}^{m-1}\int_y^\infty b_{1,k}(x,t)\partial_x^k f(x,t)\,dt + \sum_{l=0}^{n-1}\int_x^\infty c_{1,l}(s,y)\partial_y^l f(s,y)\,ds \\
&+ \int_x^\infty \int_y^\infty d_1(s,t) f(s,t)\,dt\,ds\ ,
\end{aligned}
$$

where $d_1 \in A^{(0,-1)} \cup A^{(-1,0)}$ and the first three terms on the right hand side of (3.33) are of the same form as in the expansion we seek (1.10). The last term of (3.33) is handled as follows: since this term involves the factor $d_1 \in A^{(0,-1)} \cup A^{(-1,0)}$, we decompose d_1 into two factors, one in $A^{(0,-1)}$ and the other in $A^{(-1,0)}$, and we use (3.28) for each of the corresponding integrals. As a result we obtain the expansion

$$
\begin{aligned}
(3.34)\qquad \int_x^\infty \int_y^\infty f(s,t)\,dt\,ds &= \sum_{k=0}^{m-1}\sum_{l=0}^{n-1} a_{2,k,l}(x,y)\partial_x^k\partial_y^l f(x,y) \\
&+ \sum_{k=0}^{m-1}\int_y^\infty b_{2,k}(x,t)\partial_x^k f(x,t)\,dt + \sum_{l=0}^{n-1}\int_x^\infty c_{2,l}(s,y)\partial_y^l f(s,y)\,ds \\
&+ \int_x^\infty \int_y^\infty d_2(s,t) f(s,t)\,dt\,ds\ .
\end{aligned}
$$

Here also the first three terms on the right hand side are of the right form, that is, as in (1.10), and now $d_2 \in A^{(0,-2)} \cup A^{(-1,-1)} \cup A^{(-2,0)}$. Specifically,

$$
(3.35)\qquad a_{2,k,l} \in A^{(k+1,l+1)}\ ;\quad b_{2,k} \in A^{(k+1,0)}\ ;\quad c_{2,l} \in A^{(0,l+1)}\ ,
$$

for $0 \le k \le m-1$ and $0 \le l \le n-1$. Repeating this we can make the last term be of the form $\int_x^\infty \int_y^\infty d_q(s,t)f(s,t)\,dt\,ds$ with $d_q \in \bigcup_{i=0}^q A^{(-i,-q+i)}$, for any integer $q > 0$.

We have thus proved the following theorem on the asymptotic expansion of the infinite double integral:

Theorem 3.9. *Let $f(x,y)$ be integrable on $[0,\infty) \times [0,\infty)$ and satisfy the linear p.d.e.* (1.8) *with $p_{k,l} \in A^{(k,l)}$,$0 \le k \le m$,$0 \le l \le n$ and $p_{0,0} \equiv 0$. When Assumptions 3.2, 3.4, 3.7 hold, as $x,\ y \to \infty$, $\int_x^\infty \int_y^\infty f(s,t)\,dt\,ds$ has an asymptotic expansion of the form given in* (1.10).

The asymptotic expansion (1.10) serves as the basis for the D_2-transformation. The idea is truncating the three infinite sums indexed by i in the expansion, and solving a linear system where the unknowns are the approximation to the infinite integral and the coefficients of the truncated expansions. In order to avoid the usage of too many indices, we refer to a 'diagonal' approximation, in which the above mentioned sums are all truncated in the same power of x and y. The resulting linear system is

$$D^{(m,n)}_{n_1,n_2,n_3} = \int\int_{A_r} f(s,t)\,dt\,ds + \sum_{k=0}^{m-1}\sum_{l=0}^{n-1} {x_r}^{k+1}{y_r}^{l+1}\sum_{i=0}^{n_1}\sum_{j=0}^{i}\frac{\beta^{k,l}_{i-j,j}}{{x_r}^{i-j}{y_r}^{j}}\partial_x^k\partial_y^l f(x_r,y_r) + \sum_{k=0}^{m-1} x_r^{k+1}\sum_{i=0}^{n_2}\sum_{j=0}^{i}\frac{\gamma^k_{i-j,j}}{{x_r}^{i-j}}\int_{y_r}^{\infty}\frac{\partial_x^k f(x_r,t)}{t^j}\,dt + \sum_{l=0}^{n-1} y_r^{l+1}\sum_{i=0}^{n_3}\sum_{j=0}^{i}\frac{\delta^l_{j,i-j}}{{y_r}^{i-j}}\int_{x_r}^{\infty}\frac{\partial_y^l f(s,y_r)}{s^j}\,ds\ , \tag{3.36}$$

where $r = 1,\ldots,N$, $\{(x_r,y_r)\}_{r=1}^N$ is a chosen set of points in the first quadrant, $A_r \equiv A_{x_r,y_r}$ as in (3.2), the unknowns are $D^{(m,n)}_{n_1,n_2,n_3}$ and the coefficients $\{\beta^{k,l}_{i-j,j}\}$, $\{\gamma^k_{i-j,j}\}$ and $\{\delta^l_{j,i-j}\}$, and N is the number of equations, given by

$$N = m\cdot n\cdot F(n_1) + m\cdot F(n_2) + n\cdot F(n_3) + 1\ , \tag{3.37}$$

where $F(x) = \frac{1}{2}\cdot(x+1)\cdot(x+2)$. The value $D^{(m,n)}_{n_1,n_2,n_3}$ in the solution vector represents the approximation to $\int_0^\infty\int_0^\infty f(s,t)\,dt\,ds$.

A class of exactness for the D_2-transformation. Let us look at a special class of $B^{(m,n)}$ functions: those which satisfy (1.8), with $p_{k,l}$ being constants. In this case it follows that in the asymptotic expansions on the right hand side of (1.10) only the terms with $i = 0$ are non-zero. Hence $D^{(m,n)}_{0,0,0}$ gives an exact value for $\int_0^\infty\int_0^\infty f(s,t)\,dt\,ds$ in this case. In that respect, the D_2-transformation is similar to the 2D analogue of the confluent ϵ-algorithm.

Practical implementation of the transformation. The system of equations that has to be solved in order to obtain the approximation involves some infinite 1D integrals, $\int_y^\infty \partial_x^k f(x_i,t)dt$, $\int_x^\infty \partial_y^l f(s,y_i)ds$, and double integrals $\int\int_{A_i} f(s,t)\,dt\,ds$, $i = 1,\ldots,N$. Let us refer first to $\int\int_{A_i} f(s,t)\,dt\,ds, i = 1,\ldots,N$; in order to find a satisfactory approximation to these double integrals, we split A_i, for each i, into 3 subsets, as follows:

$$A_i = A_i^{xy}\cup A_i^x\cup A_i^y\ ,\qquad i = 1,\ldots N\ ,$$

where: $A_i^{xy} = [0,x_i]\times[0,y_i]$, $A_i^x = [0,x_i]\times[y_i,\infty]$, $A_i^y = [x_i,\infty]\times[0,y_i]$. For the integral over A_i^{xy} we have used tensor product composite 6-point Gauss-Legendre rule, correct to 12 decimal digits. The integrals over A_i^x and A_i^y can be computed in several efficient ways. For example, in order to obtain an approximation to $\int_{A_i^x} f = \int_0^{x_i}\int_{y_i}^\infty f(s,t)\ dt\ ds$, we can take a sequence of finite double integrals, $\int_0^{x_i}\int_{y_i+j\xi}^{y_{i+(j+1)\xi}} f(s,t)\ dt\ ds$, $j \geq 0$, for a fixed small value of ξ, and apply the d-transformation (or any other sequence extrapolation method) to sum these finite integrals. Alternatively, the D-transformation can be used. This also refers to the infinite 1D integrals in (3.36). In many cases examined, it has been observed that the integrands, as functions of either x or y only, belong to $B^{(\overline{m})}$ for some small $\overline{m}$. In such cases, the D-transformation can be effectively applied to evaluate these infinite integrals. The actual evaluation of the different derivatives of f needed in the computation of the D_2-transformation can be done by direct differentiation, or, it can be done to any required accuracy by appropriate divided differences approximations.

4. The d_2-transformation for infinite 2D series

We start this section by presenting the $[A/S]_R$ approximant for infinite 2D sums [9] and reviewing some of its properties. See [8], [9], [10] for full details on the derivation. Further theoretical results and applications can be found in [2], [3] and [4].

The $[A/S]_R$ approximant [10]. Given a 2D series, its terms are assumed to satisfy a linear relation of the form:

$$a_{i,j} = \sum_{(k,l)\in T} \alpha_{k,l}\Delta_1^k\Delta_2^l a_{i-k,j-l} \ , \qquad i \geq I \ , \ \ j \geq J \ , \tag{4.1}$$

where T is a finite set of ordered pairs with nonnegative indices, with $(0,0) \notin T$, and $I \geq max\{i \mid (i,j) \in T\}$, $J \geq max\{j \mid (i,j) \in T\}$. For a set A of ordered pairs, we use the notation

$$A^+ = \{(i,j) \mid (i,j) \in A, \ i \geq 0, \ j \geq 0\} \ . \tag{4.2}$$

Taking

$$A = \{(i,j) \mid i < M \text{ or } j < N\}, \qquad M \geq I \ , \ \ N \geq J \ , \tag{4.3}$$

it can be shown that by performing summation by parts, we obtain:

$$\begin{aligned}\sum_{i,j=0}^{\infty} a_{i,j} = \sum_{(i,j)\in A^+} a_{i,j} &+ \sum_{(k,l)\in T, k\cdot l\neq 0} \alpha_{k,l}\Delta_1^{k-1}\Delta_2^{l-1}a_{M-k,N-l} \\ &- \sum_{(k,0)\in T}\alpha_{k,0}\sum_{j=N}^{\infty}\Delta_1^{k-1}a_{M-k,j} - \sum_{(0,l)\in T}\alpha_{0,l}\sum_{i=M}^{\infty}\Delta_2^{l-1}a_{i,N-l} \ .\end{aligned} \tag{4.4}$$

Let S and R be finite subsets of Ω^+ such that $|S| = |R| + 1$, where $|S|$ denotes the number of elements of S. The approximation $[A/S]_R$ is defined as the value Ω' in the solution vector $(\Omega', \{\beta_{i,j}\}_{(i,j)\in R})$ of the linear system of equations

$$\Omega' - \sum_{(i,j)\in R} \beta_{i,j} a_{i-k,j-l} = A_{-k,-l} \ , \qquad (k,l) \in S \ , \tag{4.5}$$

where $A_{-k,-l}$ is the partial sum of the $(-k,-l)$ translation of the A defined above. Notice that the system of equations for the computation of $[A/S]_R$ involves the need to compute the infinite 1D sums appearing in $A_{-k,-l}$. In many cases, this can be efficiently done by using the d-transformation. Numerical examples, and the generalization to higher-dimensional series can be found in [10].

The d_2-transformation. The previous transformation handles well series whose terms satisfy approximately difference equations with constant coefficients. As in the case of integrals, we seek a transformation that can handle series of a more general class, namely series whose terms satisfy difference equations with variable coefficients rather than constant ones. The generalization is the d_2-transformation which is meant for evaluating infinite double sums (1.5) whose terms satisfy a linear double difference equation of the form (1.9). As in the case of infinite integrals, the derivation of the d_2-transformation is based upon obtaining an asymptotic expansion for the remainder of the double sum. The asymptotic expansion involves infinite 1D 'boundary' series of a certain kind. In most cases, these one-dimensional sums can be well approximated by the d-transformation. In analogy to Theorem 3.9 for infinite 2D integrals whose integrands are in $B^{(m,n)}$, we now state a theorem for the sum of infinite 2D series, whose terms belong to the set of sequences $\tilde{B}^{(m,n)}$.

Theorem 4.1. *Let* $|\sum_{q,r=0}^{\infty} a_{q,r}| < \infty$, *and let* $\{a_{q,r}\}_{q,r=0}^{\infty}$ *satisfy the linear difference equation* (1.9). *Denote* $s_{k,l}(q,r) = \sum_{j=k+1}^{m}(-1)^{j+k}\Delta_1^{j-k-1}p_{j,l}(q,r)$, $k = 0,\ldots,m-1$, $l = 0,\ldots,n$ *and let* $t_l(q,r) = \sum_{j=0}^{m}(-1)^j\Delta_1^j p_{j,l}(q,r)$, $l = 0,\ldots,n$. *Assume the following:*

1. $lim_{q\to\infty}[\Delta_1^{i-1}p_{k,l}(q,r)][\Delta_1^{k-i}\Delta_2^{l}a_{q,r}] = 0$, *for* $k = i,\ldots,m$, $i = 1,\ldots,m$, $l = 0,\ldots,n$.
2. $lim_{r\to\infty}[\Delta_2^{i-1}s_{k,l}(q,r)][\Delta_1^{k}\Delta_2^{l-i}a_{q,r}] = 0$ *and* $lim_{r\to\infty}[\Delta_2^{i-1}t_l(q,r)][\Delta_2^{l-i}a_{q,r}] = 0$, *for* $i = 1,\ldots,n$, $l = i,\ldots,n$, $k = 0,\ldots,m-1$.
3. *Assumption 3.7 (stated in Section 3).*

Then, for Q, $R \to \infty$, $\sum_{q=Q}^{\infty}\sum_{r=R}^{\infty} a_{q,r}$ *has an asymptotic expansion of the form* (1.11).

The proof of this theorem is completely analogous to the proof of Theorem 3.9: the operator ∂_x is replaced by Δ_1, the operator ∂_y is replaced by Δ_2, and integration by parts is replaced by summation by parts. In a way analogous to the derivation of the D_2-transformation, we can now define the d_2-transformation. Let us make the following notations:

$$U_{Q,R}^{(i,p)} = \sum_{r=R}^{\infty} \frac{\Delta_1^p a_{Q,r}}{r^i} , \tag{4.6}$$

and

$$V_{Q,R}^{(i,p)} = \sum_{q=Q}^{\infty} \frac{\Delta_2^p a_{q,R}}{q^i} . \tag{4.7}$$

The d_2-transformation is defined as the value $d_{n_1,n_2,n_3}^{(m,n)}$ in the solution vector of the following linear system:

$$\begin{aligned} d_{n_1,n_2,n_3}^{(m,n)} &= A_{Q_s,R_s} + \sum_{k=0}^{m-1}\sum_{l=0}^{n-1} {Q_s}^{k+1}{R_s}^{l+1}\sum_{i=0}^{n_1}\sum_{j=0}^{i}\frac{\beta_{i-j,j}^{k,l}}{{Q_s}^{i-j}{R_s}^{j}}\Delta_1^k\Delta_2^l a_{Q_s,R_s} \\ &+ \sum_{k=0}^{m-1} Q_s^{k+1}\sum_{i=0}^{n_2}\sum_{j=0}^{i}\frac{\gamma_{i-j,j}^{k}}{Q_s^{i-j}}U_{Q_s,R_s}^{(j,k)} + \sum_{l=0}^{n-1} R_s^{l+1}\sum_{i=0}^{n_3}\sum_{j=0}^{i}\frac{\delta_{j,i-j}^{l}}{R_s^{i-j}}V_{Q_s,R_s}^{(j,l)} , \end{aligned} \tag{4.8}$$

where $s = 1,\ldots,N$, $\{(Q_s,R_s)\}_{s=1}^{N}$ is a chosen set of indices, A_{Q_s,R_s} denotes the corresponding partial double sum,

$$A_{Q_s,R_s} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{i,j} - \sum_{i=Q_s}^{\infty}\sum_{j=R_s}^{\infty} a_{i,j} , \tag{4.9}$$

the unknowns are $d_{n_1,n_2,n_3}^{(m,n)}$ and the coefficients $\{\beta_{i-j,j}^{k,l}\}$, $\{\gamma_{i-j,j}^{k}\}$ and $\{\delta_{j,i-j}^{l}\}$, and N is the number of equations, given by (3.37).

The value $d_{n_1,n_2,n_3}^{(m,n)}$ in the solution vector is the approximation to $\sum_{i,j=0}^{\infty} a_{i,j}$. The approximation to the infinite 1D sums in A_{Q_s,R_s}, $U_{Q_s,R_s}^{(j,k)}$ and $V_{Q_s,R_s}^{(j,l)}$ can be obtained by the d-transformation.

Remark. Application to double power series. In this case, the 'boundary' infinite 1D sums that appear in the asymptotic expansion for the remainder, turn the approximation achieved by applying the d_2-transformation into a bivariate function which does not have a simple structure. However, if one approximates the 'boundary' sums by Padé approximants, or by the d-transformation for 1D power series [12], then the d_2-transformation yields a rational approximation.

5. Numerical examples

In the tables that follow, the term 'cdd' refers to the number of *correct decimal digits*, computed by taking the logarithm of the absolute value of the approximation's error. The term 'Size' refers to the number of unknowns in the linear system solved for obtaining the approximation.

Example 5.1.

$$I = \int_0^\infty \int_0^\infty \frac{ds\, dt}{(1+s^2+t^2)^2} = \frac{\pi}{4} = 0.78539818525\ldots .$$

The integrand, $f(x,y) = \frac{1}{(1+x^2+y^2)^2}$, satisfies a first-order linear p.d.e. of the type presented in (1.8). In fact, there is more than one such equation that this function satisfies. For example, we have

$$f = -\frac{1+2x^2}{8x} f_x + \frac{1+2y^2}{8y} f_y \ .$$

In terms of equation (1.8), with $m = n = 1$, we can see that $p_{1,0} \in A^{(1,0)}$ and $p_{0,1} \in A^{(0,1)}$, hence, $f \in B^{(1,1)}$. Verifying that the other conditions of Theorem 3.9 hold is easy. In Table 5.1 (on the next page) we present the results we have obtained for this integral. In this case, it is evident that the D-transformation (for infinite 1D integrals) can be successfully applied, due to the fact that $f(x,y)$ satisfies the following two equations:

1. $f(x,y) = -\frac{1+x^2+y^2}{4x} f_x(x,y)$.
2. $f(x,y) = -\frac{1+x^2+y^2}{4y} f_y(x,y)$.

Hence, $f(x,y)$, as a function of one variable (x or y), belongs to $B^{(1)}$. In addition to that, f, as a function of one variable, satisfies all the conditions stated in the theorem which defines the 1D D-transformation, hence we expect to obtain a good approximation to the 1D integrals that appear in the asymptotic expansion. Numerical experiments verify this observation.

As input to the program we took $m = n = 1$, and $(x_0, y_0) = (1,1)$ as the point whose distance to the origin is the largest, out of all the points that are used in the system of equations.

The results in Table 5.1 illustrate the fast convergence of the D_2-transformation even with a zero degree approximation for the expansions involving the 'boundary terms'.

Faster convergence is observed for higher degree approximations, as presented in Table 5.2.

TABLE 5.1

n_1, n_2, n_3	Size	cdd	n_1, n_2, n_3	Size	cdd	n_1, n_2, n_3	Size	cdd
0,0,0	4	1.30	3,0,0	13	4.13	6,0,0	31	6.51
1,0,0	6	2.46	4,0,0	18	5.21	7,0,0	39	7.22
2,0,0	9	3.76	5,0,0	24	5.96	8,0,0	48	8.32

TABLE 5.2

n_1, n_2, n_3	Size	cdd	n_1, n_2, n_3	Size	cdd	n_1, n_2, n_3	Size	cdd
1,1,1	10	4.31	1,2,2	16	4.09	1,3,3	24	7.38
3,1,1	17	5.11	3,2,2	23	5.67	3,3,3	31	6.91
5,1,1	28	5.18	5,2,2	34	8.13	5,3,3	42	9.51

Example 5.2.

$$I = \int_0^\infty \int_0^\infty \frac{\sin(s^2+t^2)ds\,dt}{s^2+t^2} = \frac{\pi^2}{8} = 1.23370055013617\ldots .$$

This integral can be easily computed by using polar coordinates. The integrand, $f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$, satisfies the p.d.e.:

$$f = -\frac{1}{2x(x^2+y^2)}f_x - \frac{1}{2y(x^2+y^2)}f_y - \frac{1}{4xy}f_{xy} ,$$

hence $f(x,y) \in B^{(1,1)}$. It is easy to verify that $f(x,y)$ satisfies all the conditions that appear in Theorem 3.9. Notice that in terms of (1.8), we can use polynomials in inverse powers of x and y, with lower degrees than those used in (1.10). The integral in this example is considered a tough case, since the integrand oscillates rapidly as x and y grow larger. The matrix which represents the system of equations becomes ill conditioned very fast as its size grows larger. For systems of size 20×20 or so, the numerical instability becomes dominant. Despite the above difficulties, we obtain up to 4-5 digit accuracy, which is the expected accuracy, considering the truncation error in the asymptotic expansion.

At this point we mention that Sidi, [13], suggested a way to handle oscillatory integrals (he referred to the 1D case) by choosing the points x_r such that $f^{(k_i)}(x_r) = 0$. As a result, we end up with a singular matrix. However, this system can be reduced, by elimination of a few of its equations, to one that has a unique solution. The convergence properties of this modification of the D-transformation are brought in [13]. The same idea can be applied to the 2D case. The resulting system of equations will be smaller and less ill conditioned.

TABLE 5.3

n_1, n_2, n_3	Size	cdd	n_1, n_2, n_3	Size	cdd	n_1, n_2, n_3	Size	cdd
0,0,0	4	1.85	0,1,1	8	2.46	0,2,2	14	2.48
2,0,0	9	3.41	2,1,1	13	3.34	2,2,2	19	4.02
4,0,0	18	4.63	4,1,1	22	4.58	4,2,2	28	4.33

Example 5.3.

$$f(x,y) = \frac{32}{\pi^4} \sum_{m=1,2,3,\ldots} \sum_{n=1,2,3,\ldots} \frac{\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})}{mn(\frac{m^2}{a^2}+\frac{n^2}{b^2})}.$$

This example is taken from Sidi [13]. The function $f(x,y)$ is the solution to the problem

$$\begin{cases} \Delta f = -2 \ , \quad 0 < x < a \ , \ 0 < y < b \ , \\ f(0,y) = f(a,y) = f(x,0) = f(x,b) = 0 \ . \end{cases}$$

The above infinite double series converges slowly. We use the d_2-transformation to accelerate its convergence. By the form of the series' terms follows that the appropriate orders of the double difference equation should be $m = n = 2$.

There is no known explicit expression for $f(x,y)$, but it can be expressed by an infinite 1D sum [13], as follows:

$$f(x,y) = x(a-x) - \frac{8a^2}{\pi^3} \sum_{n=1,3,5,\ldots} \frac{\cosh[\frac{n\pi(2y-b)}{2a}]}{n^3\cosh(\frac{n\pi b}{2a})} \sin(\frac{n\pi x}{a}) \ ,$$

and this series converges very fast when $0 < y < b$. Using this, it can be verified that, to 15 digit accuracy, with $a = b = 2$ we have

$$f(1,1) = 0.589370826252111\ldots.$$

In Table 5.4 we present the results we have obtained, using $d^{(2,2)}_{n_1,n_2,n_3}$ with different values of n_1, n_2 and n_3, so that the size of the system is not larger than 40.

TABLE 5.4

n_1,n_2,n_3	Size	cdd	n_1,n_2,n_3	Size	cdd	n_1,n_2,n_3	Size	cdd
0,0,0	9	5.63	0,1,1	17	7.43	0,2,2	29	7.57
1,0,0	17	6.97	1,1,1	25	7.57	1,2,2	37	7.58
2,0,0	29	7.50	2,1,1	37	7.44			

In order to demonstrate the significant effect of the d_2-transformation in this case, we give some 'diagonal' partial sums of $f(1,1)$, using up to 200 terms of the original double series. Denoting the partial double sums of this series by $A_{i,j}$, we have:

$$A_{7,7} = 0.587553229\ldots \ , \quad A_{13,13} = 0.589740256\ldots \ , \quad A_{19,19} = 0.589244712\ldots \ .$$

It is evident that approximating the infinite double sum by the partial sum using 200 terms of the series is only 3 digits accurate, whereas the d_2-transformation brings us to 7-8 digit accuracy by using not more than 40 terms.

Example 5.4.

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{x^{m-1}y^{n-1}}{m^2+n^3}.$$

This example was presented in [8], where 2D versions of the t-transformation [7] were studied and compared to Chisholm approximants [1]. Sidi, in [13], also used this example for testing another class of transformations. The function $f(x,y)$ is not known analytically, and we learn about the quality of the approximations to this function by comparing to Sidi's approximations.

For fixed values of x and y, the series' terms belong to $\tilde{B}^{(1,1)}$. Hence, $d^{(1,1)}_{n_1,n_2,n_3}$ is appropriate here. Note that the given double series diverges for $|x| > 1$ or $|y| > 1$. Thus we expect to obtain better accuracy for small absolute values of x and y. Following is Table 5.5, where we present the approximations to $f(1,1)$ obtained with $d^{(1,1)}_{n_1,n_2,n_3}$, for different values of n_1, n_2, and n_3. We considered $S = 0.3149104237$, taken from [13] as the exact value.

TABLE 5.5

n_1, n_2, n_3	Size	cdd	n_1, n_2, n_3	Size	cdd
1,0,0	6	5.14	1,1,1	10	5.07
3,0,0	13	6.20	3,1,1	17	7.67
5,0,0	24	7.65	5,1,1	28	8.07
0,2,2	14	6.95	0,3,3	22	7.99
1,2,2	16	8.34	1,3,3	24	8.00

For $x = y = -0.5$ the series converges. Sidi [13] obtained the value $S = 0.38435152118 43$. In Table 5.6 we present the results using $d^{(1,1)}_{n_1,n_2,n_3}$, referring to Sidi's approximation as the correct one.

TABLE 5.6

n_1, n_2, n_3	Size	cdd	n_1, n_2, n_3	Size	cdd	n_1, n_2, n_3	Size	cdd
0,0,0	4	5.72	0,1,1	8	7.15	0,2,2	14	8.62
2,0,0	9	6.62	2,1,1	13	8.37	2,2,2	19	9.27
4,0,0	18	8.97	4,1,1	22	9.15	4,2,2	28	9.00

For $|x| > 1$ and $|y| > 1$ the series diverges. The results obtained by $d^{(1,1)}$ for $x = y = -2$ were again compared to Sidi's result, and were identical to the 5th digit.

Example 5.5. In this example we consider the power series expansion of the function

$$f(x, y) = (1 + x + 0.5y)^{-1.5}(1 + 2x + 0.2y)^{-0.5} + e^{-x-2y} ,$$

about the origin. The power series of $\tilde{f}(x, y) = (1+x+0.5y)^{-1.5}(1+2x+0.2y)^{-0.5}$ has terms which belongs to $\tilde{B}^{(1,1)}$. We examine the performance of the approximants $d^{(1,1)}$ to f, which is the sum of $\tilde{f}$ and a smooth bivariate function. The addition of the latter is expected to make a negative effect upon the performance of the transformation. f has two lines of singularity, at $1 + x + 0.5y = 0$, and at $1 + 2x + 0.2y = 0$. The point $(-\frac{3}{8}, -\frac{5}{4})$ is a multicritical point. We first look at the value of f at a point which is inside the range for which the double power series of f converges:

$$f(-0.2, -0.3) = 4.822308536399305\ldots.$$

TABLE 5.7

n_1, n_2, n_3	Size	cdd	n_1, n_2, n_3	Size	cdd	n_1, n_2, n_3	Size	cdd
1,0,0	6	2.15	1,1,1	10	4.74	1,2,2	16	5.80
3,0,0	13	4.81	3,1,1	17	5.99	3,2,2	23	7.40
5,0,0	24	7.21	5,1,1	28	7.47	5,2,2	34	7.75

For this series, the d_2-transformation works very efficiently, as is evident from Table 5.7.

Comparing the results we obtained with the partial sums of the series itself, the efficiency of the transformation is evident: the sum of the first 121 terms of the power series is

$$A_{10,10} = 4.8221925669415\ldots;$$

thus is accurate only to 5 digits, whereas the d_2-transformation is accurate to 7-8 digits using not more than 35 terms.

We have also found that the d_2-transformation is fairly efficient even beyond the radius of convergence. At the point $(x, y) = (1, 1)$,

$$f(1,1) = 0.1912084246051735\ldots ,$$

and we can still obtain an approximation which is accurate to 5-6 digits.

TABLE 5.8

n_1, n_2, n_3	Size	cdd	n_1, n_2, n_3	Size	cdd	n_1, n_2, n_3	Size	cdd
1,0,0	6	1.20	1,1,1	10	3.71	1,2,2	16	4.68
3,0,0	13	3.15	3,1,1	17	5.41	3,2,2	23	5.26
5,0,0	24	5.15	5,1,1	28	6.05	5,2,2	34	5.28

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