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NEW RESULTS IN EQUAL SUMS OF LIKE POWERS

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ABSTRACT. This paper reports on new results for the equation

$$\sum_{i=1}^{m} a_i^k = \sum_{j=1}^{n} b_j^k,$$

i.e., equal sums of like powers. Since the 1967 Lander, Parkin and Selfridge survey paper [4], few other numeric results have been published (see Elkies [6] and Ekl [3]). The present paper reports on several new smallest primitive solutions. Further, search limits have been extended in many cases, and tables of solutions are presented. Additionally, new solutions to the same class of problems in *distinct* integers have been discovered.

Introduction. Diophantine equations of the form

$$z = \sum_{i=1}^{m} a_i^k = \sum_{j=1}^{n} b_j^k$$

have long been studied [1], [2], [10]. The most comprehensive report to date is [4]. In this paper, we will update the list of known solutions to this equation by presenting several new results. All results were discovered via computer search.

Further, higher search limits will be presented for all cases.

Additionally, this paper addresses the case where all values in the solution are *distinct.*

Notations, Conventions. We will use the notations and conventions outlined in [4], that being $m \leq n$, $[k.m.n]$ as the first primitive solution and $[k.m.n]_r$ as the r^{th} primitive solution, ranked by the magnitude of the sum z. $a_i \neq b_j$ for all i, j, and all $a_i, b_j > 0$. The definition of a primitive solution is that the greatest common divisor $(a_1, \ldots, a_m, b_1, \ldots, b_n) = 1$.

Methods. The method used for the cases $[k.m.m]$, $[k.m.m+1]$ and $[k.m.m+2]$ relies on keeping a sorted, RAM resident, binary tree of values $\sum_{i=1}^{m} a_i^k$, using a modified AVL tree algorithm, precalculating k^{th} powers of integers, and utilizing large integer numeric software. See [3] for more detail.

The method used for the case $m < n+2$ is primarily a brute-force decomposition, made more efficient by eliminating decomposition possibilities by using congruential contraints, and by keeping a table of all decompositions below a fixed limit that use less than a fixed number of terms.

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Table 1: Smallest Primitive Solutions to $\sum_1^m a_i^k = \sum_1^n b_j^k$

$k.m.n$	$[k.m.n]_1$	Range [Ref]
4.1.3	$(422481) = (414560, 217519, 95800)$	[6]
4.2.2	$(159, 58) = (134, 133)$	[10]
5.1.4	$(144) = (133, 110, 84, 27)$	2.60×10^{14} [4]
5.2.3	$(14132, 220) = (14068, 6237, 5027)$	5.63×10^{20} [7]
5.3.3	$(67, 28, 24) = (62, 54, 3)$	[9]
6.1.7	$(1141) = (1077, 894, 702, 474, 402, 234, 74)$	3.16×10^{27} [4]
6.2.6	$(241, 17) = (218, 210, 118, 63^2, 42)$	1.15×10^{16} **New**
6.3.3	$(23, 15, 10) = (22, 19, 3)$	1.32×10^{19} [8]
7.1.8	$(102) = (90, 85, 83, 64, 58, 53, 35, 12)$	1.95×10^{14} [4]
7.2.8	$(33, 10) = (31, 28, 20, 15^2, 7, 6, 5)$	2.50×10^{14} [4]
7.3.5	$(96, 41, 17) = (86, 77^2, 68, 56)$	4.39×10^{18} **New**
7.4.4	$(149, 123, 14, 10) = (146, 129, 90, 15)$	1.54×10^{18} [3]
8.1.11	$(125) = (112, 106, 96, 93, 92, 70, 66, 44^2, 18, 14)$	5.96×10^{16} [4]
8.2.8	$(129, 95) = (128, 92, 86, 82, 74, 57, 55, 20)$	8.33×10^{16} **New**
8.3.8	$(50, 17, 8) = (47, 40, 38^2, 16^2, 12, 6)$	3.90×10^{13} [4]
8.4.6	$(47, 29, 12, 5) = (45, 40, 30, 26, 23, 3)$	1.17×10^{17} **New**
8.5.5	$(43, 20, 11, 10, 1) = (41, 35, 32, 28, 5)$	7.23×10^{16} [5]
9.1.14	$(66) = (63, 54, 51, 49, 38, 35, 29, 24, 21, 12, 10, 7, 2, 1)$	8.45×10^{16} **New**
9.2.12	$(21, 15) = (19^2, 17, 16, 7, 4, 3^2, 2^4)$	1.56×10^{16} [4]
9.3.9	$(38^2, 3) = (41, 23, 20^2, 18, 13^2, 12, 9)$	7.42×10^{15} **New**
9.4.9	$(38, 31, 12, 2) = (36, 32^2, 30, 15, 13, 8, 4, 3)$	3.27×10^{14} **New**
9.5.7	$(35, 26, 15^2, 12) = (33, 32, 24, 16, 14, 8, 6)$	4.73×10^{16} **New**
9.6.6	$(23, 18, 14, 13^2, 1) = (22, 21, 15, 10, 9, 5)$	4.73×10^{16} [4]
10.1.22	$(33) = (30^2, 26^2, 23, 21, 19, 18, 13^2, 12^2, 10^5, 9^2, 7, 6, 3)$	2.16×10^{16} **New**
10.2.15	$(35, 3) = (33, 32, 24, 21, 20^2, 13^3, 12, 11, 9, 7, 1^2)$	2.75×10^{15} **New**
10.3.14	$(30, 28, 4) = (31, 23, 20^2, 17^2, 16, 10, 9^3, 5, 2^2)$	8.86×10^{14} **New**
10.4.15	$(23^4) = (26, 18^5, 17^3, 15, 12, 6, 4^3)$	9.53×10^{13} **New**
10.5.16	$(20, 11, 8, 3, 1) = (18^2, 17, 16, 10, 7^2, 4^6, 2^2)$	9.53×10^{13} **New**
10.6.16	$(18, 12, 11, 10, 3, 2) = (17, 16, 13^4, 7^4, 6^4, 5, 4)$	1.66×10^{13} **New**
10.7.7	$(38, 33, 26^2, 15, 8, 1) = (36, 35, 32, 29, 24, 23, 22)$	4.27×10^{16} **New** *

*Moessner [9] had found *a* solution to [10.7.7], but it was not known if it was the miminal solution. This paper presents the previously undiscovered $[10.7.7]_1$.

Main Results. Table 1 shows the main results presented in this paper. The third column in Table 1 indicates which results are new. Further, we have listed references and the search limit achieved. For example, to look for a solution to [6.1.6], read the table for [6.1.n], which in this case is [6.1.7], and start searching in the range 3×10^{27}.

The solutions in Table 1 can be understood via the following example:

$$[6.2.6] : 241^6 + 17^6 = 218^6 + 210^6 + 118^6 + 63^6 + 63^6 + 42^6.$$

Sixth Powers. [6.2.6]: The smallest primitive solution is presented in Table 1. The smallest primitive solution in distinct integers is presented in Table 10.

[6.3.3]: 87 solutions have been found (not presented here), including $[6.3.3]_1$, published in [8] and solutions two through nine, published in [4].

[6.3.4]: Table 2 gives the 28 smallest primitive solutions for [6.3.4]. The first five were published in [4].

Table 2: Smallest Primitive solutions to [6.3.4]		
r	Solution $[6.3.4]_r$	
1	(73,58,41)	(70,65,32,15)
2	(85,62,61)	(83,69,56,52)
3	(85,74,61)	(87,71,56,26)
4	(90,88,11)	(92,78,74,21)
5	(95,83,26)	(101,28,24,23)
6	(130,44,23)	(119,108,86,38)
7	(125,114,38)	(126,104,93,68)
8	(205,113,18)	(198,148,133,39)
9	(211,123,34)	(210,134,73,39)
10	(212,164,103)	(217,130,114,8)
11	(222,34,25)	(217,156,96,68)
12	(218,167,29)	(224,107,102,65)
13	(226,110,17)	(224,143,72,34)
14	(244,123,112)	(238,180,91,72)
15	(241,172,156)	(246,145,132,56)
16	(257,155,6)	(252,181,143,114)
17	(265,147,12)	(231,221,210,114)
18	(260,218,185)	(276,152,112,25)
19	(305,85,66)	(273,267,172,122)
20	(312,241,33)	(315,228,99,2)
21	(331,234,59)	(306,294,151,95)
22	(332,243,43)	(338,177,168,95)
23	(351,265,221)	(336,309,169,73)
24	(365,137,126)	(360,234,175,133)
25	(360,265,200)	(336,318,212,169)
26	(348,325,36)	(357,276,276,82)
27	(373,288,104)	(363,292,266,120)
28	(386,113,62)	(378,260,209,88)

Table 3: Smallest Primitive solutions to [7.4.4]		
r	Solution $[7.4.4]_r$	
1	(149,123,14,10)	(146,129,90,15)
2	(194,150,105,23)	(192,152,132,38)
3	(354,112,52,19)	(343,281,46,35)

Seventh Powers. [7.3.5]: Only one solution is known, and is presented in Table 1. This smallest primitive solution was found by looking at [7.5.5].

[7.4.4]: One additional solution is known for [7.4.4] beyond the solutions given in Ekl [3]. The three smallest primitive solutions are given in Table 3.

[7.4.5]: Table 4 gives the nine smallest primitive solutions for [7.4.5]. $[7.4.5]_1$ was published in [4].

[7.5.5]: 107 primitive solutions were found to [7.5.5] (not presented here). The first 17 were published in [4]. While looking for solutions to [7.5.5], the eight solutions to [7.4.5] were found, and the one solution to [7.3.5] was found.

Eighth Powers. [8.4.6]: Only one solution is known, and is presented in Table 1. It was found by looking at [8.6.6].

Table 4: Smallest Primitive solutions to [7.4.5]		
r	Solution $[7.4.5]_r$	
1	(50,43,16,12)	(52,29,26,11,3)
2	(81,58,19,9)	(77,68,56,48,2)
3	(87,74,69,40)	(82,79,75,25,9)
4	(99,76,32,29)	(93,88,68,36,35)
5	(98,82,58,34)	(99,75,69,16,13)
6	(104,96,60,14)	(102,95,81,57,23)
7	(111,102,40,29)	(112,96,82,55,21)
8	(113,102,86,23)	(120,81,58,55,10)

Table 5: Smallest primitive solutions to [8.5.5]		
r	Solution $[8.5.5]_r$	
1	(43, 20, 11, 10, 1)	(41, 35, 32, 28, 5)
2	(42, 41, 35, 9, 6)	(45, 36, 27, 13, 8)
3	(63, 63, 31, 15, 6)	(65, 59, 48, 37, 7)
4	(75, 47, 39, 26, 6)	(67, 67, 62, 20, 11)
5	(77, 76, 71, 42, 28)	(86, 41, 36, 32, 29)
6	(90, 81, 10, 4, 3)	(92, 74, 55, 50, 37)
7	(93, 65, 65, 41, 13)	(81, 81, 79, 75, 45)
8	(89, 87, 28, 14, 14)	(96, 36, 33, 31, 24)
9	(93, 90, 32, 18, 9)	(94, 86, 71, 60, 19)
10	(104, 73, 36, 17, 3)	(103, 78, 68, 11, 9)
11	(103, 86, 58, 11, 8)	(104, 78, 69, 62, 9)
12	(108, 101, 88, 45, 1)	(116, 59, 46, 15, 3)
13	(116, 92, 79, 33, 25)	(113, 103, 60, 44, 31)
14	(123, 97, 71, 10, 2)	(125, 77, 48, 37, 26)
15	(121, 109, 71, 70, 40)	(120, 104, 99, 75, 61)
16	(127, 43, 26, 10, 3)	(123, 105, 69, 42, 14)

[8.5.5]: Table 5 gives the 16 smallest primitive solutions for [8.5.5]. $[8.5.5]_1$ was published in [5].

[8.5.6]: The 19 smallest primitive solutions were found while looking at [8.6.6]. The are in Table 6.

[8.6.6]: 204 primitive solutions were found to [8.6.6] (not presented here). $[8.6.6]_1$ was published in [4]. This search was run looking for a solution to [8.4.6]. Nineteen solutions to [8.5.6] were found during this search, three solutions to [8.5.5] were found, and one solution to [8.4.6] was found.

Ninth Powers. [9.*m*.*n*]: The smallest primitive solution was found for [9.1.14], [9.3.9], and [9.4.9], and are presented in Table 1.

[9.5.7]: The smallest primitive solution was found. It was found by looking at [9.7.7].

[9.6.6]: Table 7 gives the seven smallest primitive solutions for [9.6.6]. $[9.6.6]_1$ was in [4].

[9.7.7]: Nine solutions were found to [9.7.7] (not presented here). This search was run looking for a solution to [9.5.7], which was found.

Table 6: Smallest primitive solutions to [8.5.6]		
r	Solution $[8.5.6]_r$	
1	(36, 36, 33, 25, 21)	(38, 34, 32, 15, 15, 13)
2	(39, 33, 32, 25, 19)	(37, 35, 35, 17, 16, 20
3	(41, 21, 20, 19, 16)	(40, 31, 30, 17, 9, 8)
4	(43, 34, 24, 8, 1)	(42, 37, 28, 16, 16, 15)
5	(44, 42, 24, 17, 4)	(47, 20, 18, 8, 6, 6)
6	(49, 29, 22, 1, 1)	(47, 42, 26, 23, 17, 5)
7	(46, 46, 33, 30, 9)	(45, 45, 36, 36, 34, 32)
8	(51, 48, 39, 21, 10)	(53, 45, 25, 22, 22, 6)
9	(55, 37, 19, 19, 18)	(51, 50, 35, 26, 11, 9)
10	(58, 17, 13, 10, 7)	(56, 45, 41, 40, 8, 1)
11	(55, 53, 24, 21, 2)	(52, 52, 50, 25, 17, 7)
12	(58, 51, 17, 11, 11)	(60, 37, 34, 29, 23, 3)
13	(54, 51, 51, 43, 4)	(59, 46, 41, 30, 17, 2)
14	(58, 53, 35, 19, 17)	(61, 30, 25, 23, 16, 1)
15	(61, 29, 28, 27, 26)	(57, 52, 48, 17, 14, 5)
16	(58, 51, 49, 8, 6)	(61, 44, 32, 26, 10, 1)
17	(62, 53, 38, 32, 23)	(61, 52, 50, 34, 24, 1)
18	(59, 57, 47, 40, 8)	(62, 52, 45, 17, 15, 2)
19	(63, 62, 55, 43, 27)	(65, 59, 56, 17, 13, 10)

Table 7: Smallest primitive solutions to [9.6.6]		
r	Solution $[9.6.6]_r$	
1	(23,18,14,13,13,1)	(22,21,15,10,9,5)
2	(31,23,21,14,9,2)	(29,29,15,11,10,6)
3	(46,44,27,27,27,9)	(48,39,23,15,13,12)
4	(47,47,22,22,12,4)	(50,39,35,13,10,7)
5	(54,52,48,47,46,14)	(60,18,17,16,15,15)
6	(70,44,36,33,19,4)	(64,63,57,47,22,13)
7	(68,58,50,46,41,7)	(70,48,26,25,23,18)

Tenth Powers. $[10.m.n]$: The smallest primitive solution was found, and presented in Table 1, for each of [10.1.22], [10.2.15], [10.3.14], [10.4.15], [10.5.16], and [10.6.16].

[10.7.7]: The smallest primitive solution is presented in Table 1. A solution in distinct integers, due to Moessner [9], is presented in Table 9. It is unknown whether it is the smallest solution in distinct integers.

Euler's Conjecture. Of special interest is Euler's Conjecture. Euler conjectured that an n^{th} power can be decomposed into n n^{th} powers, but not $n-1$ or less n^{th} powers. In the current nomenclature, $[k.1.k]$ can be solved, but not $[k.1.n]$ when $n < k$. Elkies and Lander gave counterexamples (i.e., solutions of $[k.1.k-1]$) for exponents 4 and 5, respectively.

Euler's Extended Conjecture is that there are no solutions to $[k.m.n]$ where $m + n < k$. There are no known counterexamples to this conjecture. The search limits for [5.1.3], [5.2.2], and [6.2.3] are in Table 1. Search limits for other cases are in Table 9.

We can define the Euler Conjecture Number as being the minimum value of $m + n - k$ for known solutions. See Table 8.

Table 8: Euler's Conjecture Number			
Exponent	Best Solution	Delta	Reference
4	[4.1.3]	0	Elkies [6]
5	[5.1.4]	0	Lander [4]
6	[6.3.3]	0	Subba Roa [8]
7	[7.4.4]	1	Ekl [3]
8	[8.5.5]	2	Letac [5]
9	[9.6.6]	3	Lander [4]
10	[10.7.7]	4	Moessner [9]

Table 9: Balanced Equation $[k.m.m]$		
$k.m.m$	Search Limit	Comments
5.2.2	4.01×10^{30}	Conjectured - No solution
6.2.2	7.25×10^{26}	Conjectured - No solution
7.3.3	4.52×10^{23}	Conjectured - No solution
8.3.3	2.06×10^{26}	Conjectured - No solution
8.4.4	1.17×10^{17}	
9.5.5	1.36×10^{19}	
10.5.5	2.13×10^{21}	
10.6.6	3.68×10^{18}	

Table 10: Distinct Smallest Primitive Solutions to $[k.m.n]$		
$k.m.n$	Solution	Comments
6.2.6	$(302, 233) = (294, 238, 201, 168, 118, 42)$	
6.2.7	$(161, 133) = (166, 102, 87, 78, 72, 67, 48)$	
7.2.8	$(61, 53) = (63, 42, 35, 32, 29, 28, 27, 6)$	
7.3.7	$(56, 51, 46) = (60, 39, 38, 26, 20, 9, 3)$	
9.6.6	$(70, 44, 36, 33, 19, 4) = (64, 63, 57, 47, 22, 13)$	
10.7.7	$(68, 61, 55, 32, 31, 28, 1) = (67, 64, 49, 44, 23, 20, 17)$	Min?

Balanced Equations. A special case is a balanced equation; having the same number of terms on both sides, i.e. $[k.m.m]$. The search limits for some unsolved instances of this problem are shown in Table 9.

Distinct Integers. The restricted problem of only using distinct integers yields the results presented in Table 10. The only results presented are those that are different than Table 1.

The same algorithms were used for the distinct integer searches, with the additional contraints added to ensure numbers were not duplicated.

Hardware. The hardware platforms used were an HP 715 Unix workstation, with 64 M bytes of RAM, and an HP PRISM architecture RISC processor running at 50 MHz, and then an HP 735 Unix workstation, with 128 M bytes of RAM, and an HP PRIMSM architecture RISC processor running at 75 MHz. The systems each had a floating point coprocessor.

Software. Two programs were used, one for the balanced searches, and another for the brute-force decomposition searches. Both programs were written in the C language. Other than the comments in [3], no special implementation mechanisms were used.

Conclusion. Presented were new solutions to the equal sums of like powers problem. Two main unsolved problems of this form are Euler's Conjecture for exponents greater than 5, and solutions where the total number of terms is less than the exponent. Specifically and most notably, is [5.2.2] solvable?

Also of note is that numeric results for the Euler Extended Conjecture are still only known for the cases 4, 5, and 6. The search limits have been extended for some special "balanced" or "nearly balanced" cases of [7.3.4], [8.4.4], [9.4.5], and [10.5.5], but no results have been found.

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