10[33-01]—*Special functions: An introduction to the classical functions of mathematical physics*, by Nico M. Temme, John Wiley & Sons, Inc., New York, NY, 1966, xii+374 pp., 24 cm, hardcover, $54.95

Anyone who wants to learn about special functions or wants to teach a course in the subject can choose from a variety of superb books, [and], [car], [hoc], [leb], [nik], [rai], [spa], [sri]. All of these books display a personal slant on the subject; in their design and choice of material they are far more unresemblant than books on most mathematical topics. Perhaps the subject appeals especially to the iconoclast. Rainville's book stresses formal identities satisfied by the special functions, and the main tool used in obtaining these identities is what is called eponymously Sister Celine's technique (for an interesting biography of Sister Mary Celine Fasenmyer, a student of Earl Rainville, the reader should consult the whimsical and brilliant oddity [pet]). Carlson's book stresses hypergeometric functions, and its primary tool is Dirichlet averaging. Nikiforov and Uvarov's book makes much of finite difference techniques, so is able to offer a unique and highly unified theory of orthogonal polynomials of a discrete variable (though, I think, few readers will have the patience to indulge the authors's finicky derivations). Lebedev's book, a beautiful little opus from which I have taught from time to time, makes frequent and shamelessly ingenious use of multiple integration techniques to obtain special function identities, and furthermore has, for each special function discussed, a section on the applications of that function. And so on and on. In addition, there are many books treating specialized topics: orthogonal polynomials, group theoretic methods in special functions, integral transforms, asymptotics of special functions, elliptic functions, etc. And there are massive enchiridions, too: [erd], [luk].

The question is: do we need yet another book on special functions? Yes, we always need more good books. Temme's book is a very good one, indeed. Though it contains material that is pretty standard, I found intriguing items I have never seen before. The plan of the book, too, is standard. Chapter 1, on Bernoulli, Euler and Stirling numbers, begins with a fascinating puzzle that has been elucidated in a recent and celebrated paper [bor]:

We all know that

$$\frac{\pi}{2} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}.$$

Taking 50 000 terms of the sum gives the astonishing result

$$2\sum_{n=1}^{50\ 000} \frac{(-1)^{n-1}}{2n-1}$$
$$= 1.5707\ \overline{8}6326\ 79489\ \overline{7}6192\ 31321\ \overline{1}9163\ 975\overline{20\ 5}2098\ 58\overline{331\ 46876}$$

where I have placed a bar over the figures that are in error. Why astonishing? Experience tells us that if one digit of an answer is in error, all the following will be, too. What can account for the strange distribution of erroneous digits in the above figure? A sly idea: to begin the book with a problem so intriguing and involving it would vivify the most enervated reader. In this chapter Temme points out that the same phenomenon occurs with Euler's series for ln 2 and reveals that the mystery is explicable in terms of the properties of Euler numbers and of a little-known procedure called Boole's summation method, an analog of Euler's summation formula which uses Euler, rather than Bernoulli, polynomials:

Let f have $k \geq 1$ continuous derivatives. Then

$$f(1) = \frac{1}{2} \sum_{i=0}^{k-1} \frac{E_i(1)}{i!} [f^{(i)}(1) + f^{(i)}(0)] + R_k,$$

$$R_k = \frac{1}{2(k-1)!} \int_0^1 f^{(k)}(x) E_{k-1}(x) dx,$$

where $E_k(x)$ is the Euler polynomial.

The chapter concludes with an abundance of challenging and interesting exercises, a feature characteristic of the book.

Rather than devote Chapter 2 to a class of special functions, the author declares time out to explain an assortment of techniques that are useful in obtaining results in special functions, for instance, integrating series termwise, interchanging the order of integration in double integrals, expanding integrals asymptotically, Watson's lemma, the saddle point method.

Chapter 3 introduces the Gamma function. In an unusual move, the author states, but does not prove, the Bohr-Mollerup theorem: *the gamma function is the only positive logarithmically convex function with $f(1) = 1$ that satisfies the recursion formula $f(x + 1) = xf(x)$.* As far as I know, this theorem is not stated nor proved in other books on special functions. However, the proof of the theorem is neither lengthy nor demanding [con], and I would like to have seen it here. The author lists the usual properties of $\Gamma(z)$, but sometimes includes unusual proofs, for instance, Legendre's duplication formula is proved from the integral representation of the Beta function. Having developed the theory of the Euler-Maclaurin summation procedure, the author has established the foundation for an elaborate and satisfying explanation of Stirling's formula. This chapter contains the most appealing treatment of the Gamma function available in an introductory text.

In Chapter 4, in an obviously eclectic mood, the author strikes out into the territory of differential equations, but it makes a great deal of sense to do so. Almost all the special functions are solutions of differential equations. He talks about the wave equation—separating variables—then about differential equations in the complex plane, namely, second order differential equations with three singular points, including the hypergeometric equation. He explains how to develop solutions in the neighborhood of singular points, e.g., the method of Frobenius. An understanding of the hypergeometric function, the subject of Chapter 5, depends on this sedulously prepared infrastructure.

Succeeding chapters treat, in a fairly traditional manner, orthogonal polynomials, the confluent hypergeometric function, Legendre functions, Bessel functions. In Chapter 10 the author discusses separating the wave equation in various coordinate systems; this, too, is an unusual touch, and he shows how the various special functions previously discussed arise out of this process.

The author's special area of expertise is the confluent hypergeometric function, and, in particular, the incomplete gamma functions, and, still more in particular, the asymptotic theory of these functions. This makes him eminently qualified to discuss, in Chapter 11, special statistical distribution functions, since most of these functions are special cases of the confluent hypergeometric functions. No other elementary book contains a description of the uniform asymptotic expansion of these functions, a very valuable feature.

Chapter 12 contains a brief treatment of elliptic functions, and the final chapter, Chapter 13, discusses the numerical computation of special functions, including the use of recurrence relations, Miller's algorithm, and techniques for computing with continued fractions. Most other books maintain an attitude of sublime indifference to the actual computation of special functions.

All in all, the book is a superb achievement. It is beautifully and clearly written, the material is cannily organized and each topic takes its carefully designated place in the overall scheme of things. The idea of first describing an assembly of techniques that will facilitate the analysis of the special functions to follow is a very fruitful one, and it makes the theory look much more coherent than in other books, which often have the ad hoc quality of a potpourri of special functions thrown together with no unifying pattern underlying the exposition.

I think the book would be ideal as a text in a one semester or even a two quarter course. The author has included applications, but not so many as to encumber the book, a fault I found in the otherwise admirable Lebedev.

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Jet Wimp
Department of Mathematics and Computer Science
Drexel University
Philadelphia, PA 19104
E-mail address: `jwimp@mcs.drexel.edu`