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POWER SERIES WITH RESTRICTED COEFFICIENTS AND A ROOT ON A GIVEN RAY

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ABSTRACT. We consider bounds on the smallest possible root with a specified argument ϕ of a power series $f(z) = 1 + \sum_{n=1}^{\infty} a_i z^i$ with coefficients a_i in the interval $[-g, g]$. We describe the form that the extremal power series must take and hence give an algorithm for computing the optimal root when $\phi/2\pi$ is rational. When $g \geq 2\sqrt{2} + 3$ we show that the smallest disc containing two roots has radius $(\sqrt{g} + 1)^{-1}$ coinciding with the smallest double real root possible for such a series. It is clear from our computations that the behaviour is more complicated for smaller g. We give a similar procedure for computing the smallest circle with a real root and a pair of conjugate roots of a given argument. We conclude by briefly discussing variants of the beta-numbers (where the defining integer sequence is generated by taking the nearest integer rather than the integer part). We show that the conjugates, λ, of these pseudo-beta-numbers either lie inside the unit circle or their reciprocals must be roots of $[-1/2, 1/2)$ power series; in particular we obtain the sharp inequality $|\lambda| \leq 3/2$.

1. INTRODUCTION

We are interested in studying the shape of the zero-free region for power series with restricted coefficients by finding the smallest root of such a power series that can lie along a specified ray.

Given a $g > 0$ we let $\mathcal{F}_g$ denote the set of $[-g, g]$ power series

$$\mathcal{F}_g := \left\{ f(z) = 1 + \sum_{i=1}^{\infty} a_i x^i \ : \ a_i \in [-g, g] \right\}.$$

For a given argument ϕ we let $\mathcal{J}_g(\phi)$ denote the set of positive real numbers α such that $\alpha e^{i\phi}$ is a root of a power series f_α in $\mathcal{F}_g$, and define $r_g(\phi)$ to be the infimum of this set. Because of symmetry $(u \mapsto \pm u, \pm \bar{u})$, we can restrict our attention to ϕ in $[0, \pi/2]$. For a general (not necessarily symmetrical) interval I we similarly use $r_I(\phi)$ to denote the smallest root with argument ϕ possible for a power series with lead coefficient one and remaining coefficients a_i in I.

Solomyak [3] has extensively studied the corresponding problem for the intervals $I = [0, 1]$ in connection with conjugates of beta-numbers; in several places we shall refer the reader to his excellent manuscript when the proof of the corresponding

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result requires only minor adaptations. We should remark that the related problem of zero-free regions for integer polynomials has been considered by Odlyzko and Poonen [2] in the case of $\{0,1\}$ coefficients and by Yamamoto [4] for norm-bounded polynomials; the regions they obtain clearly having a different, more fractal looking, appearance than ours.

We first note the following sharp bounds on $r_g(\phi)$:

Theorem 1. *For all $g > 0$ and ϕ in $[0, \pi/2]$*

$$\frac{1}{g+1} \leq r_g(\phi) \leq \frac{1}{\sqrt{g+1}}$$

with equality achieved for $\phi = 0$ and $\pi/2$ respectively.

Of course the angle $\phi = 0$ should really be regarded quite separately from the remaining arguments $(0, \pi/2]$ (since we are dealing with real power series and vanishing at a u thus entails vanishing at $\bar{u}$ it is readily seen that $r_g(\phi)$ tends to a real double root and not $r_g(0)$). Hence omitting zero we might hope to improve the lower bound slightly. For $g > 1$ this is certainly true:

Theorem 2. *For all ϕ in $(0, \pi)$*

$$r_g(\phi) > \frac{1}{\sqrt{g}+1}.$$

For $g \geq 2\sqrt{2}+3$

$$\lim_{\phi \to 0} r_g(\phi) = \frac{1}{\sqrt{g}+1},$$

where $(\sqrt{g}+1)^{-1}$ is the smallest double root of a power series in $\mathcal{F}_g$; namely

$$f(x) = 1 - (2\sqrt{g}+1)x + g\sum_{i=2}^{\infty} x^i.$$

More precisely, for $g \geq 2\sqrt{2}+3$ and ϕ in $(0, \pi)$, $r_g(\phi)$ is the positive real root of

$$1 - g\sum_{i=2}^{\infty} \left| \frac{\sin(i-1)\phi}{\sin \phi} \right| z^i = 0.$$

For $g < 2\sqrt{2}+3$ the location of the smallest value $r_g(\phi)$ (and hence the radius of the smallest disc containing two roots of a power series in $\mathcal{F}_g$) will generally occur away from zero and seems much harder to determine.

A standard compactness argument shows that the infimum is always achieved. We next show that the series for the minimal root must take a very specific form.

Theorem 3. *For a given $g > 0$ and argument ϕ, there exists a unique β in $\mathcal{J}_g(\phi)$ such that, for some θ in $(0, \pi/2)$, the coefficients of the corresponding power series $1 + \sum_{j=1}^{\infty} b_j x^j$ in $\mathcal{F}_g$ satisfy*

$$b_j = \begin{cases} g & \text{if } j\phi - \theta \in (0, \pi) \pmod{2\pi}, \\ -g & \text{if } j\phi - \theta \in (-\pi, 0) \pmod{2\pi}. \end{cases}$$

Moreover

$$\beta = r_g(\phi)$$

and the coefficients of any additional power series $\tilde{f}_\beta$ in $\mathcal{F}_g$ with a root at $\beta e^{i\phi}$ must be of this form (taking the same θ).

Notice that if $\phi/2\pi$ is irrational, then the corresponding series for $r_g(\phi)e^{i\phi}$ in $\mathcal{F}_g$ is certainly unique (with at most one coefficient, namely a_J where J must satisfy $J\phi = \theta \pmod{\pi}$, not taking the value of an end point $\pm g$). If $\phi/2\pi = t/s$ is rational, then (by the discreteness of the arguments $n_i\phi$) we can assume that $\theta = J\phi \pmod{\pi}$ for some J and the series will not be unique unless all the remaining $a_{J+js} = a_J = \pm g$ (and $a_{J+s/2+js} = -a_J$ if k is even).

However in the rational case $\phi = 2\pi t/s$, setting $l = s$ if s is odd and $s/2$ if s is even, we observe that by setting

$$A_j := \left(\sum_{i=0}^{\infty} a_{j+is} r_\phi^{is}\right)(1 - r_\phi^s), \qquad j = 1, \dots s, \ s \text{ odd},$$

$$A_j := \left(\sum_{i=0}^{\infty} a_{j+is} r_\phi^{is} - a_{j+s/2+is} r_\phi^{is+s/2}\right)(1 + r_\phi^{s/2}), \qquad j = 1, \dots s/2, \ s \text{ even},$$

we can replace any optimal series $1 + \sum a_i x^i$ by a series $1 + \sum A_i x^i$ in $\mathcal{F}_g$ where the A_i are periodic with period s and $A_{j+s/2} = -A_j$ if s is even. Notice that in the rational case we can therefore (on multiplying the series by $(1 + (-1)^s x^l)$) replace the infinite series by finite polynomials of the form

$$p(x) := 1 + (-1)^s (g+1) x^l + \sum_{i=1}^{l-1} A_i x^i. \tag{1}$$

Henceforth we shall regard the periodic extremal series as being the canonical form when $\phi/2\pi$ is rational, and will use $A_j(\phi)$ to denote its coefficients, and the coefficients of the unique optimal series when $\phi/2\pi$ is irrational.

We are often forced to single out a set of awkward angles ϕ;

$$\mathcal{U}_g := \{\phi : A_j(\phi) = \pm g \text{ for all } j\},$$

including those ϕ with rational $\phi/2\pi$ and a unique series. Although for a given g it is difficult to decide whether there are rational $\phi/2\pi$ with ϕ in $\mathcal{U}_g$, they certainly can occur. For example when $g = \sqrt{3}$ the point $\pi/4$ is in $\mathcal{U}_{\sqrt{3}}$ with unique extremal sequence $1 - \sqrt{3}(x - x^2 - x^3 - x^4)/(1 + x^4)$; indeed it appears from Figure 3 (see the end of the paper) that $r_{\sqrt{3}}(\pi/4) = (\sqrt{3} - 1)\sqrt{2}/2$ may well be the minimum.

For ϕ not in $\mathcal{U}_g$ we shall define

$$J(\phi) := \min\{j : A_j(\phi) \neq \pm g\}.$$

For ϕ in $\mathcal{U}_g$ we can similarly define $J(\phi)$ to be the smallest j such that $\theta \equiv j\phi \pmod{2\pi}$ can be taken as the argument of the dividing line in Theorem 3; where $J(\phi)$ is potentially ∞ for some irrationals.

2. Other intervals and polynomial versions

Although we have concentrated upon fixed symmetric intervals many of the results can be easily extended to a broader class of power series (with varying intervals) and to polynomials (constructed from a given set of exponents):

Given a set S of exponents $\mathcal{E}_S = \{0 < n_1 < n_2 < \cdots\}$ and intervals $I_i = [u_i, v_i]$ each containing zero, we consider the power series

$$\mathcal{F}_S := \left\{ f(z) = 1 + \sum_{n_i \in \mathcal{E}_S} a_i z^{n_i} \ : \ a_i \in [u_i, v_i] \right\},$$

the set $\mathcal{J}_S(\phi)$ of positive real α for which there is a root $\alpha e^{i\phi}$ of a power series g_α in $\mathcal{F}_S$ and define $r_S(\phi)$ to be the infimum of this set. Solomyak [3] considers in detail the case when the intervals are all $[0,1]$. We prove the following generalisation of Theorem 3, in particular recovering Solomyak's Structure Theorem 3.3 for the intervals $[0,1]$:

Theorem 4. *If $\mathcal{J}_S(\phi) \neq \emptyset$, then there exists a unique β in $\mathcal{J}_S(\phi)$ for which there is a θ in $(0,\pi/2)$ such that the coefficients of the corresponding power series $1+\sum_{j=1}^{\infty} b_j x^{n_j}$ in $\mathcal{F}_S$ satisfy*

$$b_j = \begin{cases} v_i & \text{if } n_j\phi - \theta \in (0,\pi) \pmod{2\pi}, \\ u_i & \text{if } n_j\phi - \theta \in (-\pi,0) \pmod{2\pi}. \end{cases}$$

Moreover

$$\beta = r_S(\phi)$$

and any additional power series $\tilde{f}_\beta$ in $\mathcal{F}_S$ with a root at $\beta e^{i\phi}$ must be of this form.

3. The computations

When $\phi/2\pi = t/s$ is rational, $0 < \phi < \pi$, Theorem 3 and the polynomial form (1) provide a method for computing $r_g(\phi)$. Appealing to Theorem 4 we actually give here the algorithm to find $r_I(\phi)$, the smallest root with argument ϕ of a power series having lead coefficient one and remaining coefficients in I, for any fixed interval $I := [m-g, m+g]$ containing zero. In this more general setting we can still assume that the coefficients of the extremal power series are periodic, $a_{i+js} = A_i$, with $(A_j - m) = -(A_{j+s/2} - m)$ if s is even.

For a trial $1 \le J \le l$ one assigns coefficients

$$A_i := m - g \operatorname{sign}\left(\frac{\sin(J-i)\phi}{\sin J\phi}\right), \quad 1 \le i \le l-1,\ i \neq J,$$

and, using the vanishing of the real and imaginary parts at $re^{i\phi}$ to eliminate A_J (as in the proof of Theorem 4), solves the resulting equation

$$(1+g-m) + \sum_{i=0}^{l-1}\left(\frac{m}{(1+(-1)^s x^l)}\left(\frac{\sin(J-i)\phi}{\sin J\phi}\right) - \frac{g}{(1-x^l)}\left|\frac{\sin(J-i)\phi}{\sin J\phi}\right|\right)x^i = 0,$$

for a root $0 < r < 1$; increasing J until one reaches an r that yields

$$A_J - m = -\frac{m(1-r^l)r^{1-J}}{(1-2r\cos\phi + r^2)}\left(\frac{\sin\phi}{\sin J\phi}\right) + g\sum_{\substack{i=1 \\ i\neq J}}^{l-1} r^{i-J}\operatorname{sign}\left(\frac{\sin(J-i)\phi}{\sin J\phi}\right)\left(\frac{\sin i\phi}{\sin J\phi}\right)$$

with $|A_J - m| \le g$, and hence $r = r_I(\phi)$.

The graphs at the end of the paper illustrate the results for the symmetrical intervals $m=0$, $g = 1/2, 1, \sqrt{3}, 2\sqrt{2}+3$ and the corresponding values of $J(\phi)$ and $A_{J(\phi)}$ for $g=1$. For $g=1$ the smallest value we encountered was $r_1(203\pi/684) = .63560642...$, the precise minimum (giving the radius of the smallest disc containing two roots of a $[-1,1]$ power series) appearing to lie between $203\pi/684$ and $249\pi/839$. Figures 7 and 8 show $r_I(\phi)$ for the one-sided intervals $I=[0,1]$ and $[-1,0]$. For $[0,1]$ the smallest value we found was $r_I(229\pi/310) = .73295789...$ (the minimum apparently lying between $229\pi/310$ and $376\pi/509$). For $[-1,0]$ the minimum appears to be $r_I(2\pi/3) = (1/2)^{1/3}$ corresponding to the series $1 - \sum_{j=1}^{\infty} x^{3j}$

(this latter value is not the radius of the smallest disc with two roots, $1-\sum_{j=1}^{\infty} x^{2j}$ giving $(1/2)^{1/2}$, but is potentially the smallest with three roots).

4. Some general properties

Although $r_g(0)$ is necessarily a discontinuity we do otherwise have continuity in the value of $r_g(\phi)$. The curve however is certainly not smooth, with maxima at all the ϕ in $\pi\mathbf{Q}\setminus\mathcal{U}_g$. We state several such properties below (the proofs of Proposition 1 are readily reconstructed from the corresponding $[0,1]$ results of Solomyak [3] and hence are omitted):

Proposition 1. (i) *The function* $\phi\mapsto r_g(\phi)$ *is continuous on* $(0,\pi)$.
(ii) *If* ϕ *is in* $\pi\mathbf{Q}\setminus\mathcal{U}_g$, *then* $r_g(\phi)$ *is not smooth, more precisely there is a sector with vertex* $r_g(\phi)e^{i\phi}$ *and angle greater than* π *outside of the curve.*
(iii) *If* ϕ/π *is irrational, not in* $\mathcal{U}_g$, *and not a Liouville number, then the curve has a tangent at the point* $r_g(\phi)e^{i\phi}$.

Away from $\mathcal{U}_g$ the behaviour of $J(\phi)$ is quite predictable:

Proposition 2. (i) *If* ϕ *is not in* $\mathcal{U}_g$ *and* $\phi/2\pi$ *is irrational, then for sufficiently small* $\delta=\delta(\phi,g)>0$

$$J(\theta)=J(\phi)\qquad\forall\theta\in(\phi-\delta,\phi+\delta).$$

(ii) *If* $\phi/2\pi=t/s$ *is rational, set* $l=s$ *or* $s/2$ *as* s *is odd or even, and define (if possible) non-negative integers* n,m *such that*

$$\operatorname{sign}(\sin J(\phi)\phi)A_{J(\phi)}(\phi)\in g\left(1-2r_g(\phi)^{ml},1-2r_g(\phi)^{(m+1)l}\right)$$

or

$$-\operatorname{sign}(\sin J(\phi)\phi)A_{J(\phi)}(\phi)\in g\left(1-2r_g(\phi)^{nl},1-2r_g(\phi)^{(n+1)l}\right).$$

Then for a suitably small $\delta=\delta(\phi,g)>0$

$$J(\theta)=\begin{cases} J(\phi)+ml & \forall\theta\in(\phi,\phi+\delta),\\ J(\phi)+nl & \forall\theta\in(\phi-\delta,\phi).\end{cases}$$

Conversely, suppose that there is an interval $(\phi,\phi+\delta)$ *(respectively* $(\phi-\delta,\phi)$*) with* $J(\theta)=J$ *constant on the interval and set*

$$A^+=\lim_{\theta\to\phi^+}A_J(\phi)\qquad(\textit{respectively } A^-=\lim_{\theta\to\phi^-}A_J(\phi)).$$

(a) *If* $\phi/2\pi$ *is irrational, then* $J(\phi)=J$ *and* $A_J(\phi)=A^+$ *(respectively* A^-*).*
(b) *If* $\phi/2\pi$ *is rational (with* $l=s$ *or* $s/2$ *as* s *is odd or even), then* $J(\phi)=J_1<l$ *where* $J=J_1+tl$ *and*

$$A_{J_1}(\phi)=A^+r^{tl}(1-r^l)-g\ \operatorname{sign}(\sin J_1\phi)(1-r^{tl}-r^{(t+1)l})$$

(respectively

$$A_{J_1}(\phi)=A^-r^{tl}(1-r^l)+g\ \operatorname{sign}(\sin J_1\phi)(1-r^{tl}-r^{(t+1)l})\Big),$$

where $r=r_g(\phi)=\lim_{\theta\to\phi}r_g(\theta)$.

Notice that (even if the value of $J(\theta)$ remains constant in an interval about ϕ) the value of $A_{J(\theta)}(\theta)$ is discontinuous at rational ϕ; observe also that the value of $J(\theta)$ away from elements in $\mathcal{U}_g$ remains constant around rationals if $A_J(\phi)$ is in $(-g(1-2r_g(\phi)^l), g(1-2r_g(\phi)^l))$ (i.e. far enough away from the endpoint $\pm g$ to absorb the necessary discontinuity). These discontinuities at the rationals are clearly visible in Figure 5 together with the necessary jump from $J=1$ to 4 at $\pi/3$.

Theorem 2 relies on the fact that $J(\phi)=1$ for all $\phi>0$ if $g \geq 2\sqrt{2}+3$. For $g<2\sqrt{2}+3$ we must certainly have at least two different values of J (since $J(\phi)\geq 2$ as $\phi \to 0$ and $J(\phi)=1$ as $\phi \to \pi/2$); in fact it seems plausible that J is always unbounded when $g \leq 2\sqrt{2}+3$.

Notice that for rational $\phi/2\pi$ in addition to elements in $\mathcal{U}_g$ we are forced also to avoid ϕ with $A_J(\phi)=\pm(1-2r_g(\phi)^{ln})$ for a non-negative integer n (for which potentially $\lim_{\theta\to\phi^\pm} A_{J(\theta)}(\theta)=\pm g$). Similarly observe that if ϕ is in $\mathcal{U}_g$ and $\phi/2\pi$ rational, then ϕ cannot be the end point of interval with constant J (since (b) forces $|A_{J_1}(\phi)|<g$).

5. Three roots on a circle

For a given argument ϕ Theorem 3 gave us a way to characterize the radius $r_g(\phi)$ of the smallest circle containing two roots $re^{i\phi}$,$re^{-i\phi}$. One natural extension would to be to ask for the radius of the smallest circle $\tilde{r}_g(\phi)$ containing the three roots $r, re^{i\phi}, re^{-i\phi}$. In this case there is a Structure Theorem resembling Theorem 3:

Theorem 5. *For a given $g>0$ and angle ϕ in $(0,\pi)$ there is a unique $\beta>0$ with $\beta, \beta e^{i\phi}, \beta e^{-i\phi}$ all roots of a power series*

$$f_\beta(x) = 1 + \sum_{i=1}^{\infty} b_i x^i$$

in $\mathcal{F}_g$ such that for two arguments $\theta_1 \leq \theta_2$ the coefficients of f_β satisfy

$$b_j := \begin{cases} -g & \text{if } j\phi \in (\theta_2 - 2\pi, \theta_1) \pmod{2\pi}, \\ g & \text{if } j\phi \in (\theta_1, \theta_2) \pmod{2\pi}. \end{cases}$$

Moreover

$$\beta = \tilde{r}_g(\phi)$$

and any series f_β must be of this form.

It is not hard to see that a Theorem 4 style generalisation holds for power series with coefficients b_j lying in varying intervals $[u_i, v_i]$ containing zero.

Now in the rational case $\phi/2\pi = t/s$, $s \geq 3$, we can once again reduce to a periodic series, hence reduce to a polynomial

$$\tilde{p}(x,\phi) := 1 - (1+g)x^s + \sum_{i=1}^{s-1} A_i x^i,$$

and again obtain an algorithm for computing $\tilde{r}_g(\phi)$:

For a trial pair of integers $1 \leq I < J < s$ one solves the polynomial

$$1 - (g+1)x^s - g\sum_{j=1}^{s-1} \left| \frac{\sin\frac{1}{2}(I-j)\phi \sin\frac{1}{2}(J-j)\phi}{\sin\frac{1}{2}I\phi \sin\frac{1}{2}J\phi} \right| x^j$$

(that is the polynomial that would result if $I\phi$ and $J\phi$ gave the correct θ_1 and θ_2 (mod 2π) and we eliminated A_I, A_J from the equations $\tilde{p}(r,\phi)=0$, $\tilde{p}(re^{i\phi},\phi)=0$) for the real root $x=r$ ($0 < r < 1$), changing I, J until one reaches a pair that, with

$$A_j := -g \operatorname{sign}\left(\frac{\sin(\frac{1}{2}(I-j)\phi)\sin(\frac{1}{2}(J-j)\phi)}{\sin\frac{1}{2}I\phi\sin\frac{1}{2}J\phi}\right), \quad j \neq I, J,$$

yields "missing coefficients" $|A_I|, |A_J| \leq g$, where

$$\begin{aligned} A_I &= \frac{\sin J\phi - \sin 0}{\sin J\phi - \sin I\phi}((g+1)r^s - 1)r^{-I} - \sum_{\substack{j=1\\ j\neq I,J}}^{s-1} A_j \frac{\sin J\phi - \sin j\phi}{\sin J\phi - \sin I\phi} r^{j-I}, \\ A_J &= \frac{\sin I\phi - \sin 0}{\sin I\phi - \sin J\phi}((g+1)r^s - 1)r^{-J} - \sum_{\substack{j=1\\ j\neq I,J}}^{s-1} A_j \frac{\sin I\phi - \sin j\phi}{\sin I\phi - \sin J\phi} r^{j-J}, \end{aligned}$$

when $\sin(I\phi) \neq \sin(J\phi)$ (if $\sin(I\phi) = \sin(J\phi)$ one simply replaces the sines by cosines in the formulae for A_I and A_J).

Figure 9 shows $\tilde{r}_g(\phi)$ for $g=1$, Figure 10 showing the successful values of I, J against ϕ in this case. The smallest value we encountered was $\tilde{r}_1(221\pi/497) = .71615109...$, the minimum lying between $\phi = 4\pi/9$ and $221\pi/497$.

Clearly power series with coefficients in totally positive intervals (such as $[0,1]$) can have no positive real roots. Similarly for totally negative intervals it becomes uninteresting to ask for roots r and $re^{i\phi}$ with $r>0$. For example a power series with lead coefficient one and remaining coefficients in $[-1,0]$ cannot have such a pair of roots when $\phi/2\pi$ is irrational, and when $\phi/2\pi = t/s$ is rational the smallest r is simply $(1/2)^{1/s}$ with extremal series $1-\sum_{j=1}^{\infty} x^{sj}$ (since the equations resulting from vanishing at r and $re^{i\phi}$ clearly require $\cos(n\phi)=1$ for any non-zero coefficients a_n in the series). Hence for a general interval I it is perhaps more natural to define $\tilde{r}_I(\phi)$ to be the smallest r such that there is a power series with coefficients in I and three roots $-r$, $-re^{i\phi}, -re^{-i\phi}$. We give the corresponding curves for $I=[0,1]$ and $[-1,0]$ in Figures 11 and 12. The smallest value found for $[0,1]$ was $\tilde{r}_I(7\pi/58) = .79794300...$, the minimum lying between $7\pi/58$ and $108\pi/895$. For $[-1,0]$ the minimum appears to be $\tilde{r}_I(\pi/2) = (1/2)^{1/4}$ from $1-\sum_{j=1}^{\infty} x^{4j}$.

Unfortunately it is no longer clear that this approach necessarily leads us to the smallest disc containing three roots or what is the correct extension of this to four or more roots. Concerning $R_g(k)$, the radius of the smallest disc containing k roots of a $[-g,g]$ power series, one may obtain the following bounds

$$\left(1+\frac{1}{k}\right)^{-1/2} \frac{1}{(g^2k+1)^{1/2k}} \leq R_g(k) \leq \frac{1}{(g+1)^{1/k}};$$

the lower bound a consequence of Jensen's Theorem (see [1]), the upper bound arising from the power series $1-g\sum_{j=1}^{\infty} x^{jk}$.

Alternatively one could ask for $r_g(\phi,k)$ the smallest value of α such that $\alpha e^{i\phi}$ is a k-fold root of a series in $\mathcal{F}_g$. In [1] we gave a procedure for computing $r_g(k,0)$ and it is clear that $r_g(k,\phi) \geq r_g(k,0)$. It is also easy to see that $r_g(k,\pi/2) = \sqrt{r_g(k,0)}$, but it is not clear whether $\pi/2$ remains the worst argument as when $k=1$.

6. Pseudo-beta-numbers, an application

For any real δ we can define a variant of the integer part function

$$[x]_\delta := \mathbf{Z} \cap (x-(1-\delta), x+\delta]$$

and for a real θ and δ a mapping $T_\theta(x,\delta) : \mathbf{R} \to [-\delta, 1-\delta)$ by

$$T_\theta(x,\delta) : x \mapsto \theta x - [\theta x]_\delta.$$

For a given δ and $\theta > 1$ we can define a sequence of integers $d = d(\theta,\delta) = (d_i)_{i=1}^\infty$ by

$$d_j := \left[\theta T_\theta^{j-1}(1,\delta)\right]_\delta.$$

Writing

$$\theta = \sum_{i=1}^{k} d_i \theta^{-(i-1)} + T_\theta^k(1,\delta)\theta^{-(k-1)}$$

this sequence can be thought of as giving us a "δ-θ-expansion of 1":

$$1 = \sum_{j=1}^{\infty} d_j \theta^{-j}.$$

We shall call the number θ a δ-beta-number (respectively a simple δ-beta-number) if the sequence d is eventually periodic (respectively finite). The most natural cases to consider are of course $\delta = 0$ (the traditional beta-numbers) and $\delta = 1/2$ (the analogues where one takes the nearest integer rather than the integer part).

Notice that if $d = d_1 \dots d_k \overline{d_{k+1} \dots d_{k+m}}$, then $1/\theta$ is a root of

$$1 - \sum_{j=1}^{\infty} d_j z^j = 1 - \sum_{j=1}^{k} d_j z^j - (1-z^m)^{-1} \sum_{j=k+1}^{k+m} z^j$$

and so an algebraic integer, all of whose conjugates $1/\lambda$ with $|\lambda| > 1$ must also be roots of

$$1 - \sum_{j=1}^{\infty} d_j z^j = (1-\theta z)\left(1 + \sum_{j=1}^{\infty} T_\theta^j(1,\delta) z^j\right)$$

and hence roots of power series with coefficients $T_\theta^j(1,\delta)$ in $[-\delta, 1-\delta)$. Thus for $\delta = 1/2$ the value of $r_{1/2}(\phi)$ illustrated in Figure 1 yields a bound $|\lambda| \le r_{1/2}(\phi)^{-1}$ for any conjugates λ of a 1/2-beta-number having argument ϕ. Solomyak [3] has shown for $\delta = 0$ that the set of zeroes of $[0,1]$-power series is in fact exactly the closure of the set of reciprocals of the conjugates of the standard beta-numbers. It is not clear to what extent this remains true for these more general pseudo-beta-numbers. However we still certainly obtain upper bounds on the conjugates from studying the roots of power series with appropriately restricted coefficients:

Theorem 6. *If λ is a conjugate of a δ-beta-number, with $0 \le \delta < 1$, then*

$$|\lambda| \le \begin{cases} 1+\delta & \text{if } 1/2 < \delta < 1, \\ \frac{1}{2}(1-\delta+\sqrt{5+2\delta+\delta^2}) & \text{if } 0 \le \delta \le 1/2. \end{cases}$$

Further, this inequality is best possible for $0 \le \delta \le 1/2$.

7. The proofs

Proof of Theorem 1. The lower bound is trivial:

If α is a root of a series in $\mathcal{F}_g$ then

$$1 = \left|\sum_{i=1}^{\infty} a_i \alpha^i\right| \le g \sum_{i=1}^{\infty} |\alpha|^i = \frac{g|\alpha|}{1-|\alpha|}.$$

For the upper bound we show that for any angle ϕ there is a power series in $\mathcal{F}_g$ with a root at $(g+1)^{-1/2}e^{i\phi}$.

For a given argument ϕ in $[0, \pi/2]$ we define

$$\alpha = \alpha(\phi) := \arccos\left(\frac{\sqrt{1+g}}{1+\frac{1}{2}g}\cos\phi\right)$$

and set

$$h_g(z,\phi) := 1 - g\sum_{j=1}^{\infty}\cos(j\alpha)z^j \in \mathcal{F}_g.$$

Now for $|z| < 1$ we can write

$$h_g(z,\phi) = 1 - \frac{1}{2}g\sum_{j=1}^{\infty}\left((ze^{i\alpha})^j + (ze^{-i\alpha})^j\right) = \frac{(1+g)z^2 - (2+g)(\cos\alpha)z + 1}{z^2 - 2(\cos\alpha)z + 1},$$

and $h_g(z,\phi)$ plainly has the required zero at $z = (1+g)^{-1/2}e^{i\phi}$.

To see that the upper bound cannot be improved at $\pi/2$ observe that vanishing of the real part of a series in $\mathcal{F}_g$ at ir amounts to r being a root of a power series $f(z^2)$ with f in $\mathcal{F}_g$. □

We postpone the proof of Theorem 2 until after the proof of Theorem 4. Theorem 3 is a special case of Theorem 4.

Proof of Theorem 4. We first show that such a configuration of coefficients would lead to the extremal $r_S(\phi)$:

If $\alpha = re^{i\phi}$ is a root of $g_\alpha(z) = 1 + \sum_{n_i\in\mathcal{E}_S} a_i z^{n_i}$, then separating real and imaginary parts, we have

$$1 + \sum_{n_i\in\mathcal{E}_S} a_i\cos(n_i\phi)r^{n_i} = 0, \qquad \sum_{n_i\in\mathcal{E}_S} a_i\sin(n_i\phi)r^{n_i} = 0.$$

For $\sin\theta \neq 0$ we set

$$S_\theta := \{j : n_j\phi = \theta (\mathrm{mod}\ \pi)\}$$

and use the second equation to eliminate any $a_j, j \in S_\theta$, from the first;

$$1 + \sum_{i\notin S_\theta}^{\infty} a_i\frac{\sin(\theta - n_i\phi)}{\sin\theta}r^{n_i} = 0.$$

Clearly then if

$$T_1 = \{i : n_i\phi - \theta \in (0,\pi)\}, \qquad T_2 = \{i : n_i\phi - \theta \in (-\pi, 0)\}$$

r can be no smaller than the smallest positive real root of

$$1 - \sum_{i\in T_1}\left|v_i\frac{\sin(\theta - n_i\phi)}{\sin J\phi}\right|x^{n_i} - \sum_{n_i\in T_2}\left|u_i\frac{\sin(\theta - n_i\phi)}{\sin J\phi}\right|x^{n_i} = 0$$

and that this is plainly achieved for a configuration of the type given in the statement of the theorem and for no other. Here $J\phi = \theta (\text{mod } \pi)$.

It remains to show that the extremal polynomial achieves this form (this is in fact already clear if there is an $n_I \in S_J$ with $a_{n_I} \neq u_I$ or v_I; since if there was an a_{n_L}, $n_L \notin S_J$, not in the claimed optimal position we could perturb a_{n_L} very slightly to reduce r at the cost of a new a_{n_I} still within the required interval).

We shall need a couple of lemmas:

Lemma 1. *We suppose that $g(x) = 1 + \sum_{n_i \in \mathcal{E}_S} a_i x^{n_i}$ in $\mathcal{F}_S$ is a power series with a root at $w = re^{i\phi}$ with r minimal, then $g(x)$ has at least one non-zero coefficient at an endpoint $a_i = u_i$ or v_i.*

Proof. Suppose that $g(x)$ has all its non-zero a_i in (u_i, v_i). Note that $g(x)$ cannot be a polynomial; otherwise for some suitably small $A > 1$, $g(Az)$ would still be in $\mathcal{F}_S$ contradicting the minimality of the root. If $g(x)$ has infinitely many terms we let a_I, a_J denote the first non-zero coefficients with $\sin(n_I - n_J)\phi \neq 0$. Such coefficients must exist, since if $\sin(n_i - n_j)\phi = 0$ for all the non-zero a_i, a_j then (for the real part of f to vanish) $\cos n_i\phi = \pm 1$ for all i and we can construct a power series

$$\tilde{g}(x) := 1 + \sum_{\cos n_i\phi = 1} u_i x^{n_i} + \sum_{\cos n_i\phi = -1} v_i x^{n_i}$$

with a smaller root.

Hence for any $u = Re^{i\lambda}$ setting

$$\beta_I(u) := \frac{R\sin(\lambda - n_J\phi)}{r^{n_I}\sin(n_I - n_J)\phi}, \qquad \beta_J(u) := \frac{R\sin(n_I\phi - \lambda)}{r^{n_J}\sin(n_I - n_J)\phi}$$

we have

$$\beta_I w^{n_I} + \beta_J w^{n_J} = u.$$

In particular we can take

$$u := \sum_{i=N}^{\infty} a_i w^{n_i}$$

with N so large that the corresponding $\beta_j(u) < \min\{a_j - u_j, v_j - a_j\}$ for $j = I, J$ and

$$\tilde{g}(x) := \sum_{i<N, i\neq I,J} a_i x^{n_i} + (a_I + \beta_I)x^{n_I} + (a_J + \beta_J)x^{n_J}$$

is now a polynomial in $\mathcal{F}_S$ with a root at w and all its non-zero coefficients in (u_i, v_i) (in contradiction to the above). □

Lemma 2. *We suppose that $f(x) = 1 + \sum_{n_i \in \mathcal{E}_S} a_i x^{n_i}$ in $\mathcal{F}_S$ is a power series with a root at the minimal $w = re^{i\phi}$.*

(i) *If $\phi/2\pi = r/s$ is rational, then*

$$a_i = \begin{cases} u_i & \text{if } n_i = js \text{ for some } j, \\ v_i & \text{if } n_i = js + s/2 \text{ for some } j \text{ if } s \text{ is even.} \end{cases}$$

(ii) *If $a_J \in (u_J, v_J)$, then*

$$a_i = \begin{cases} v_i & \text{if } (n_i - n_J)\phi \in (0,\pi) \text{ and } n_J\phi \in (0,\pi) \\ & \text{or } (n_i - n_J)\phi \in (-\pi, 0) \text{ and } n_J\phi \in (-\pi,0) \ (\text{mod } 2\pi), \\ u_i & \text{if } (n_i - n_J)\phi \in (-\pi,0) \text{ and } n_J\phi \in (0,\pi) \\ & \text{or } (n_i - n_J)\phi \in (0,\pi) \text{ and } n_J\phi \in (-\pi,0) (\text{mod } 2\pi). \end{cases}$$

(iii) *If* $a_J = v_j$, *then*

$$a_i = \begin{cases} v_i & \text{if } n_i\phi \in (n_J\phi, \pi) \text{ and } n_J\phi \in (0,\pi) \\ & \text{or } n_i\phi \in (-\pi, n_J\phi) \text{ and } n_J\phi \in (-\pi, 0) \pmod{2\pi}, \\ u_i & \text{if } n_i\phi \in (-n_J\phi, 0) \text{ and } n_J\phi \in (0,\pi) \\ & \text{or } n_i\phi \in (0, -n_J\phi) \text{ and } n_J\phi \in (-\pi, 0) \pmod{2\pi}. \end{cases}$$

(iv) *If* $a_J = u_j$, *then*

$$a_i = \begin{cases} v_i & \text{if } n_i\phi \in (-\pi, n_J\phi - \pi) \text{ and } n_J\phi \in (0,\pi) \\ & \text{or } n_i\phi \in (-n_J\phi, \pi) \text{ and } n_J\phi \in (-\pi, 0) \pmod{2\pi}, \\ u_i & \text{if } n_i\phi \in (0, n_J\phi) \text{ and } n_J\phi \in (0,\pi) \\ & \text{or } n_i\phi \in (n_J\phi, 0) \text{ and } n_J\phi \in (-\pi, 0) \pmod{2\pi}. \end{cases}$$

Proof. (i) If there exists an $n_I = js$ with $a_I \neq u_I$, then there is a small positive δ, $0 < \delta < (a_I - u_I)r^{n_I}$, such that

$$\tilde{g}(x) := 1 + \sum_{n_i \in \mathcal{E}_S \setminus \{n_I\}} (1-\delta)a_i x^{n_i} + ((1-\delta)a_I - \delta r^{-n_I})x^{n_I}$$

is a series in $\mathcal{F}_S$ vanishing at α but with all its non-zero coefficients a_i strictly inside the intervals (u_i, v_i) in contradiction to Lemma 1.

Similarly if there is an $n_I = js + s/2$ with $a_I \neq v_I$, then for $0 < \delta < (v_I - a_I)r^{n_I}$ we form

$$\tilde{g}(x) = 1 + \sum_{n_i \in \mathcal{E}_S \setminus \{n_I\}} (1-\delta)a_i x^{n_i} + ((1-\delta)a_I + \delta r^{-n_I})x^{n_I}.$$

(ii)–(iv) Given two I, J with $(n_I - n_J)\phi \notin \pi\mathbf{Z}$ it is readily seen that there exist

$$\alpha_I := -\frac{1}{r^{n_I}} \frac{\sin n_J\phi}{\sin(n_I - n_J)\phi}, \quad \alpha_J := \frac{1}{r^{n_J}} \frac{\sin n_I\phi}{\sin(n_I - n_J)\phi}$$

such that

$$\alpha_I w^{n_I} + \alpha_J w^{n_J} = 1.$$

Hence for any $0 < \delta < 1$ the series

$$\tilde{g}(z) := \sum_{n_i \in \mathcal{E}_S \setminus \{n_I, n_J\}} (1-\delta)a_i z^{n_i} + ((1-\delta)a_I - \delta\alpha_I)z^{n_I} + ((1-\delta)a_J - \delta\alpha_J)z^{n_J}$$

will certainly have a root at α and we shall gain the by now familiar contradiction to Lemma 1 if (for a suitably small positive δ) we can make an adjustment that puts

$$((1-\delta)a_I - \delta\alpha_I)z^{n_I} \in (u_I, v_I), \quad ((1-\delta)a_J - \delta\alpha_J)z^{n_J} \in (u_J, v_J).$$

That is, if $a_J \in (u_J, v_J)$, or if $a_J = v_J$ and $\alpha_J > 0$, or if $a_J = u_J$ and $\alpha_J < 0$ we cannot have

$$\alpha_I > 0 \text{ and } a_I \neq u_I \quad \text{or} \quad \alpha_I < 0 \text{ and } a_I \neq v_I.$$

The rest is just a matter of checking the signs of $\sin n_J\phi$, $\sin n_I\phi$ and $\sin(n_I - n_J)\phi$. □

Theorem 4 readily follows from the latter lemma with the angle θ marking the line of transition from angles $n_j\phi$ with $a_j = u_j$ and those with $a_j = v_j$. □

Proof of Theorem 2. From the inequality

$$\left|\frac{\sin m\phi}{\sin\phi}\right| = \left|\frac{e^{im\phi}-e^{-im\phi}}{e^{i\phi}-e^{-i\phi}}\right| = \left|\sum_{j=0}^{m-1} e^{(2j\phi)i}\right| < m$$

for ϕ in $(0,\pi/2)$, it is clear that any real root $0<r<1$ of

$$1-g\sum_{i=2}^{\infty}\left|\frac{\sin(i-1)\phi}{\sin\phi}\right| x^i = 0 \tag{2}$$

must satisfy

$$1 = g\sum_{i=2}^{\infty}\left|\frac{\sin(i-1)\phi}{\sin\phi}\right| r^i \le g\sum_{i=2}^{\infty}(i-1)r^i = g\frac{r^2}{(1-r)^2}$$

and hence

$$r > \frac{1}{\sqrt{g}+1}. \tag{3}$$

Now if $J(\phi)=1$ (that is one can take $\theta=\phi$ in Theorem 3) then, setting the coefficients of the extremal series to satisfy $A_i := -g\ \mathrm{sign}(\sin(1-i)\phi/\sin\phi)$ for $(i-1)\phi \not\equiv 0 \pmod{\pi}$ and, in the manner of the proof of Theorem 4, using the vanishing of the real and imaginary parts of the power series to eliminate the remaining coefficients, we obtain an equation of the form (2) and $r = r_g(\phi)$. If $J(\phi)\ne 1$, then r still provides a lower bound for $r_g(\phi)$ (since, by Theorem 4, r becomes extremal if we weaken the problem by allowing the coefficient A_1 to lie in some suitably larger interval).

To show that $J(\phi)=1$ when $g\ge 2\sqrt{2}+3$ it remains only to check that the value of the missing coefficient A_1 required to cause vanishing of the power series at $re^{i\phi}$ satisfies $|A_1|\le g$. Using the vanishing of the imaginary part of the power series (arbitrarily assigning values $|A_i|\le g$ for any $i>1$ with $(i-1)\phi\equiv 0 \pmod{\pi}$) and (2) we have

$$|A_1| = \left|\sum_{i=2}^{\infty} A_i\frac{r^i\sin i\phi}{r\sin\phi}\right| \le g\sum_{i=2}^{\infty}\left|\frac{\sin i\phi}{\sin\phi}\right| r^{i-1} = \frac{1}{r^2}-g \le 2\sqrt{g}+1\le g$$

for $\sqrt{g}\ge\sqrt{2}+1$.

It is easy to check that $z = 1/(\sqrt{g}+1)$ is a double root of

$$1-(2\sqrt{g}+1)z + g\frac{z^2}{(1-z)}$$

and from the form of the series it must actually be the smallest double root (see Theorem 4 of [1]).

Proof of Proposition 2. Recall that for an angle θ we have $J(\theta)=J$ iff the root $r=r(\theta,J)$, $0<r<1$, of

$$1-g\sum_{i=1,i\ne J}^{\infty}\left|\frac{\sin(J-i)\theta}{\sin J\theta}\right| x^i = 0$$

gives

$$A_J(\theta) = \sum_{i=1}^{\infty} \operatorname{sign}\left(\frac{\sin(J-i)\theta}{\sin J\theta}\right) \frac{\sin i\theta}{\sin J\theta} r^{i-J}$$

in $[-g, g]$.

Now if $\phi/2\pi$ is irrational and $|A_J(\phi)| < g$, then we can take θ sufficiently close to ϕ that

$$\operatorname{sign}\left(\frac{\sin(J-i)\theta}{\sin J\theta}\right) = \operatorname{sign}\left(\frac{\sin(J-i)\phi}{\sin J\phi}\right)$$

for all $j < N$, with N sufficiently large that $r(\theta)$ is also close enough to $r(\phi)$ that $A_J(\theta) \approx A_J(\phi)$ still satisfies $|A_J(\theta)| < g$.

If $\phi/2\pi = t/s$ rational (with $l = s$ or $s/2$ as s is odd or even) and $\theta > \phi$ (respectively $\theta < \phi$) we set

$$J_1 = J + ml \quad (\text{respectively } J_1 = J + nl).$$

Oberve that for θ sufficiently close to ϕ we still have $r(\theta) \approx r(\phi)$ and

$$\operatorname{sign}\left(\frac{\sin(J_1-j)\theta}{\sin J_1\theta}\right) = \operatorname{sign}\left(\frac{\sin(J_1-j)\phi}{\sin J_1\phi}\right), \quad j < N,\ j \not\equiv J (\operatorname{mod} l),$$

while for $j = J + il < N$ we have

$$\operatorname{sign}\left(\frac{\sin(J_1-j)\theta}{\sin J_1\theta}\right) \sin(j\theta) \approx \begin{cases} -\operatorname{sign}(\sin J\phi)\sin J\phi, & i < m \text{ (resp. } i > n),\\ \operatorname{sign}(\sin J\phi)\sin J\phi, & i > m \text{ (resp. } i < n). \end{cases}$$

Hence we obtain

$$\frac{A_J(\phi)}{1-r^l} \approx A_{J_1}(\theta) r^{ml} - g \operatorname{sign}(\sin J\phi)\left(\frac{1 - r^{lm} - r^{l(m+1)}}{1-r^l}\right)$$

(respectively

$$\frac{A_J(\phi)}{1-r^l} \approx A_{J_1}(\theta) r^{nl} + g \operatorname{sign}(\sin J\phi)\left(\frac{1 - r^{ln} - r^{l(n+1)}}{1-r^l}\right)\Bigg)$$

with $A_{J_1}(\theta)$ safely inside $(-g, g)$ when $A_J(\phi)$ lies in the stated range and θ is sufficiently close to ϕ.

Properties (a), (b) follow from a similar appeal to continuity.

Proof of Theorem 5. Given two arguments ω_1, ω_2 in $(0, \pi)$ we let

$$S(\omega_1, \omega_2) := \left\{ j : \frac{1}{2} j\phi \equiv \omega_1 \quad \text{or} \quad \frac{1}{2} j\phi \equiv \omega_2 \right\}.$$

Hence if $f = 1 + \sum_{i=1}^{\infty} a_i x^i$ in $\mathcal{F}_g$ vanishes at r and $re^{i\phi}$ we can use the equations

$$1 + \sum_{i=1}^{\infty} a_i r^i = 0, \quad 1 + \sum_{i=1}^{\infty} a_i \cos(i\phi) r^i = 0, \quad \sum_{i=1}^{\infty} a_i \sin(i\phi) r^i = 0$$

to eliminate any terms a_j with j in $S(\omega_1,\omega_2)$ obtaining

$$1+\sum_{j\notin S(\omega_1,\omega_2)} a_j\left(\frac{\sin(\omega_1-\frac{1}{2}j\phi)\sin(\omega_2-\frac{1}{2}j\phi)}{\sin\omega_1\sin\omega_2}\right)r^j=0.$$

Hence r can be no smaller than the real root of

$$1-g\sum_{j\notin S(\omega_1,\omega_2)}\left|\frac{\sin(\omega_1-\frac{1}{2}j\phi)\sin(\omega_2-\frac{1}{2}j\phi)}{\sin\omega_1\sin\omega_2}\right|x^j$$

achieved for a configuration

$$a_j=-g\ \mathrm{sign}\left(\frac{\sin(\omega_1-\frac{1}{2}j\phi)\sin(\omega_2-\frac{1}{2}j\phi)}{\sin\omega_1\sin\omega_2}\right)$$

of the type given in the lemma (with $\theta_1=2\omega_1,\theta_2=2\omega_2$) and no other.

It remains to show that the series f_r for $r=\tilde{r}_g(\phi)$ must be of this form. A slight adjustment in Lemma 1 shows that an extremal series f_r must always contain at least one $a_j=\pm g$.

The following variant of Lemma 2 then completes the proof (with θ_1 and θ_2 marking the point in the arguments $j\phi$ (mod 2π) where a_j first changes from negative to positive and from positive to negative respectively). □

Lemma 3. *Suppose j_1, j_2 are two integers such that $\frac{1}{2}j_1\phi\equiv\lambda_1$ (mod π) and $\frac{1}{2}j_2\phi\equiv\lambda_2$ (mod π) satisfy $0<\lambda_1<\lambda_2<\pi$.*

(i) *If $\phi/2\pi=r/s$ is rational, then $a_j=-g$ for $j\equiv 0$ (mod s).*

(ii) *If $a_{j_2}\neq -g$ and $a_{j_1}\neq -g$, then*

$$a_j=g\quad for\ \frac{1}{2}j\phi\in(\lambda_1,\lambda_2)\cup(\lambda_1+\pi,\lambda_2+\pi)\ (mod\ 2\pi).$$

(iii) *If $a_{j_2}\neq -g$ and $a_{j_1}\neq g$, then*

$$a_j=-g\quad for\ \frac{1}{2}j\phi\in(0,\lambda_1)\cup(\pi,\lambda_1+\pi)\ (mod\ 2\pi).$$

(iv) *If $a_{j_2}\neq g$ and $a_{j_1}\neq -g$, then*

$$a_j=g\quad for\ \frac{1}{2}j\phi\in(\lambda_2,\pi)\cup(\lambda_2+\pi,2\pi)\ (mod\ 2\pi).$$

The proof is similar to that of Lemma 2 and relies on our ability (given any three exponents $\vec{n}=(n,m,r)$), to construct a real polynomial

$$\begin{aligned}p(x,\vec{n}):=&\left(\frac{\sin\frac{1}{2}m\phi\sin\frac{1}{2}r\phi}{\sin\frac{1}{2}(n-m)\phi\sin\frac{1}{2}(n-r)\phi}\right)\left(\frac{x}{r}\right)^n\\&-\left(\frac{\sin\frac{1}{2}n\phi\sin\frac{1}{2}r\phi}{\sin\frac{1}{2}(n-m)\phi\sin\frac{1}{2}(m-r)\phi}\right)\left(\frac{x}{r}\right)^m\\&+\left(\frac{\sin\frac{1}{2}m\phi\sin\frac{1}{2}n\phi}{\sin\frac{1}{2}(n-r)\phi\sin\frac{1}{2}(m-r)\phi}\right)\left(\frac{x}{r}\right)^r\end{aligned}$$

which takes the value 1 at $x = r$ and $x = re^{i\phi}$. If an a_j with $\frac{1}{2}j\phi$ in the given interval did not take the stated value, then for a sufficiently small δ we could perturb the supposed extremal power series f_r

$$\tilde{f}_r(x) = 1 + (1-\delta)(f_r(x) - 1) - \delta p(x; j_1, j_2, j)$$

to obtain a new extremal series vanishing at r and $re^{i\phi}$ but no coefficient $\tilde{a}_j = \pm g$ contradicting the minimality of r. □

Proof of Theorem 6. Observing that any root α of a power series with coefficients a_i in $[-\delta, 1-\delta]$ (and hence $|a_i - (1/2 - \delta)| \leq 1/2$) satisfies

$$\left|1 + (1/2 - \delta)\frac{\alpha}{1-\alpha}\right| \leq \frac{1}{2}\frac{|\alpha|}{1-|\alpha|}$$

we obtain

$$|\alpha|^{-1} \leq \begin{cases} 1+\delta & \text{if } 1/2 < \delta < 1, \\ \frac{1}{2}(1 - \delta + \sqrt{5 + 2\delta + \delta^2}) & \text{if } 0 \leq \delta \leq 1/2 \end{cases}$$

(with equality achieved for the series

$$1 - \delta \sum_{i=1}^{\infty} z^i, \quad 1 + (1-\delta)\sum_{i=1}^{\infty} x^{2i-1} - \delta \sum_{i=1}^{\infty} x^{2i}$$

respectively). This then gives an upper bound on $|\lambda|$ for any conjugates λ of a δ-beta-number, $0 \leq \delta < 1$. For $0 \leq \delta \leq 1/2$ we show that this latter bound is best possible:

For large integers k, N and $M \approx \delta N$ we take $\theta > 1$ to be the real root of

$$f := z^{2k+1} - Nz^{2k} - (N-M)\sum_{i=1}^{k} z^{2i-1} + M\sum_{i=0}^{k-1} z^{2i}.$$

Writing

$$\theta = N + (N-M)\sum_{i=1}^{k}\theta^{-(2i-1)} - M\sum_{i=1}^{k}\theta^{-2i}$$

and observing that (for large enough N)

$$-\delta\theta < \sum_{i=0}^{j}\left((N-M)\theta^{-2i} - M\theta^{-(2i+1)}\right) < (1-\delta)\theta,$$

$$-\delta\theta < -M + \sum_{i=1}^{j}\left((N-M)\theta^{-(2i-1)} - M\theta^{-2i}\right) < (1-\delta)\theta$$

for $0 \leq j \leq k-1$, it is not hard to see that θ has a finite expansion $d = N$, $N-M, -M, \ldots, N-M, -M, 0, \ldots$ and hence is a simple δ-beta-number.

If N and M are chosen to have a prime p with $p|N, M$ but $p^2 \nmid N, M$, then f is irreducible (by Eisenstein's criterion) and it is not hard to see that as $N, k \to \infty$, with $M/N \to \delta$ the polynomial f must have a root $1/\lambda$ with

$$\lambda \to -\frac{(\sqrt{5 + 2\delta + \delta^2} - (1-\delta))}{2(1+\delta)}. \quad \square$$

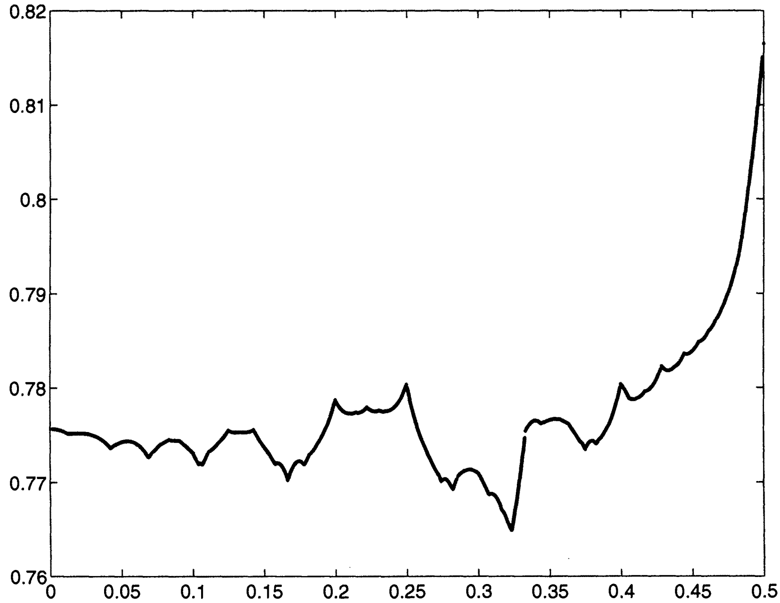

FIGURE 1. $r_g(\phi)$ against ϕ/π for $g = 1/2$.

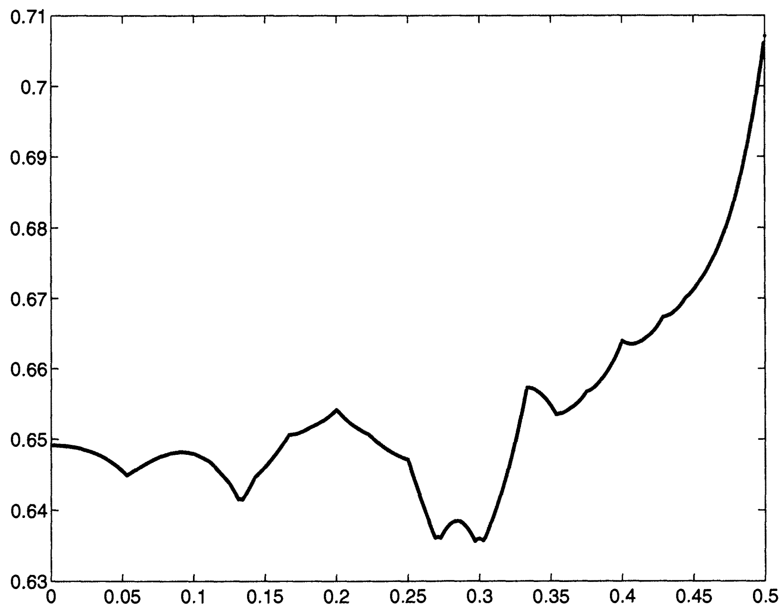

FIGURE 2. $r_g(\phi)$ against ϕ/π for $g = 1$.

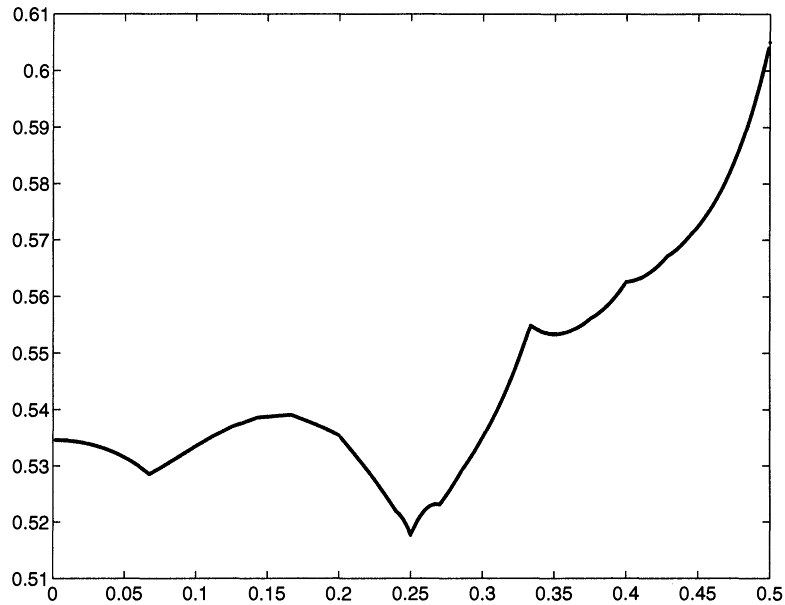

FIGURE 3. $r_g(\phi)$ against ϕ/π for $g = 3^{1/2}$.

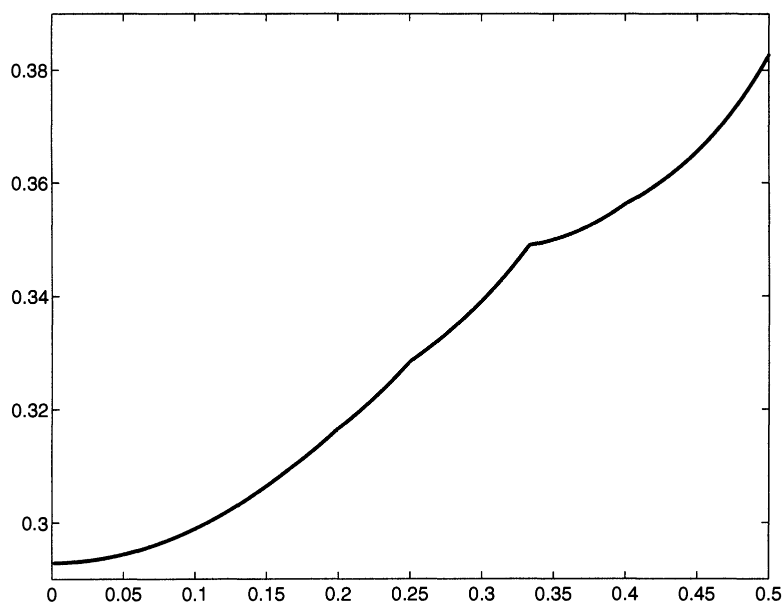

FIGURE 4. $r_g(\phi)$ against ϕ/π for $g = 2^{3/2} + 3$.

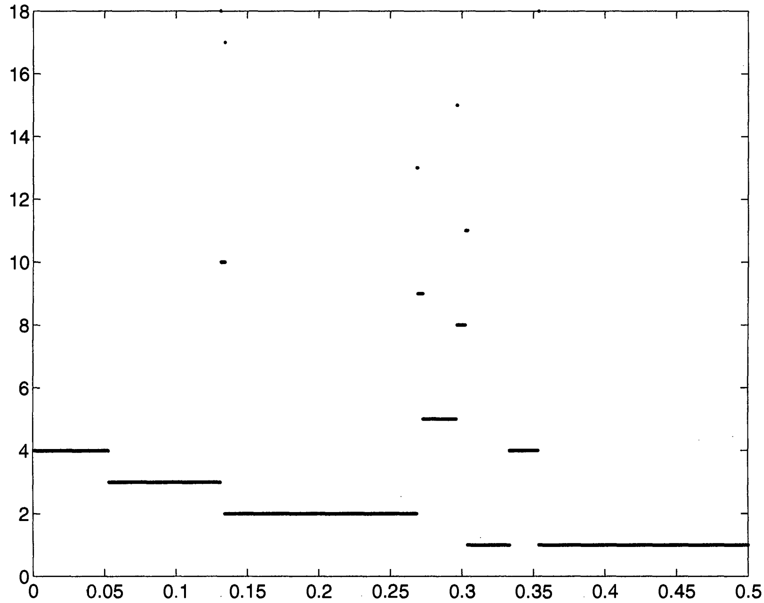

FIGURE 5. $J(\phi)$ against ϕ/π for $g = 1$.

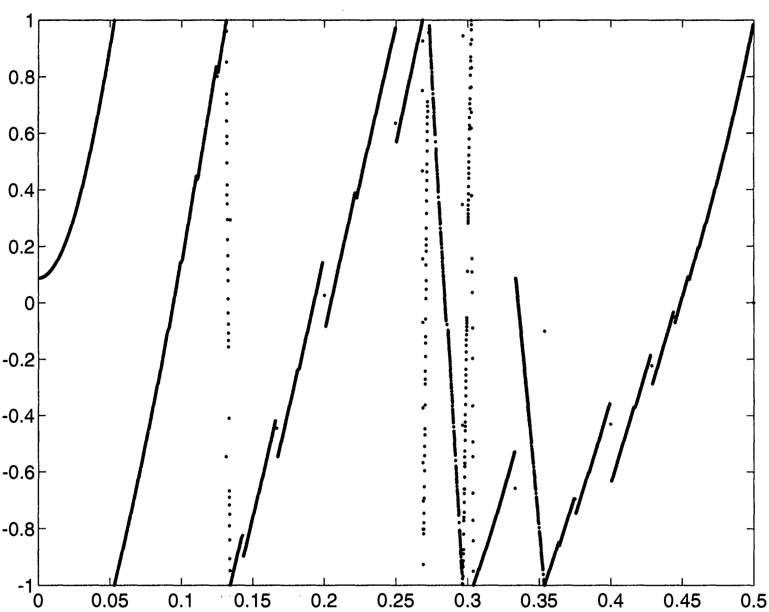

FIGURE 6. $A_{J(\phi)}$ against ϕ/π for $g = 1$.

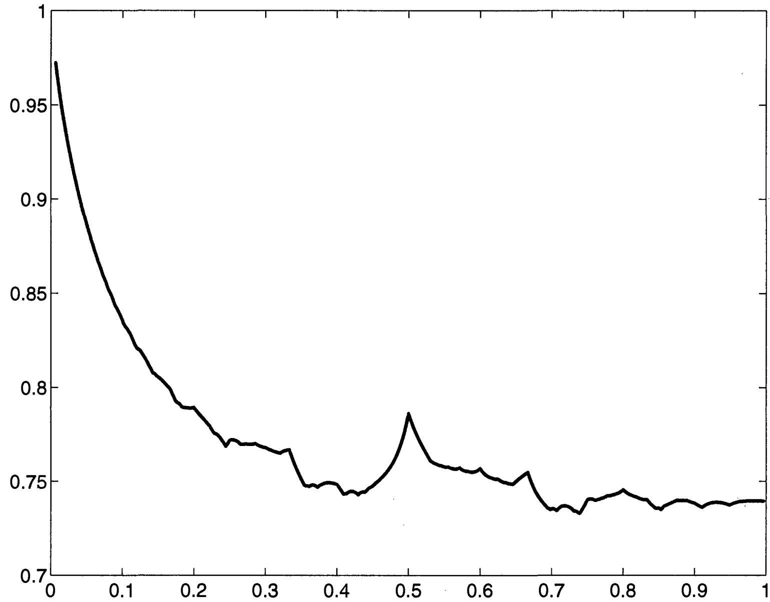

FIGURE 7. $r_I(\phi)$ against ϕ/π for $I = [0, 1]$.

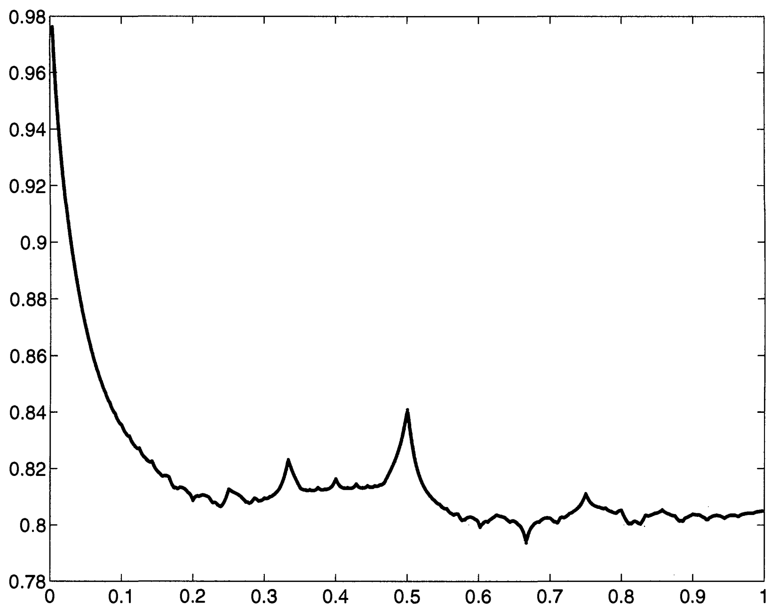

FIGURE 8. $r_I(\phi)$ against ϕ/π for $I = [-1, 0]$.

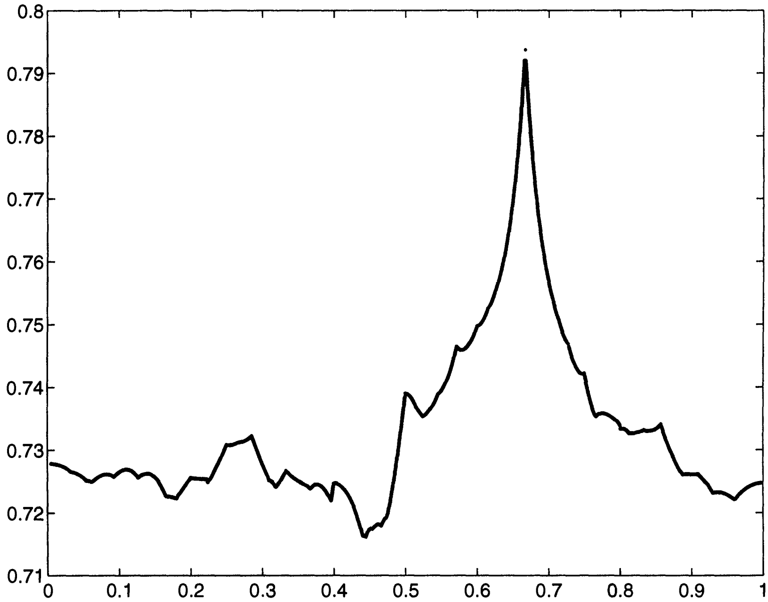

FIGURE 9. $\tilde{r}_g(\phi)$ against ϕ/π for $g = 1$.

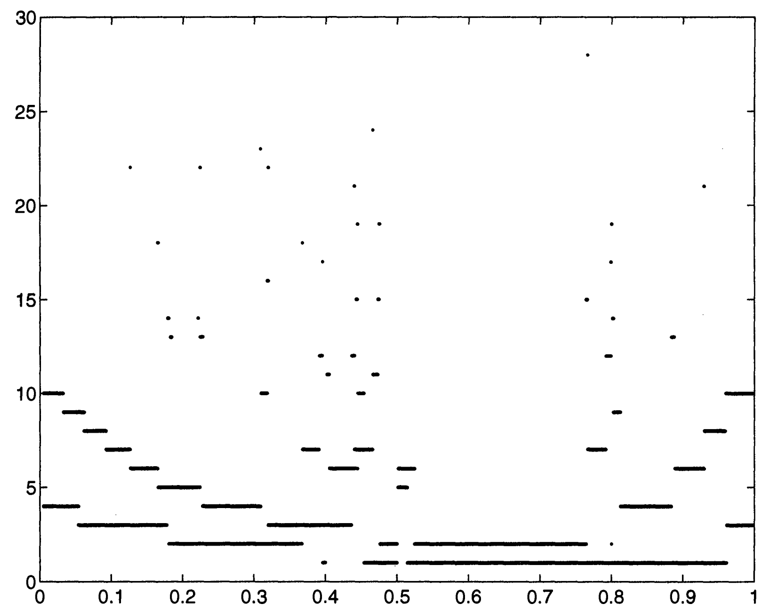

FIGURE 10. I, J against ϕ/π for $g = 1$.

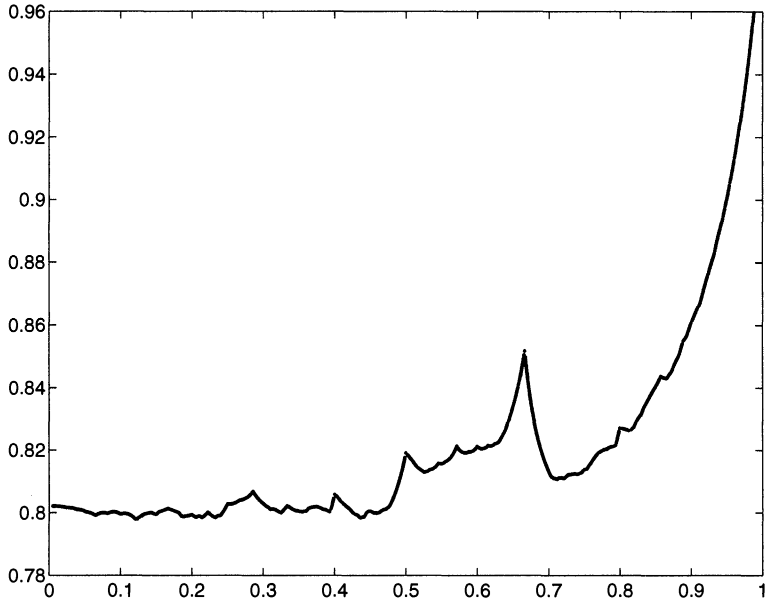

FIGURE 11. $\tilde{r}_I(\phi)$ against ϕ/π for $I = [0, 1]$.

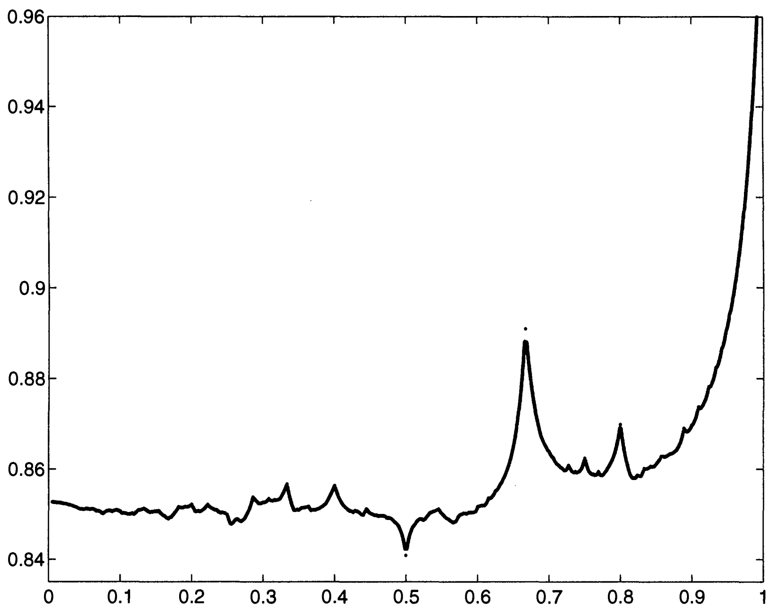

FIGURE 12. $\tilde{r}_I(\phi)$ against ϕ/π for $I = [-1, 0]$.

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