A NEGATIVE-NORM LEAST SQUARES METHOD
FOR REISSNER-MINDLIN PLATES

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ABSTRACT. In this paper a least squares method, using the minus one norm developed by Bramble, Lazarov, and Pasciak, is introduced to approximate the solution of the Reissner-Mindlin plate problem with small parameter t, the thickness of the plate. The reformulation of Brezzi and Fortin is employed to prevent locking. Taking advantage of the least squares approach, we use only continuous finite elements for all the unknowns. In particular, we may use continuous linear finite elements. The difficulty of satisfying the inf-sup condition is overcome by the introduction of a stabilization term into the least squares bilinear form, which is very cheap computationally. It is proved that the error of the discrete solution is optimal with respect to regularity and uniform with respect to the parameter t. Apart from the simplicity of the elements, the stability theorem gives a natural block diagonal preconditioner of the resulting least squares system. For each diagonal block, one only needs a preconditioner for a second order elliptic problem.

1. INTRODUCTION

The numerical solution of the Reissner-Mindlin plate model has been discussed by many authors. Many schemes developed early on, using standard finite element methods, are known to possess a locking problem for very thin plates. Using a Helmholtz decomposition, Brezzi and Fortin [11] derived a reformulation of the Reissner-Mindlin plate model, which fundamentally removed the reason for the locking. Another advantage of this reformulation is that the four unknowns are decoupled into two elliptic problems and one mixed saddle point problem. Brezzi and Fortin gave a scheme based on this for which they proved error estimates independent of the plate thickness t for transversal displacement, rotations and shear stresses. The main part of the reformulation is a problem similar to the Stokes problem, so there are some disadvantages when discretizing it directly. First, a discrete saddle point problem has to be solved. Second, the bubble function subspace, B_3, is used in addition to the linear finite elements, because one cannot independently choose finite element subspaces; the discrete "inf-sup" condition has to be satisfied. Finally, the preconditioning of the resulting system is not simple. Brenner [9] developed a multigrid W-cycle iteration, and proved that when the number of smoothings is large enough, the convergence rate of the iteration does not depend on t or h, the mesh size. How many smoothings are required is not clear.

A number of authors tried to develop locking-free schemes which involve only the primitive variables. (Arnold and Falk [1], Duran and Liberman [13], Onate, Zarate and Flores [15], Brezzi, Fortin and Stenberg [12], ...) Each of them used the same framework, a pair of subspaces and a reduction operator [2]. The key point of the analysis is to make the discrete system of the two primitive unknowns equivalent to a discretization of the Brezzi-Fortin reformulation, so that it can be locking free. The choice of the subspace of the rotation is implicitly restricted by the stability condition on the mixed saddle point problem of the Brezzi-Fortin reformulation. The choice of the reduction operator and the subspace of the displacement is restricted by the discrete Helmholtz decomposition, so that the extra unknowns can be eliminated. Therefore the pair of spaces has to be rather complicated. The simplest pair of subspaces is the one given by Onate, Zarate and Flores [15], but Arnold and Falk [2] showed that this method does not converge in the classical sense. Although it is ideal to be able to work on the primitive variables only, as far as computing time is concerned working on the two extra unknowns of the Brezzi-Fortin reformulation does not cost more. In fact, the two elliptic problems can be solved relatively in little time; the solution technique of the Stokes-like problem is also well known.

The other approach is a stabilization method (e.g., [12]). The major measures taken to prevent locking in this approach are the following: (a) A projector which brings the discrete rotation function and the gradient of the displacement function into the same subspace, and (b) the replacement of t^{-2} by $(t^2 + \alpha h^2)^{-1}$, which is, in a certain sense, equivalent to approximating a thin plate by a thicker plate but keeping the error introduced by this within the magnitude of the discretization error. This approach does produce simpler schemes.

Later, Hughes and Franca [14] and Stenberg [18] developed stabilized mixed formulations which are consistant to the PDE. Therefore they can use balanced elements designed only for interpolation properties. In [18], Stenberg proposed a scheme which is simpler than the one in [14]. The shear stress is removed from both the formulation and the analysis. One possible disadvantage of this approach is that a high regularity assumption on the transversal displacement is needed. But for a plate with partially clamped–partially free boundary conditions, the displacement function may not even be in H^2 because of the corners of the domain. Besides, the multigrid preconditioners directly based on the formulation of these methods are complicated (cf. [16]), if compared to the diagonal preconditioners.

Arnold, Falk and Winther [4] worked on the preconditioning of these types of schemes. They reported that until now, no attempt to precondition the primitive variable discretizations has been successful. The small parameter t also creates trouble for preconditioning. It appears that even though one can discretize the Reissner-Mindlin plate model with the primitive unknowns only, a locking-free reformulation has to be used for preconditioning, and so additional unknowns have to be used. The paper [4] used a different locking-free reformulation with the shear stress as the new unknown and set up a framework for diagonal preconditioning of the discretization methods described in the paragraphs above. This reformulation also has five scalar unknowns in a saddle point system, but not decoupled. Two of the diagonal blocks of the preconditioner use standard elliptic problem preconditioners. The other one is a preconditioner for $I - t^2 \mathbf{grad}div$, which they call $H(div)$ preconditioning. The method given in [3] and [4] is restricted to the case in which the pair of subspaces are such that the discrete Helmholtz decomposition holds.

In this paper, we develop a least squares discretization based on the reformulation of Brezzi and Fortin. A discrete version of $\| \cdot \|_{-1}$ norm [6], [7] is employed for the least squares method to weight the system correctly. The major advantage of least squares methods is that one does not have to satisfy the discrete "inf-sup" condition and can choose the subspaces for different variables independently. Therefore, continuous elements (e.g., linear finite elements) may be used here for all the unknowns. The error estimates are optimal with respect to regularity and are uniform in t. Instead of leading to a saddle point problem, the resulting least squares system is symmetric and positive definite. The stability theorem of the least squares form provides a natural way to precondition the system diagonally. For each block, one only needs to have a preconditioner for a second order elliptic problem, a task which is well understood.

The rest of the paper is organized as follows. Section 2 describes the Reissner-Mindlin plate model and the reformulation of Fortin and Brezzi. Section 3 gives the least squares discretization. A stability result on the least squares form is proved in Section 4. The error estimate under full regularity is proved in Section 5. Implementation and numerical calculations are discussed and presented in Sections 6 and 7.

2. Reissner-Mindlin plate model and its reformulations

We introduce now some notation for some standard function spaces. Let $L^2(\Omega)$ be the square integrable functions on a domain $\Omega \subset \mathbf{R}^2$. $\mathbf{L}^2(\Omega) = L^2(\Omega) \times L^2(\Omega)$. The space $\widehat{L}^2(\Omega)$ is the space of L^2 functions with mean value zero; i.e., $p \in \widehat{L}^2(\Omega)$ implies that $\int_\Omega p = 0$. Let $H^1(\Omega)$ be the usual Sobolev space of functions whose first derivatives are square integrable. The subspace $H_0^1(\Omega)$ consists of those functions $u \in H^1(\Omega)$ such that $u = 0$ on $\partial\Omega$. $\widehat{H}^1(\Omega)$ denotes the space of H^1 functions, p, with $\int_\Omega p = 0$. $\mathbf{H}_0^1(\Omega) = H_0^1(\Omega) \times H_0^1(\Omega)$. $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$ with respect of $L^2(\Omega)$. $\mathbf{H}^{-1}(\Omega)$ is the dual space of $\mathbf{H}_0^1(\Omega)$ with respect of $\mathbf{L}^2(\Omega)$. The Reissner-Mindlin model of a thin plate with clamped boundary and thickness t is given as the solution of the following problem.

Find $(\omega, \phi) \in H_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$ such that

$$(2.1) \quad a(\phi, \psi) + \lambda t^{-2}(\phi - \nabla\omega, \psi - \nabla\mu) = (g, \mu), \qquad \forall(\mu, \psi) \in H_0^1(\Omega) \times \mathbf{H}_0^1(\Omega),$$

here $(\cdot, \cdot)$ denotes the $\mathbf{L}^2$ inner product, and

$$a(\phi, \psi) = \frac{E}{12(1 - \nu^2)} \int_\Omega [(\frac{\partial\phi_1}{\partial x} + \nu\frac{\partial\phi_2}{\partial y})\frac{\partial\psi_1}{\partial x} + (\nu\frac{\partial\phi_1}{\partial x} + \frac{\partial\phi_2}{\partial y})\frac{\partial\psi_2}{\partial y}$$
$$+ \frac{1 - \nu}{2}(\frac{\partial\phi_1}{\partial y} + \frac{\partial\phi_2}{\partial x})(\frac{\partial\psi_1}{\partial y} + \frac{\partial\psi_2}{\partial x})]dxdy.$$

The constants and variables are:
  ϕ: rotation of vertical fiber;
  ω: transversal displacement;
  g: transversal loading;
  E: Young's modulus;
  ν: Poisson's ratio;
  $\lambda = \frac{Ek}{2(1+\nu)}$ for $0 < k < 1$.

By Korn's inequality, there exist constants $C_1 > C_0 > 0$, such that for any $\phi \in \mathbf{H}_0^1(\Omega)$,

$$C_0 a(\phi, \phi) \leq (\nabla \phi_1, \nabla \phi_1) + (\nabla \phi_2, \nabla \phi_2) \leq C_1 a(\phi, \phi).$$

Brezzi and Fortin reformulated the Reissner-Mindlin plate problem [11] to avoid the locking phenomenon. By Helmholtz' Theorem, there exist $(r, p) \in H_0^1(\Omega) \times \widehat{H}^1(\Omega)$ such that

$$(2.2) \qquad \lambda t^{-2}(\nabla \omega - \phi) = \nabla r + \mathbf{curl}\, p.$$

Here $\mathbf{curl}\, p = (\frac{\partial p}{\partial y}, -\frac{\partial p}{\partial x})$. Using (2.2), one can easily show that (2.1) is equivalent to the following problem.

Find $(r, \phi, p, \omega) \in H_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \times \widehat{H}^1(\Omega) \times H_0^1(\Omega)$, such that

$$(2.3) \qquad (\nabla r, \nabla \mu) = (g, \mu), \qquad \forall \mu \in H_0^1(\Omega),$$

$$(2.4) \qquad a(\phi, \psi) - (\mathbf{curl}\, p, \psi) = (\nabla r, \psi), \qquad \forall \psi \in \mathbf{H}_0^1(\Omega),$$

$$(2.5) \qquad -(\phi, \mathbf{curl}\, q) - t^2(\mathbf{curl}\, p, \mathbf{curl}\, q) = 0, \qquad \forall q \in \widehat{H}^1(\Omega),$$

$$(2.6) \qquad (\nabla \omega, \nabla s) = (\phi + t^2 \nabla r, \nabla s), \qquad \forall s \in H_0^1(\Omega).$$

If not otherwise mentioned, we assume that Ω is a convex polygonal domain. Under this assumption, Arnold and Falk [1] proved the following a priori estimates and regularity results:

$$(2.7) \qquad \| r \|_1 + \| \phi \|_2 + \| p \|_1 + t \| p \|_2 + \| \omega \|_1 \leq C \| g \|_{-1},$$

and

$$(2.8) \qquad \| r \|_2 + \| \omega \|_2 \leq C \| g \|_0, \qquad \text{if } g \in L^2(\Omega).$$

A slight modification of the proof in [1] leads to the following improved estimate for ω:

$$(2.9) \qquad \| \omega \|_2 \leq C(\| g \|_{-1} + t^2 \| g \|_0).$$

It is obvious that (2.3) is an independent Dirichlet problem. Once (2.4) and (2.5) are solved for ϕ and p, (2.6) is also easy. The problem is thus reduced to solving the system of (2.4) and (2.5).

For the purpose of approximating the solution of this system we reformulate it as follows. Let $\mathbf{x}^\perp = (x_2, -x_1)$ for $\mathbf{x} \in \mathbf{L}^2(\Omega)$. Then

$$\mathbf{curl}\, p = \nabla p^\perp.$$

We introduce new variables $\mathbf{f}, \mathbf{u}$ and $\mathbf{v}$ by $\mathbf{f}^\perp = \nabla r, \mathbf{u}^\perp = \phi$, and $\mathbf{v}^\perp = \psi$. Then (2.4) and (2.5) can be rewritten as follows: Find $(p, \mathbf{u}) \in \widehat{H}^1(\Omega) \times \mathbf{H}_0^1(\Omega)$, such that

$$(2.10) \qquad \tilde{a}(\mathbf{u}, \mathbf{v}) - (\nabla p, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega),$$

$$(2.11) \qquad (\nabla \cdot \mathbf{u}, q) - t^2(\nabla p, \nabla q) = 0, \qquad \forall q \in \widehat{H}^1(\Omega),$$

where $\tilde{a}(\mathbf{u}, \mathbf{v}) = a(\phi, \psi)$.

By examining the proof of (2.7) in [1], we have the following estimate:

$$(2.12) \qquad \| \mathbf{u} \|_2 + \| p \|_1 + t \| p \|_2 \leq C \| \mathbf{f} \| \leq C(\| \mathbf{u} \|_2 + \| p \|_1),$$

for $\mathbf{f} \in \mathbf{L}^2(\Omega)$.

Define

$$A : \mathbf{H}_0^1(\Omega) \longrightarrow \mathbf{H}^{-1}(\Omega), \qquad (A\mathbf{u}, \mathbf{v}) = \tilde{a}(\mathbf{u}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega),$$

and

$$\Delta : \widehat{H}^1(\Omega) \longrightarrow (\widehat{H}^1(\Omega))^*, \qquad (\Delta p, q) = -(\nabla p, \nabla q), \quad \forall q \in \widehat{H}^1(\Omega).$$

The formal version of (2.10) and (2.11) is the following:

$$(2.13) \qquad\qquad A\mathbf{u} - \nabla p = \mathbf{f}, \qquad \text{in } \Omega,$$

$$(2.14) \qquad\qquad \nabla \cdot \mathbf{u} + t^2 \Delta p = 0, \qquad \text{in } \Omega,$$

$$(2.15) \qquad\qquad \mathbf{u} = 0, \quad \frac{\partial p}{\partial n} = 0, \qquad \text{on } \partial\Omega.$$

3. A LEAST SQUARES DISCRETIZATION

In this section we consider a least squares method solving (2.10) and (2.11). Assume that Ω is quasi-uniformly triangulated. The parameter h is the diameter of the largest of the triangles. Let $V_h \subset H_0^1(\Omega)$ and $\Pi_h \subset \widehat{H}^1(\Omega)$ be the subspaces consisting of continuous piecewise polynomials of degree $m-1$, with $m \geq 2$, and let $\mathbf{V}_h = V_h \times V_h \in \mathbf{H}_0^1(\Omega)$. One can choose to use different orders of polynomials for $\mathbf{V}_h$ and Π_h. In our numerical computations we use the continuous linear finite elements and hence $m = 2$.

First we need to define some discrete operators:

$$(3.1) \qquad A_h \;\; : \;\; \mathbf{H}_0^1(\Omega) \longrightarrow \mathbf{V}_h, \qquad (A_h\mathbf{u}, \mathbf{v}) = \tilde{a}(\mathbf{u}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_h.$$

$$(3.2) \qquad \nabla_h \;\; : \;\; L^2(\Omega) \longrightarrow \mathbf{V}_h, \qquad (\nabla_h p, \mathbf{v}) = -(p, \nabla \cdot \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_h.$$

$$(3.3) \qquad \triangle_h \;\; : \;\; \widehat{H}^1(\Omega) \longrightarrow \Pi_h, \qquad (\triangle_h p, q) = -(\nabla p, \nabla q), \quad \forall q \in \Pi_h.$$

$$(3.4) \qquad \nabla_h \cdot \;\; : \;\; \mathbf{H}_0^1(\Omega) \longrightarrow \Pi_h, \qquad (\nabla_h \cdot \mathbf{u}, q) = -(\mathbf{u}, \nabla q), \quad \forall q \in \Pi_h.$$

Define the discrete norm of order minus one, for $\mathbf{v} \in \mathbf{H}^{-1}(\Omega)$, by

$$(3.5) \qquad\qquad \| \mathbf{v} \|_{-1,h} = \sup_{\mathbf{w} \in \mathbf{V}_h} \frac{(\mathbf{v}, \mathbf{w})}{\| \mathbf{w} \|_1}.$$

It is easy to prove the following lemma [6].

Lemma 3.1. *For any $\mathbf{v} \in \mathbf{V}_h$,*

$$(3.6) \qquad\qquad \| \mathbf{v} \|_{-1,h}^2 = (T_h\mathbf{v}, \mathbf{v}),$$

where $T_h : \mathbf{H}^{-1}(\Omega) \longrightarrow \mathbf{V}_h$ is defined by $(\nabla T_h\mathbf{z}, \nabla \mathbf{x}) = (\mathbf{z}, \mathbf{x}), \forall \mathbf{x} \in \mathbf{V}_h$. Therefore, if $(\cdot, \cdot)_{-1,h}$ is the inner product induced by $\| \cdot \|_{-1,h}$, then

$$(\mathbf{v}, \mathbf{w})_{-1,h} = (T_h\mathbf{v}, \mathbf{w}), \qquad \forall \mathbf{v}, \mathbf{w} \in \mathbf{V}_h.$$

Let $B_h : \mathbf{H}^{-1}(\Omega) \longrightarrow \mathbf{V}_h$ be an easily computable and spectrally equivalent preconditioner for T_h. That is,

$$(3.7) \qquad C_0(T_h\mathbf{v}, \mathbf{v}) \leq (B_h\mathbf{v}, \mathbf{v}) \leq C_1(T_h\mathbf{v}, \mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{V}_h,$$

for some positive constants $C_0 < C_1$ independent of h. There is an extensive literature concerning the theory and techniques for building such preconditioners, cf. [5], [10].

Our least squares method will be based on the following bilinear form on $\widehat{H}^1(\Omega) \times \mathbf{H}_0^1(\Omega)$:

$$
\begin{aligned}
\langle\langle(p,\mathbf{u}),(q,\mathbf{v})\rangle\rangle \quad = \quad & (B_h(A_h\mathbf{u} - \nabla_h p), A_h\mathbf{v} - \nabla_h q) \\
(3.8) \qquad + \quad & h^2(A\mathbf{u} - \nabla p, A\mathbf{v} - \nabla q)_h \\
+ \quad & \gamma(\nabla_h \cdot \mathbf{u} + t^2 \triangle_h p, \nabla_h \cdot \mathbf{v} + t^2 \triangle_h q)_M.
\end{aligned}
$$

Here γ is a parameter to be adjusted for balancing and preconditioning the form and $(\cdot,\cdot)_M$ is an inner product, equivalent to the L^2 inner product on Π_h with constants independent of h, chosen to avoid computing the inverse of the mass matrix. The details will be discussed when we describe the implementation. Set $(\mathbf{u},\mathbf{v})_h = \sum_{e\in\Gamma_h}\int_e \mathbf{u}\cdot\mathbf{v}$, the elementwise inner product. When $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbf{L}^2(\Omega)$, $(\mathbf{u},\mathbf{v})_h = (\mathbf{u},\mathbf{v})$.

The least squares problem is the following: Find $(P,\mathbf{U}) \in \Pi_h \times \mathbf{V}_h$, such that

$$(3.9) \qquad \langle\langle(P,\mathbf{U}),(Q,\mathbf{V})\rangle\rangle = (B_h\mathbf{f}, A_h\mathbf{V} - \nabla_h Q) + h^2(\mathbf{f}, A\mathbf{V} - \nabla Q)_h,$$

for all $(Q,\mathbf{V}) \in \Pi_h \times \mathbf{V}_h$.

4. STABILITY

First we prove a stability result on the least squares bilinear form (3.8). For the rest of the paper, C will stand for a constant independent of both h or t which may take on different values at different occurrences.

Theorem 4.1. *For any* $(P,\mathbf{U}) \in \Pi_h \times \mathbf{V}_h$,

$$
\begin{aligned}
\| \mathbf{U} \|_1 + \| P \| &+ t \| P \|_1 + t^2 \| \triangle_h P \| \\
&\le C\{\| A_h\mathbf{U} - \nabla_h P \|_{-1,h} + h \| A\mathbf{U} - \nabla P \|_h + \| \nabla_h \cdot \mathbf{U} + t^2 \triangle_h P \|\} \\
&\le C\langle\langle(P,\mathbf{U}),(P,\mathbf{U})\rangle\rangle^{\frac{1}{2}} \\
&\le C(\| \mathbf{U} \|_1 + \| P \| + t \| P \|_1 + t^2 \| \triangle_h P \|),
\end{aligned}
$$

where $\| \cdot \|_h = (\cdot,\cdot)_h^{\frac{1}{2}}$.

Proof. We prove the first inequality; the second and third are obvious. It is well known that for all $p \in \widehat{L}^2(\Omega)$,

$$(4.1) \qquad\qquad \| p \| \le \sup_{\mathbf{v}\in\mathbf{H}_0^1(\Omega)} \frac{(p, \nabla\cdot\mathbf{v})}{\| \mathbf{v} \|_1}.$$

This result may be found in Girault and Raviart [17]. Fix $P \in \Pi_h$, $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$. Let $\mathbf{w} \in \mathbf{V}_h$ satisfy

$$h \| \mathbf{w} \|_1 + \| \mathbf{v} - \mathbf{w} \| \le Ch \| \mathbf{v} \|_1.$$

Then

$$
\begin{aligned}
| (P, \nabla\cdot\mathbf{v}) | \quad = \quad & | (P, \nabla\cdot(\mathbf{v}-\mathbf{w})) + (P, \nabla\cdot\mathbf{w}) | \\
\le \quad & | (\nabla P, \mathbf{v}-\mathbf{w}) | + | (\nabla_h P, \mathbf{w}) |,
\end{aligned}
$$

and

$$| (\nabla P, \mathbf{v}-\mathbf{w}) | \le \| \nabla P \| \| \mathbf{v}-\mathbf{w} \| \le Ch \| \nabla P \|_h \| \mathbf{v} \|_1,$$

$$| (\nabla_h P, \mathbf{w}) | \le \| \nabla_h P \|_{-1,h} \| \mathbf{w} \|_1 \le C \| \nabla_h P \|_{-1,h} \| \mathbf{v} \|_1.$$

Hence

$$\| P \| \leq C(\| \nabla_h P \|_{-1,h} + h \| \nabla P \|_h).$$
$$\leq C(\| A_h \mathbf{U} - \nabla_h P \|_{-1,h} + h \| A\mathbf{U} - \nabla P \|_h + \| \mathbf{U} \|_1).$$

Here we have used the inverse property $h \| \mathbf{U} \|_k \leq C \| \mathbf{U} \|_{k-1}$ for $\mathbf{U} \in \mathbf{V}_h$.

For $\mathbf{U} \in \mathbf{V}_h$, according to Korn's inequality,

$$\begin{aligned} C \| \mathbf{U} \|_1^2 &\leq \tilde{a}(\mathbf{U}, \mathbf{U}) = (A_h \mathbf{U}, \mathbf{U}) \\ &= (A_h \mathbf{U} - \nabla_h P, \mathbf{U}) + (\nabla_h P, \mathbf{U}) \\ &= (A_h \mathbf{U} - \nabla_h P, \mathbf{U}) - (P, \nabla_h \cdot \mathbf{U}) \\ &= (A_h \mathbf{U} - \nabla_h P, \mathbf{U}) - (P, \nabla_h \cdot \mathbf{U} + t^2 \triangle_h P) + t^2 (P, \triangle_h P). \end{aligned}$$

Thus

$$C \| \mathbf{U} \|_1^2 + t^2 \| P \|_1^2$$
$$\leq \| A_h \mathbf{U} - \nabla_h P \|_{-1,h} \| \mathbf{U} \|_1 + \| P \| \| \nabla_h \cdot \mathbf{U} + t^2 \triangle_h P \|.$$

Combining the last inequality and the estimate for $\|P\|$ above, using the Schwarz inequality and some simple manipulations, we get

$$\| \mathbf{U} \|_1 + \| P \| + t \| P \|_1$$
$$\leq C\{\| A_h \mathbf{U} - \nabla_h P \|_{-1,h} + h \| A\mathbf{U} - \nabla P \|_h + \| \nabla_h \cdot \mathbf{U} + t^2 \triangle_h P \|\}.$$

Now it is easy to see that the first inequality of the theorem holds and the theorem is proved. $\square$

This first theorem implies that the least squares form (3.8) is positive definite and hence the least squares problem (3.9) has a unique solution.

5. ERROR ESTIMATES

Theorem 5.1. *Let $(p, \mathbf{u})$ be the solution of (2.10) and (2.11), and $(P, \mathbf{U})$ be the solution of (3.9). Assume that $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $(p, \mathbf{u}) \in (H^m(\Omega) \cap \widehat{H}^1(\Omega)) \times (\mathbf{H}^m(\Omega) \cap \mathbf{H}_0^1(\Omega))$, and that $\mathbf{V}_h$ and Π_h consist of continuous piecewise polynomials of degree $m - 1$, $m \geq 2$. Then*

$$\| \mathbf{u} - \mathbf{U} \|_1 + \| p - P \| + t \| p - P \|_1 \leq Ch^{m-1}(\| \mathbf{u} \|_m + \| p \|_{m-1} + t \| p \|_m).$$

In particular, for $m = 2$,

$$\| \mathbf{u} - \mathbf{U} \|_1 + \| p - P \| + t \| p - P \|_1 \leq Ch \| \mathbf{f} \|.$$

Proof. Now

(5.1) $$\tilde{a}(\mathbf{u}, \mathbf{v}) - (\nabla p, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega),$$

so that

(5.2) $$(A_h \mathbf{u} - \nabla_h p, \mathbf{V}) = (\mathbf{f}, \mathbf{V}), \qquad \forall \mathbf{V} \in \mathbf{V}_h.$$

By (5.1) and density,

(5.3) $$(A\mathbf{u} - \nabla p, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{L}^2(\Omega).$$

Moreover, for any $Q \in \Pi_h \subset \widehat{H}^1(\Omega)$,

$$(\nabla_h \cdot \mathbf{u} + t^2 \triangle_h p, Q) = -(\mathbf{u}, \nabla Q) - t^2(\nabla p, \nabla Q) = (\nabla \cdot \mathbf{u}, Q) - t^2(\nabla p, \nabla Q) = 0,$$

and therefore

$$(5.4) \qquad \qquad \nabla_h \cdot \mathbf{u} + t^2 \triangle_h p = 0.$$

It follows from (3.9), (5.2), (5.3) and (5.4) that

$$(5.5) \qquad \qquad \langle\langle (e, \mathbf{E}), (Q, \mathbf{V}) \rangle\rangle = 0, \qquad \forall (Q, \mathbf{V}) \in \Pi_h \times \mathbf{V}_h,$$

where $e = p - P$ and $\mathbf{E} = \mathbf{u} - \mathbf{U}$. Define $\tilde{P} \in \Pi_h$ by

$$(5.6) \qquad \qquad (\nabla \tilde{P}, \nabla Q) = (\nabla p, \nabla Q), \qquad \forall Q \in \Pi_h.$$

Now $\tilde{P}$ is the $\widehat{H}^1(\Omega)$ projection of p into Π_h. Thus, since for some $\bar{P} \in \Pi_h$,

$$(5.7) \qquad \qquad \| p - \bar{P} \|_1 \leq Ch^{l-1} \| p \|_l, \qquad l \leq m,$$

we have

$$(5.8) \qquad \qquad \| p - \tilde{P} \|_1 \leq \| p - \bar{P} \|_1 \leq Ch^{l-1} \| p \|_l, \qquad l \leq m.$$

Moreover, by the standard duality argument (Aubin-Nitsche trick), we have

$$(5.9) \qquad \qquad \| p - \tilde{P} \| \leq Ch^{m-1} \| p \|_{m-1}.$$

Choose $\tilde{\mathbf{U}} \in \mathbf{V}_h$ such that

$$(5.10) \qquad \qquad \| \mathbf{u} - \tilde{\mathbf{U}} \| + h \| \mathbf{u} - \tilde{\mathbf{U}} \|_1 < Ch^m \| \mathbf{u} \|_m,$$

and let

$$(\tilde{e}, \tilde{\mathbf{E}}) = (\tilde{P} - P, \tilde{\mathbf{U}} - \mathbf{U}).$$

By the stability theorem and (5.5) we have that,

$$
\begin{aligned}
\| \tilde{\mathbf{E}} \|_1 + \| \tilde{e} \| + t \| \tilde{e} \|_1 &\leq C \langle\langle (\tilde{e}, \tilde{\mathbf{E}}), (\tilde{e}, \tilde{\mathbf{E}}) \rangle\rangle^{1/2} \\
&= C \langle\langle (\tilde{P} - p, \tilde{\mathbf{U}} - \mathbf{u}), (\tilde{e}, \tilde{\mathbf{E}}) \rangle\rangle^{1/2}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\| \tilde{\mathbf{E}} \|_1 + \| \tilde{e} \| + t \| \tilde{e} \|_1 &\leq C \langle\langle (\tilde{e}, \tilde{\mathbf{E}}), (\tilde{e}, \tilde{\mathbf{E}}) \rangle\rangle^{1/2} \\
&\leq C \langle\langle (\tilde{P} - p, \tilde{\mathbf{U}} - \mathbf{u}), (\tilde{P} - p, \tilde{\mathbf{U}} - \mathbf{u}) \rangle\rangle^{1/2} \\
&\leq C \{ \| A_h(\tilde{\mathbf{U}} - \mathbf{u}) - \nabla_h(\tilde{P} - p) \|_{-1,h} \\
&\quad + h \| A(\tilde{\mathbf{U}} - \mathbf{u}) - \nabla(\tilde{P} - p) \|_h \\
&\quad + \| \nabla_h \cdot (\tilde{\mathbf{U}} - \mathbf{u}) + t^2 \triangle_h (\tilde{P} - p) \| \}.
\end{aligned}
$$

But

$$\| A_h(\tilde{\mathbf{U}} - \mathbf{u}) \|_{-1,h} \leq C \| \tilde{\mathbf{U}} - \mathbf{u} \|_1 \leq Ch^{m-1} \| \mathbf{u} \|_m,$$

$$\| \boldsymbol{\nabla}_h(\tilde{P} - p) \|_{-1,h} \leq \| \tilde{P} - p \| \leq Ch^{m-1} \| p \|_{m-1},$$

$$h \| \boldsymbol{\nabla}(\tilde{P} - p) \|_h \leq h \| \tilde{P} - p \|_1 \leq Ch^{m-1} \| p \|_{m-1}.$$

In [7], it is shown that

$$h \| A(\tilde{\mathbf{U}} - \mathbf{u}) \|_h \leq Ch^{m-1} \| \mathbf{u} \|_m.$$

For $m = 2$, the inequality above is trivial. Because of the definition of $\tilde{P}$,

$$\triangle_h \tilde{P} = \triangle_h p,$$

and thus we have
$$\| \nabla_h \cdot (\tilde{\mathbf{U}} - \mathbf{u}) + t^2 \triangle_h (\tilde{P} - p) \|$$
$$= \| \nabla_h \cdot (\tilde{\mathbf{U}} - \mathbf{u}) \| \leq C \| \tilde{\mathbf{U}} - \mathbf{u} \|_1 \leq C h^{m-1} \| \mathbf{u} \|_m .$$
Therefore,

(5.11) $$\| \tilde{\mathbf{E}} \|_1 + \| \tilde{e} \| + t \| \tilde{e} \|_1 \leq C h^{m-1}(\| \mathbf{u} \|_m + \| p \|_{m-1}).$$

By (5.10), (5.8), (5.9), (5.11), and the triangle inequality,
$$\| \mathbf{E} \|_1 + \| e \| + t \| e \|_1 \leq C h^{m-1}(\| \mathbf{u} \|_m + \| p \|_{m-1} + t \| p \|_m).$$
This proves the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

Next we give an error estimate theorem for the solution of the complete system.

Theorem 5.2. *Let* $(r, \boldsymbol{\phi}, p, \omega) \in H_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \times \widehat{H}^1(\Omega) \times H_0^1(\Omega)$ *be the solution of system* (2.3), (2.4), (2.5) *and* (2.6). $(r_h, \boldsymbol{\phi}_h, p_h, \omega_h) \in V_h \times \mathbf{V}_h \times \Pi_h \times V_h$, $\mathbf{u}_h^\perp = \boldsymbol{\phi}_h$, *and* $\mathbf{f}_h^\perp = \nabla r_h$, *such that*

(5.12) $$(\nabla r_h, \nabla \mu) = (g, \mu),$$

(5.13) $$\langle\langle (p_h, \mathbf{u}_h), (Q, \mathbf{V}) \rangle\rangle = (B_h \mathbf{f}_h, A_h \mathbf{V} - \nabla_h Q) + h^2(\mathbf{f}_h, A\mathbf{V} - \nabla Q)_h,$$

(5.14) $$(\nabla \omega_h, \nabla s) = (\boldsymbol{\phi}_h + t^2 \nabla r_h, \nabla s),$$

for all $\mu \in V_h$, $(Q, \mathbf{V}) \in \Pi_h \times \mathbf{V}_h$ *and* $s \in V_h$. *Then*
$$\| r - r_h \|_1 \leq C h \| g \|,$$
$$\| r - r_h \| + \| \mathbf{u} - \mathbf{u}_h \|_1 + \| p - p_h \| + t \| p - p_h \|_1 \leq C h \| g \|_{-1},$$
$$\| \omega - \omega_h \|_1 \leq C h(\| g \|_{-1} + t^2 \| g \|).$$

Proof. The first estimate for $r - r_h$ is standard. By the duality argument we also have
$$\| r - r_h \| \leq C h \| g \|_{-1} .$$
Let $(\tilde{\mathbf{u}}, \tilde{p})$ be the solution of (2.10) and (2.11) with $\mathbf{f}$ replaced by $\mathbf{f}_h$. Then,

(5.15) $$\quad \tilde{a}(\mathbf{u} - \tilde{\mathbf{u}}, \mathbf{v}) - (\nabla(p - \tilde{p}), \mathbf{v}) = (\mathbf{f} - \mathbf{f}_h, \mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega),$$

(5.16) $$(\nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}), q) - t^2 (\nabla(p - \tilde{p}), \nabla q) = 0, \qquad \forall q \in \widehat{H}^1(\Omega).$$

Let $\mathbf{v} = \mathbf{u} - \tilde{\mathbf{u}}$ and $q = p - \tilde{p}$ in the two equations. Adding them together we have,
$$\tilde{a}(\mathbf{u} - \tilde{\mathbf{u}}, \mathbf{u} - \tilde{\mathbf{u}}) + t^2(\nabla(p - \tilde{p}), \nabla(p - \tilde{p})) = (\mathbf{f} - \mathbf{f}_h, \mathbf{u} - \tilde{\mathbf{u}}),$$
from which it follows that,
$$\| \mathbf{u} - \tilde{\mathbf{u}} \|_1 + t \| p - \tilde{p} \|_1 \leq C \| \mathbf{f} - \mathbf{f}_h \|_{-1} .$$
In order to bound the L^2 norm of $p - \tilde{p}$ we use (4.1). Thus
$$\begin{aligned} \| p - \tilde{p} \| &\leq C \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{(\nabla(p - \tilde{p}), \mathbf{v})}{\| \mathbf{v} \|_1} \\ &= C \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{(\mathbf{f} - \mathbf{f}_h, \mathbf{v}) - \tilde{a}(\mathbf{u} - \tilde{\mathbf{u}}, \mathbf{v})}{\| \mathbf{v} \|_1} \\ &\leq C(\| \mathbf{f} - \mathbf{f}_h \|_{-1} + \| \mathbf{u} - \tilde{\mathbf{u}} \|_1). \end{aligned}$$

But

(5.17) $\| \mathbf{f} - \mathbf{f}_h \|_{-1} = \| \nabla(r - r_h) \|_{-1} \leq \| r - r_h \| \leq Ch \| g \|_{-1} \,.$

Hence

(5.18) $\| \mathbf{u} - \tilde{\mathbf{u}} \|_1 + \| p - \tilde{p} \| + t \| p - \tilde{p} \|_1 \leq C \| \mathbf{f} - \mathbf{f}_h \|_{-1} \leq Ch \| g \|_{-1} \,.$

By the previous theorem,

(5.19) $\| \mathbf{u}_h - \tilde{\mathbf{u}} \|_1 + \| p_h - \tilde{p} \| + t \| p_h - \tilde{p} \|_1 \leq Ch \| \mathbf{f}_h \| \leq Ch \| g \|_{-1} \,.$

Now (5.18) and (5.19) imply that

(5.20) $\| \mathbf{u}_h - \mathbf{u} \|_1 + \| p_h - p \| + t \| p_h - p \|_1 \leq Ch \| g \|_{-1},$

which completes the proof of the next to last inequality of the theorem.

Finally, we need to prove the estimate for $\omega - \omega_h$. Let $\tilde{\omega}$ be the solution of

(5.21) $(\nabla \tilde{\omega}, \nabla s) = (\phi_h + t^2 \nabla r, \nabla s), \qquad \forall s \in H_0^1(\Omega).$

Then $\tilde{\omega} \in H^2(\Omega)$ and

(5.22) $\| \omega - \tilde{\omega} \|_1 \leq \| \phi - \phi_h \| \leq Ch \| g \|_{-1} \,.$

Let $\theta = \tilde{\omega} - t^2 r$, $\theta_h = \omega_h - t^2 r_h$, then θ and θ_h are solutions of the problems

$$(\nabla \theta, \nabla s) = (\phi_h, \nabla s), \qquad \forall s \in H_0^1(\Omega),$$

and

$$(\nabla \theta_h, \nabla s) = (\phi_h, \nabla s), \qquad \forall s \in V_h,$$

respectively. (Here we used the fact that both r_h and ω_h are in the same subspace V_h.) It is standard that

$$\| \theta - \theta_h \|_1 \leq Ch \| \nabla \cdot \phi_h \| \leq Ch \| g \|_{-1},$$

and

(5.23) $\| \tilde{\omega} - \omega_h \|_1 \leq \| \theta - \theta_h \|_1 + t^2 \| r - r_h \|_1 \leq Ch(\| g \|_{-1} + t^2 \| g \|).$

The proof is completed by combining (5.22) and (5.23).   □

If we replace $\mathbf{f}_h^{\perp} = \nabla r_h$ by $\mathbf{f}_h^{\perp} = \nabla_h r_h$ in this theorem, the theorem holds. In fact, the only thing that needs to be verified is (5.17) and (5.19). We observe that

$$\begin{aligned}
\| \mathbf{f} - \mathbf{f}_h \|_{-1} &= \| \nabla r - \nabla_h r_h \|_{-1} \\
&\leq C \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{(\nabla r - \nabla_h r_h, \mathbf{v})}{\| \mathbf{v} \|_1} \\
&= C \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{(\nabla r - \nabla_h r_h, \mathbf{v} - \mathbf{v}_h) + (\nabla r - \nabla_h r_h, \mathbf{v}_h)}{\| \mathbf{v} \|_1} \\
&\leq C(\| r \|_1 h + \| r - r_h \|) \\
&\leq Ch \| g \|_{-1} \,.
\end{aligned}$$

Here $\mathbf{v}_h \in \mathbf{V}_h$ is chosen to satisfy

$$h \| \mathbf{v}_h \|_1 + \| \mathbf{v} - \mathbf{v}_h \| \leq Ch \| \mathbf{v} \|_1 \,.$$

It is obvious that $\| \nabla_h r_h \| \leq \| \nabla r_h \| \leq \| \nabla r \|.$

Although this replacement seems unnecessary and computationally uneconomical, it might be needed to prove an error estimate under weaker regularity assumptions.

6. Implementation

First we set up some terminology. A partial differential equation can be described as

$$Lw = z,$$

where $w \in H_E$ is the solution, H_E is a Sobolev space, $z \in F$ is the right hand side, $F \subset H_E^*$, H_E^* is the dual space of H_E with respect to L^2 and $L : H_E \to F$ is the differential operator.

A conforming finite element subspace W_h is a finite dimensional subspace of H_E. Let $\{\varphi^1, \cdots, \varphi^m\}$ be a basis of W_h. If a function $w \in W_h$ is an approximate solution, a correction to be added to an approximate solution, or a test function in a weak formulation, we usually compute its coefficient vector $w_c = (\alpha_j)$ under the basis $\{\varphi^1, \cdots, \varphi^m\}$, defined by $w = \sum_{j=1}^m \alpha_j \varphi^j$. For a function z in W_h or F, which is the right hand side, a residual, or an image of L, we usually need to compute its dual vector $z_d = (\beta^i)$, with $\beta^i = (\varphi^i, z)$. In the weak formulation

$$(Lw, v) = (z, v), \qquad \forall v \in H_E,$$

where Lw and z are in the dual space of H_E. So, it is only natural to consider their dual vectors. Although the L^2 projections of Lw and z into W_h are in W_h, they still represent functionals on W_h, not functions in W_h. And it still does not make sense to consider their coefficient vectors under a basis.

It is well known that, as a matrix action, a preconditioner takes a dual vector and produces a coefficient vector.

1. The computation of the bilinear form

$$(B_h(A_h \mathbf{U} - \nabla_h P), A_h \mathbf{V} - \nabla_h Q)$$

has been discussed in [7]. For the convenience of the readers, we include a discussion about it here. Let $\{\xi^1, \cdots, \xi^n\}$ be a basis of $\mathbf{V}_h$, and $\{\eta^1, \cdots, \eta^k\}$ be a basis of $\Pi_h \oplus \mathbf{R}$. For $\mathbf{U}, \mathbf{V} \in \mathbf{V}_h$ and $P, Q \in \Pi_h \oplus \mathbf{R}$, $\mathbf{U}_c, \mathbf{V}_c$ and P_c, Q_c are the coefficient vectors of $\mathbf{U}, \mathbf{V}$ and P, Q, respectively. To compute $(B_h(A_h\mathbf{U} - \nabla_h P), A_h\mathbf{V} - \nabla_h Q)$, first we need to compute the entries of the the dual vector of $A_h\mathbf{U} - \nabla_h P$:

$$(\xi^i, A_h\mathbf{U} - \nabla_h P) = \widetilde{a}(\xi^i, \mathbf{U}) + (\nabla \cdot \xi^i, P) = (K\mathbf{U}_c)_i + (SP_c)_i,$$

where $K \doteq (\widetilde{a}(\xi^i, \boldsymbol{\xi}^j))$ is the stiffness matrix, $S = ((\nabla \cdot \xi^i, \eta^j))$.

Let $\mathbf{w}_c$ be the coefficient vector of $\mathbf{w} = B_h(A_h\mathbf{U} - \nabla_h P)$, then $\mathbf{w}_c = B_1(K\mathbf{U}_c + SP_c)$, where B_1 is the matrix corresponding to B_h. Now it is easy to see that

$$
\begin{aligned}
&(B_h(A_h\mathbf{U} - \nabla_h P), A_h\mathbf{V} - \nabla_h Q) \\
&\quad = \widetilde{a}(\mathbf{w}, \mathbf{V}) + (\nabla \cdot \mathbf{w}, Q) \\
&\quad = \widetilde{a}(\mathbf{V}, \mathbf{w}) + (Q, \nabla \cdot \mathbf{w}) \\
&\quad = \mathbf{V}_c^T K \mathbf{w}_c + Q_c^T S^T \mathbf{w}_c \\
&\quad = (\mathbf{V}_c^T K + Q_c^T S^T) B_1 (K\mathbf{U}_c + SP_c) \\
&\quad = \begin{pmatrix} \mathbf{V}_c \\ Q_c \end{pmatrix}^T \begin{pmatrix} K \\ S^T \end{pmatrix} B_1 \begin{pmatrix} K & S \end{pmatrix} \begin{pmatrix} \mathbf{U}_c \\ P_c \end{pmatrix}.
\end{aligned}
$$

The matrix $\mathcal{A}_1 = \begin{pmatrix} K \\ S^T \end{pmatrix} B_1 \begin{pmatrix} K & S \end{pmatrix}$ is usually a full matrix, which is not easily assembled. But its action on a coefficient vector of the form $\begin{pmatrix} \mathbf{U}_c \\ P_c \end{pmatrix}$ is clearly computable.

2. Now we consider the computation of

$$(\nabla_h \cdot \mathbf{U} + t^2 \triangle_h P, \nabla_h \cdot \mathbf{V} + t^2 \triangle_h Q)_M.$$

The inner product $(\cdot, \cdot)_M$ is defined as follows.

Suppose that $R \in \Pi_h$ is given in the form of $\nabla_h \cdot \mathbf{U}$ or $\triangle_h P$. Then it is only convenient to compute its dual vector. To get the coefficient vector, one has to multiply the dual vector by the inverse of the mass matrix $M = ((\eta^i, \eta^j))$.

Let $\mathbf{U}_c, \mathbf{V}_c$ and P_c, Q_c be the coefficient vectors of $\mathbf{U}, \mathbf{V} \in \mathbf{V}_h$ and $P, Q \in \Pi_h$, respectively, as above. It is clear that the dual vector of $\nabla_h \cdot \mathbf{U} + t^2 \triangle_h P$ is given by

$$(\eta^i, \nabla_h \cdot \mathbf{U} + t^2 \triangle_h P) = (S^T \mathbf{U}_c - t^2 K_p P_c)_i,$$

where S is as defined before, $K_p = ((\nabla \eta^i, \nabla \eta^j))$. Similarly, the dual vector of $\nabla_h \cdot \mathbf{V} + t^2 \triangle_h Q$ is given by

$$(\eta^i, \nabla_h \cdot \mathbf{V} + t^2 \triangle_h Q) = (S^T \mathbf{V}_c - t^2 K_p Q_c)_i.$$

If we want to compute the L^2 inner product

$$(\nabla_h \cdot \mathbf{U} + t^2 \triangle_h P, \nabla_h \cdot \mathbf{V} + t^2 \triangle_h Q)$$
$$= (S^T \mathbf{U}_c - t^2 K_p P_c)^T M^{-1} (S^T \mathbf{V}_c - t^2 K_p Q_c),$$

at some point we have to use M^{-1}, which is not a pleasant thing to do. But M^{-1} is spectrally equivalent to $h^{-2} I$, i.e.,

$$C h^{-2} \chi^T \chi \leq \chi^T M^{-1} \chi \leq C h^{-2} \chi^T \chi$$

for any $\chi \in \mathbf{R}^k$. Therefore, we define the new inner product $(\cdot, \cdot)_M$ on $(\Pi_h \oplus \mathbf{R}) \times (\Pi_h \oplus \mathbf{R})$ by

$$(R, R)_M = h^{-2} R_d^T R_d.$$

It is easy to see that

(6.1) $$C(R, R) \leq (R, R)_M \leq C(R, R).$$

Now

$$\gamma(\nabla_h \cdot \mathbf{U} + t^2 \triangle_h P, \nabla_h \cdot \mathbf{V} + t^2 \triangle_h Q)_M$$
$$= \gamma h^{-2} (S^T \mathbf{U}_c - t^2 K_p P_c)^T (S^T \mathbf{V}_c - t^2 K_p Q_c)$$

is easily computable. Because of (6.1), in the proof of the theorems, we can switch between $(\cdot, \cdot)$ and $(\cdot, \cdot)_M$ any time we want to. The matrix corresponding to the right hand side of the last equality is

$$\mathcal{A}_2 = \gamma h^{-2} \begin{pmatrix} S \\ -t^2 K_p \end{pmatrix} \begin{pmatrix} S^T & -t^2 K_p \end{pmatrix}.$$

For the other term in the least squares bilinear form (3.8), $h^2(A\mathbf{U} - \nabla P, A\mathbf{V} - \nabla Q)$, if linear elements are used, the matrix is

$$\mathcal{A}_3 = h^2 \begin{pmatrix} 0 & 0 \\ 0 & K_p \end{pmatrix}.$$

Therefore the matrix corresponding to the form (3.8) is

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3.$$

3. Next we consider the preconditioning of the least squares bilinear form (3.8). Recall the stability theorem: For any $(\mathbf{U}, P) \in \mathbf{V}_h \times \Pi_h$,

$$\| \mathbf{U} \|_1^2 + \| P \|^2 + t^2 \| P \|_1^2 + t^4 \| \triangle_h P \|^2 \asymp \langle\langle (P, \mathbf{U}), (P, \mathbf{U}) \rangle\rangle,$$

where $\asymp$ means that the two sides are spectrally equivalent. This tells us that the system can be preconditioned diagonally. We already have a preconditioner of the form

$$D(\mathbf{U}, \mathbf{U}) = \| \mathbf{U} \|_1^2,$$

namely B_h, which is used in the least squares formulation. For the other component, P,

$$C(\| P \|^2 + t^2 \| P \|_1^2 + t^4 \| \triangle_h P \|^2)$$
$$\leq (P - t^2 \triangle_h P, P - t^2 \triangle_h P) = ((I - t^2 \triangle_h)^2 P, P)$$
$$\leq C(\| P \|^2 + t^2 \| P \|_1^2 + t^4 \| \triangle_h P \|^2).$$

Hence, what we need is a preconditioner of the operator $(I - t^2 \triangle_h)^2$ on Π_h. Of course, even if B is a preconditioner of $I - t^2 \triangle_h$, B^{-2} does not have to be spectrally equivalent to $(I - t^2 \triangle_h)^2$. Fortunately, it is proved that such a kind of B does exist [8]. A multigrid V-cycle with variable numbers of smoothing is one of the good choices. Besides, when t is small, $(I - t^2 \triangle_h)^2$ is not really ill conditioned. The numerical experiments suggest that even a V-cycle with constant 2 smoothing will do a very good job. To summarize, we have

$$\| \mathbf{U} \|_1^2 + \| P \|^2 + t^2 \| P \|_1^2 + t^4 \| \triangle_h P \|^2 \asymp (B_h^{-1} \mathbf{U}, \mathbf{U}) + (B_P^{-2} P, P),$$

where B_P is a preconditioner of $I - t^2 \triangle_h$, such that B_P^2 is a preconditioner of $(I - t^2 \triangle_h)^2$. If B_2 is the matrix corresponding to B_P, then the matrix of the diagonal preconditioner is

$$\mathcal{B} = \left(\begin{array}{cc} B_1 & 0 \\ 0 & B_2^2 \end{array} \right)$$

7. NUMERICAL EXPERIMENTS

Numerical experiments are done for different plate thicknesses on the unit square $\Omega = [0, 1] \times [0, 1]$. First, the domain is divided into $2^J \times 2^J$ squares of size $h \times h$, with $h = 2^{-J}$. Then, each small square is divided into two triangles by the NW-SE diagonal. Continuous piecewise linear elements are used for all the unknowns. As is expected, the numerical solution does converge to the exact solution at the optimal rate without locking. The standard multigrid method is used to compute the solutions of the two elliptic problems. The conjugate gradient method with diagonal preconditioning is used to solve (5.13). The hard work is the tuning of the preconditioner.

In general, the condition number of a block diagonally preconditioned multivariable problem depends heavily on the weighting of the diagonal blocks. Even though the condition number is proved to be independent of the mesh size h, the constants can be large and the iteration can behave very poorly if the subsystems are

unbalanced or the parameters of the preconditioner do not match the parameters of the original problem. For the Reissner-Mindlin plate model (2.1)

$$(7.1) \quad a(\boldsymbol{\phi}, \boldsymbol{\psi}) + \lambda t^{-2}(\boldsymbol{\phi} - \nabla\omega, \boldsymbol{\psi} - \nabla\mu) = (g, \mu), \qquad \forall (\mu, \boldsymbol{\psi}) \in H_0^1(\Omega) \times \mathbf{H}_0^1(\Omega),$$

we define $\widetilde{t}$ and $\widetilde{g}$ by

$$\widetilde{t}^{-2} = \frac{12(1 - \nu^2)\lambda t^{-2}}{E}, \qquad\qquad \widetilde{g} = \frac{12(1 - \nu^2)g}{E},$$

and then consider the equivalent dimensionless problem

$$(7.2) \quad \overline{a}(\boldsymbol{\phi}, \boldsymbol{\psi}) + \widetilde{t}^{-2}(\boldsymbol{\phi} - \nabla\omega, \boldsymbol{\psi} - \nabla\mu) = (\widetilde{g}, \mu), \qquad \forall (\mu, \boldsymbol{\psi}) \in H_0^1(\Omega) \times \mathbf{H}_0^1(\Omega).$$

The meaning of $\overline{a}(\cdot, \cdot)$ is clear in the context. Instead of four parameters $E, \lambda, \nu,$ and t, we now need to deal with only two parameters ν and $\widetilde{t}$. The parameter γ in the form (3.8) needs to be tuned to balance the subsystems corresponding to the two equations of the Stokes-like problem. For the rest of this section, we will use $a(\cdot, \cdot)$ for $\overline{a}(\cdot, \cdot)$, t for $\widetilde{t}$, and g for $\widetilde{g}$, just to simplify the notation.

In the diagonal preconditioner $\begin{pmatrix} B_h & 0 \\ 0 & B_P^2 \end{pmatrix}$, $B_h : \mathbf{H}_0^1(\Omega) \longrightarrow \mathbf{V}_h$ is a multigrid V-cycle preconditioner for the operator $-\triangle$ on $\mathbf{H}_0^1(\Omega)$. The number of smoothings can be one, but we chose two because we found that the case of two smoothings is more efficient in terms of total work. B_P is a multigrid V-cycle preconditioner for $I - t^2\triangle_h$ on Π_h. We also choose two smoothings in practice, although more smoothing is needed on the coarse grids to prove that B_P^{-2} is spectrally equivalent to $(I - t^2\triangle_h)^2$. The numerical experiment shows that three smoothings work better when $h << t$.

Let X be the solution of $\mathcal{A}X = F$, X_m be the iterates, $R^m = F - \mathcal{A}X_m$ be the residual. Then there are $0 < C_0 < C_1$, independent of h, such that

$$C_0(\mathcal{A}(X - X_m), X - X_m) \le (\mathcal{B}R^m, R^m) \le C_1(\mathcal{A}(X - X_m), X - X_m),$$

and so

$$\frac{C_0}{C_1} \frac{(\mathcal{B}R^m, R^m)}{(\mathcal{B}R^0, R^0)} \le \frac{(\mathcal{A}(X - X_m), X - X_m)}{(\mathcal{A}(X - X_0), X - X_0)} \le \frac{C_1}{C_0} \frac{(\mathcal{B}R^m, R^m)}{(\mathcal{B}R^0, R^0)}.$$

Therefore

$$\frac{(\mathcal{B}R^m, R^m)}{(\mathcal{B}R^0, R^0)} \le \varepsilon$$

is used to stop the iteration. In the conjugate gradient method, $(\mathcal{B}R^m, R^m)$ is computed as part of the iteration, so the error estimator is free of cost.

To reveal the real error reduction rate and condition number of the preconditioned system, we choose $\varepsilon = 10^{-8}$. It is well known that the error reduction in the first few iterations does not reflect the condition number of the system. Too large a value of ε might produce an unreliable estimate of the condition number.

Table 1 reports the iteration numbers of the conjugate gradient iterations solving (5.13) for mesh size $h = 1/8, 1/16, 1/32, 1/64, 1/128$ and plate thickness $t = 0, 0.001, 0.01, 0.1, 1.0$, with $\nu = 0.25, \gamma = 0.36, \varepsilon = 10^{-8}$. The condition number of $\mathcal{B}\mathcal{A}$ can be estimated by

$$K(\mathcal{B}\mathcal{A}) \le ((1 + \theta)/(1 - \theta))^2, \qquad \theta = \varepsilon^{\frac{1}{m}},$$

where m is the iteration number. For all the cases in the table, $K(\mathcal{B}\mathcal{A}) \in (7.4, 26.8)$. The condition numbers are uniformly bounded, although they are larger when

TABLE 1. Iteration numbers of the PCG, $\nu = 0.25, \gamma = 0.36, \varepsilon = 10^{-8}$

$h =$	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
$t = 0.000$	24	26	30	34	35
$t = 0.001$	24	26	30	34	35
$t = 0.010$	25	27	28	34	39
$t = 0.100$	29	42	41	40	41
$t = 1.000$	33	40	43	45	47

$h \ll t$. The reason for this is reflected in the proof of the stability theorem, which we use to construct the diagonal preconditioner; the constants change when $t \neq 0$.

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