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REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

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8[65Kxx, 90Cxx]—*Primal-dual interior-point methods*, by Stephen J. Wright, SIAM, Philadelphia, PA, 1997, xx+289 pp., 25 cm, softcover, $37.00

During the last twelve years, a minor explosion has occurred in the literature on optimization, caused by the impact of Karmarkar's projective algorithm for linear programming. Over a thousand papers have been published on the resulting methods, called interior-point algorithms. Early papers provided extensions and clarification of Karmarkar's original method, and some preliminary (and mixed) computational experiments. The connection to classical barrier methods in nonlinear (and linear programming) was established. Two key early papers were Megiddo [2] and Renegar [4]; Megiddo proposed a symmetric primal-dual path-following approach, while Renegar analyzed a simple primal-only path-following method that required only $O(\sqrt{n}\ln(1/\epsilon))$ iterations to attain precision ϵ in the worst case, compared to Karmarkar's $O(n\ln(1/\epsilon))$; here n is the number of inequality constraints.

Monteiro and Adler and independently Kojima, Mizuno, and Yoshise realized Megiddo's program by developing $O(\sqrt{n}\ln(1/\epsilon))$ symmetric primal-dual algorithms. Significantly, both showed how these methods could be extended beyond linear programming: Monteiro and Adler treated convex quadratic programming, while Kojima et al. dealt with the monotone linear complementarity problem, which subsumes linear and convex quadratic programming. Implementations of these primal-dual methods, differing from the theoretical versions by taking much longer steps, began to be developed. The work of Lustig, Marsten, and Shanno was particularly significant here; they showed a practical way to deal with the difficulty of not having an initial strictly feasible iterate from which to start, and exhibited very favorable computational results. Mehrotra introduced a predictor-corrector variant whose ideas have been adopted by all successful primal-dual codes. For a recent review of the state of the art in computational experiments, see the survey article by Lustig, Marsten, and Shanno [1]. At the same time as these developments, Nesterov and Nemirovskii were investigating a very general theoretical approach to certain convex programming problems using primal algorithms based on so-called self-concordant barrier functions; their work culminated in the brilliant but technically demanding monograph [3].

The field of interior-point methods has now reached a fairly mature state, and it is clearly time that some generally accessible books dealt with the subject. A few texts in operations research or mathematical programming include some material on Karmarkar's method or the (primal or dual) affine-scaling method, but it now seems clear that the methods of choice for linear programming and the extensions mentioned above are primal-dual methods. Several books have appeared dealing with interior-point methods in the last year or so, e.g., those of Roos, Terlaky, and

Vial [5], Saigal [6], and Vanderbei [7]. The current book of Stephen Wright, concentrating on primal-dual methods, provides a highly readable and up-to-date account of the field. It is accessible to numerical analysts and applied mathematicians with some background in optimization, and is also suitable for an advanced graduate class. Before discussing it in depth, let me make some general remarks about the two main classes of methods for linear programming: simplex and interior-point algorithms.

The feasible region of a linear programming problem is a convex polyhedron. Dantzig's well-known simplex method exploits the fact that the extremum of a linear function over such a set must be attained at an extreme point; it iterates among these extreme points, moving along the edges of the feasible region. It has proven remarkably efficient over the last fifty years, as problem sizes have risen from a few tens to those involving tens of thousands of (equality) constraints and hundreds of thousands of (nonnegative) variables. It is a totally astounding fact that the number of iterations required typically remains a small multiple of the number of constraints, possibly multiplied by a term that is logarithmic in the number of variables, although the performance degrades as the problems reach the largest scale indicated above. Theoretical studies on the diameter of polytopes and on probabilistic analyses of variants of the simplex method give a partial, but far from convincing, justification for its efficiency. Consideration of small examples suggests that it might be more efficient to travel through the interior (or relative interior) of the feasible region to reach an optimal solution. While several such methods were tried (and found wanting) in the fifties, this view is the basis of the new interior-point methods. These also exhibit outstanding (and partially inexplicable) behavior by using sophisticated techniques to generate efficient search directions, and seem to require a number of iterations rising extremely slowly, from ten or so for the smallest problems to a hundred or so for the largest. Each iteration is much more expensive than a simplex iteration (depending very much on efficient sparse linear algebra techniques and the structure of the problem), paving the way for serious competition between the two classes of methods. At present, it appears that interior-point methods are slightly slower for small and medium-sized problems, comparable on large problems, and superior (depending on the structure of the problem) for very large scale problems.

Wright's book starts with an excellent introduction, describing the main classes of primal-dual methods and the issues that will be dealt with in future chapters. Chapter 2 gives some background on linear programming and interior-point methods. It is generally clear, but not necessarily to one with no prior linear programming exposure. The development relies on the Karush-Kuhn-Tucker optimality conditions, which are discussed further in an appendix. In particular, the central path, which plays a fundamental role in interior-point methods, is defined and its existence established. It is somewhat surprising that the main theorem here does not prove that there is a unique solution to the central path equations for each positive τ, nor is it shown (a simple consequence of the implicit function theorem) that the set of solutions forms a path. An excellent historical overview of the development of interior-point methods is given. Chapter 3 then provides a good informal treatment of complexity theory. There is some confusion of rational and real-number complexity; the author suggests ensuring that the coefficient matrix A has full row rank by performing a QR factorization, but this will destroy rationality of the entries.

Chapters 4, 5 and 6 describe the main methods of the book; potential-reduction, path-following, and infeasible-interior-point algorithms. Each of these chapters provides a beautifully written and concise development of the basic motivation, properties, and convergence behavior of these methods, with technical lemmas well planned and motivated so that the reader is led to a clear understanding of the subject. The only change I would like is for the equations defining the search directions to be repeated, rather than continually referring back to Chapter 1. Also, Theorem 6.1 on the last class of methods really needs an explicit assumption that the primal and dual linear programming problems do have feasible solutions.

The next three chapters give important further refinements and extensions of the basic methods. Chapter 7 deals with superlinear convergence, a subject to which the author has made significant contributions. This rather technical subject is very cleanly presented, including treatment of both theoretical and practical algorithms with fast asymptotic convergence. Then Chapter 8 shows how various extensions of linear programming, in particular linear complementarity and quadratic programming problems, can be dealt with using the same basic tools and algorithms. There is a short discussion of the very active area of semidefinite programming also. Chapter 9 discusses refinements of the methods that permit the detection of infeasibility. Unfortunately, the proof of Theorem 9.1 in the appendix is not quite complete, and the proof of Theorem 9.3 refers to Theorem 2.2, which should be Corollary 2.2 and does not quite apply. Nevertheless, the topic is well treated.

Finally, the author covers practical aspects, in particular Mehrotra's predictor-corrector algorithm, and details of implementations, in the final two chapters. These give excellent background for the issues that really determine the effectiveness of primal-dual interior-point methods for solving large-scale problems in practice.

Two appendices give some background results and proofs that are postponed from the main development for clarity and an excellent listing of software available. This reviewer has to compliment the author particularly on his efforts to make the book as up-to-date as possible (there are several references to papers that appeared in 1996) and his intention to maintain a web site keeping the list of available software current. (The author has already performed a singular service to the interior-point methods community by running a web site for dissemination of announcements of new papers and meetings and maintaining an archive of preprints.)

While discussing only a subset of interior-point methods (although, as indicated above, primal-dual methods are by far the most important in practice), and confining himself in the main to linear programming for simplicity, the author provides an exceptionally broad and clear treatment of this very significant area of optimization. Many authors concentrate on polynomial complexity issues, theoretical discussion of asymptotic convergence, or practical aspects of implementations. Steve Wright gives a very balanced and excellently written treatment of all these topics. I highly recommend this book to both specialists in these methods and general readers interested in this fascinating area of computational mathematics.

References

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9[65-01, 65-04]—*Computational mathematics in engineering and applied science: ODEs, DAEs, and PDEs*, by William E. Schiesser, CRC Press, Boca Raton, FL, 1994, xii+587 pp., 26 cm, $74.95

This is an engineering book on scientific computation, a book on computing by Fortran programming. More than half of the contents are source codes (with many comment lines), data files, and output information. There are no convergent theories.

The main stream of the book is the method of lines. The basic idea in the method of lines for PDEs is to replace the spatial derivatives with algebraic approximations, thereby leaving only the time derivatives. The procedure produces an ODE at each spatial grid, and the resulting system of ODEs can be solved by a known ODE solver. Basically, the method of lines is a systematical way of using ODE solvers to integrate PDEs.

The methodology is illustrated through detailed examples in Fortran 77 coding. The underlying mathematics of the methods are not discussed in detail, and theoretical analyses are not provided. Effectiveness and validity of the methods are discussed through observation from the output of the computation, and the error analyses of computed solutions are presented numerically as part of the example applications.

All examples are coded in the same format and presented in the following structure: (1) start from an example which includes a differential equation and initial and/or boundary conditions; (2) explain the coding, list all subroutines and data files, and comment on their purposes; (3) list some output and draw some conclusions from computing experience. Almost every example calls for some library routines, usually, differential routines and integration routines. Instead of writing an entire code from the very beginning, the author proposes to use quality library routines which have achieved the status of international standards. However, the author does not recommend the uninformed use of library routines, since some knowledge of differential equation characteristics and numerical methods will invariably lead to more effective use of existing packages.

The intention of the book is not to provide the state-of-art methods, rather it is to introduce some practical methods proved to be effective in dealing with some typical problems. Therefore, the book is a source of some practical tools for numerical methods of ODEs and PDEs. Because of its elementary contents, the book is also suitable for beginners.