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ACCELERATED SPECTRAL APPROXIMATION

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ABSTRACT. A systematic development of higher order spectral analysis, introduced by Dellwo and Friedman, is undertaken in the framework of an appropriate product space. Accelerated analogues of Osborn's results about spectral approximation are presented. Numerical examples are given by considering an integral operator.

1. INTRODUCTION

In [10] Osborn considered numerical solution of an eigenvalue problem for a compact operator T on a complex Banach space X and obtained error estimates for the approximation of eigenvalues, eigenvectors and spectral subspaces, when a sequence (T_n) of compact operators approximates T in a collectively compact manner. In [11] Vainikko obtained similar results under (discrete) regular approximation. Subsequently, numerical solutions of eigenvalue problems for compact as well as noncompact operators have been studied extensively ([1], [3], [4], [6], [7], [9]).

In [5] Dellwo and Friedman developed a new approach to the spectral approximation of a compact operator by solving a polynomial eigenvalue problem of a higher degree. The eigenvalue problem associated with the qth degree operator polynomial was referred to as the qth order spectral analysis of T, $q = 1, 2, \ldots$. They proved that, if λ is a nonzero eigenvalue of T of algebraic multiplicity m and ascent l, then the qth order spectral analysis provides sets $\sigma_{q,n}$ of approximate spectra associated with λ, which satisfy the order relationship

$$\max_{\mu \in \sigma_{q,n}} |\lambda - \mu|^l = O\left(\frac{1}{|\lambda|^q} \|(T - T_n)^q T_n\|\right).$$

Several numerical examples were considered to illustrate the effectiveness of higher order spectral analysis. However, the exact nature of the set $\sigma_{q,n}$ was not specified.

In this paper an attempt is made to develop a methodology for a systematic study of higher order spectral analysis. We transform a polynomial eigenvalue problem associated with a higher order spectral analysis to an equivalent ordinary eigenvalue problem in an appropriate product space. We thus obtain error estimates for accelerated approximation of eigenvalues, eigenvectors and spectral subspaces in exactly the same fashion as the ordinary spectral approximation. We consider a cluster Λ of nonzero eigenvalues of T of total algebraic multiplicity $m < \infty$ and show

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that the qth order spectral analysis provides exactly m eigenvalues $\lambda_{q,n,1}, \dots, \lambda_{q,n,m}$ (counted according to their algebraic multiplicities) near the cluster Λ. If $\hat{\lambda}$ and $\hat{\lambda}_{q,n}$ denote the weighted averages of the eigenvalues in Λ and of their approximations $\lambda_{q,n,1}, \dots, \lambda_{q,n,m}$, respectively, and if $\epsilon < \min\{|\lambda| : \lambda \in \Lambda\}$, then

$$|\hat{\lambda} - \hat{\lambda}_{q,n}| \leq \frac{C}{\epsilon^{q-1}} \|(T - T_n)^q{}_{|R(P)}\|,$$

where C is a constant independent of n and q. This gives an accelerated analogue of Osborn's result for the approximation of the arithmetic mean $\hat{\lambda}$. We also prove that

$$|\hat{\lambda} - \hat{\lambda}_{q,n}| = O\left(\frac{1}{\epsilon^{q-1}} \|(T - T_n)^q T_n\|\right).$$

This estimate improves upon the result of Dellwo and Friedman quoted earlier. If Λ consists of a single eigenvalue λ of ascent $l > 1$, then error estimates for the approximation of λ by individual eigenvalues $\lambda_{q,n,1}, \dots, \lambda_{q,n,m}$ is obtained by taking the lth root of the above-mentioned error estimates. This slower convergence is illustrated in the last section by considering an integral operator. We give similar estimates for the approximation of eigenvectors and spectral subspaces as well. Results analogous to the improved error estimates given in Theorems 3 and 4 of [10] will be given in another paper. The methodology developed in this paper can be used to obtain accelerated analogues of various spectral refinement schemes which will be discussed in subsequent papers.

In Section 2, we give improved versions of results from [10] for the sake of completeness and for use in the subsequent sections. In Section 3, we develop a framework for higher order spectral analysis and obtain accelerated analogues of the results in [10] for the approximation of a cluster of eigenvalues, eigenvectors and spectral subspaces of a bounded linear operator.

2. Preliminaries

Throughout this paper X will denote a complex Banach space and $BL(X)$ the Banach space of all bounded linear operators on X along with the operator norm. For T in $BL(X)$, let $\sigma(T)$ and $\rho(T)$ denote the spectrum and the resolvent set of T, respectively. We consider a nonempty subset Λ of $\sigma(T) \setminus \{0\}$ which is separated from the rest of the spectrum of T and from 0 by a simple closed positively oriented rectifiable curve Γ lying in $\rho(T)$. Let $\ell(\Gamma)$ denote the length of Γ. For $z \in \rho(T)$, we let

$$R(z) = (T - zI)^{-1},$$

so that

$$P = -\frac{1}{2\pi i} \int_{\Gamma} R(z)\, dz$$

is the spectral projection associated with T and Λ. We assume that rank $P = m < \infty$. Then Λ consists of eigenvalues $\lambda_1, \dots, \lambda_m$ of T, counted according to their algebraic multiplicities. For nonzero subspaces Y and Z of X, let

$$\delta(Y, Z) = \sup\{\operatorname{dist}(y, Z) : y \in Y, \|y\| = 1\}.$$

Then

$$\hat{\delta}(Y, Z) = \max\{\delta(Y, Z), \delta(Z, Y)\}$$

is known as the **gap** between Y and Z. For $T \in BL(X)$, we denote by $R(T)$ and $N(T)$ the range space and the null space of T, respectively.

In this section we consider a sequence (T_n) in $BL(X)$ which satisfies
(H1) : $(\|T_n\|)$ is a bounded sequence,
(H2) : $\|(T - T_n)T\| \to 0$ and $\|(T - T_n)T_n\| \to 0$ as $n \to \infty$.
While it may be possible to weaken these hypotheses to some extent, they allow a simplicity in the presentation and are easy to check in several important examples. For $z \in \rho(T_n)$, we let

$$R_n(z) = (T_n - zI)^{-1}.$$

Lemma 2.1 (Nair [9]). *Let E be a closed subset of $\rho(T) \setminus \{0\}$ and $\delta = \min\{|z| : z \in E\}$. Then there is a constant c_1 such that*

$$\max_{z\in E} \|R(z)\| \leq c_1.$$

Let n_0 be a positive integer such that

$$\|(T - T_n)^2\| < \delta^2 \text{ and } c_1\|(T - T_n)T_n\| \leq \frac{\delta}{2}$$

for all $n \geq n_0$. Then $E \subset \rho(T_n)$ and

$$\max_{z\in E} \|R_n(z)\| \leq 2c_1 \left[1 + \frac{\|T - T_n\|}{\delta}\right] \leq c_2$$

for some constant c_2 and all $n \geq n_0$.

For a proof of this result we refer to [9]. Letting $E = \Gamma$ in Lemma 2.1, we see that $\Gamma \subset \rho(T_n)$ for all large n. Let

$$P_n = -\frac{1}{2\pi i} \int_\Gamma R_n(z)\, dz$$

denote the spectral projection associated with T_n and Γ. It can be seen, as in the proof of Theorem 3.1 of [9], that

$$\|(P - P_n)P\| \to 0 \text{ and } \|(P - P_n)P_n\| \to 0 \text{ as } n \to \infty.$$

Hence rank $P_n =$ rank $P = m$ for all large n (cf. [9], Proposition 2.2.).

Theorem 2.2 (Osborn [10]). *For all large n,*

$$\hat{\delta}(R(P), R(P_n)) \leq \frac{\ell(\Gamma)}{\pi} c_1 c_2 \min\left\{\|(T - T_n)_{|R(P)}\|, \|(T - T_n)_{|R(P_n)}\|\right\},$$

where c_1 and c_2 are as in Lemma 2.1 with $E = \Gamma$.

Proof. The proof of Theorem 1 of [10] shows that

$$\delta(R(P), R(P_n)) \leq \frac{\ell(\Gamma)}{2\pi} c_1 c_2 \|(T - T_n)_{|R(P)}\|.$$

Since, with $\delta = \min\{|z| \in \Gamma\}$,

$$\|(T - T_n)_{|R(P)}\| \leq \|(T - T_n)P\| \leq \frac{\ell(\Gamma)}{2\pi\delta} c_1 \|(T - T_n)T\| \to 0,$$

as $n \to \infty$, we take n_0 so large that $\delta(R(P), R(P_n)) \leq 1/2$.

As $\dim R(P_n) = \dim R(P) < \infty$, by a result given by Kato [8],

$$\delta(R(P_n), R(P)) \leq \frac{\delta(R(P), R(P_n))}{1 - \delta(R(P), R(P_n))} \leq 2\delta(R(P), R(P_n)).$$

Thus

$$\hat{\delta}(R(P), R(P_n)) \leq \frac{\ell(\Gamma)}{\pi} c_1 c_2 \|(T - T_n)_{|R(P)}\|.$$

By interchanging the roles of T and T_n, we obtain

$$\hat{\delta}(R(P_n), R(P)) \leq \frac{\ell(\Gamma)}{\pi} c_1 c_2 \|(T_n - T)_{|R(P_n)}\|.$$

□

Since rank $P_n = m$ for all large n, $\sigma(T_n) \cap \operatorname{Int}\Gamma$ consists of m eigenvalues $\lambda_{n,1}, \ldots, \lambda_{n,m}$ of T_n, counted according to their algebraic multiplicities. Let

$$\hat{\lambda}_n = \frac{\lambda_{n,1} + \cdots + \lambda_{n,m}}{m}$$

denote their arithmetic mean.

Theorem 2.3 (Osborn [10]). *For all large n, the maps*

$$A_n = P_{n|R(P)} : R(P) \to R(P_n) \text{ and } B_n = P_{|R(P_n)} : R(P_n) \to R(P)$$

are isomorphisms, $\|A_n^{-1}\| \leq 2, \|B_n^{-1}\| \leq 2$ and

$$|\hat{\lambda} - \hat{\lambda}_n| \leq 2 \min \left\{ \|P_n\| \, \|(T - T_n)_{|R(P)}\|, \, \|P\| \, \|(T - T_n)_{|R(P_n)}\| \right\}.$$

If $\Lambda = \{\lambda\}$ and the ascent of λ equals l, then for each $j = 1, \ldots, m$,

$$|\lambda - \lambda_{n,j}|^l \leq 2 \min \left\{ c_n \|P_n\| \, \|(T - T_n)_{|R(P)}\|, \, d_n \|P\| \, \|(T - T_n)_{|R(P_n)}\| \right\},$$

where

$$\begin{aligned} c_n &= \sum_{k=0}^{l-1} \|\lambda I_{|R(P)} - A_n^{-1} T_n A_n\|^{l-1-k} \, \|\lambda I_{|R(P)} - T_{|R(P)}\|^k, \\ d_n &= \sum_{k=0}^{l-1} \|\lambda I_{|R(P_n)} - T_{n|R(P_n)}\|^{l-1-k} \, \|\lambda I_{|R(P_n)} - B_n^{-1} T B_n\|^k. \end{aligned}$$

Proof. The argument given in the proof of Theorem 2 of [10] shows that A_n is bijective and $\|A_n^{-1}\| \leq 2$ for all large n. The same argument shows that B_n is bijective and $\|B_n^{-1}\| \leq 2$ for all large n. Define $\hat{T} = T_{|R(P)}$ and $\hat{T}_n = A_n^{-1} T_n A_n$. Then

$$\begin{aligned} |\hat{\lambda} - \hat{\lambda}_n| &= \frac{1}{m} |\text{trace } (\hat{T} - \hat{T}_n)| \leq \|\hat{T} - \hat{T}_n\| \\ &= \sup\{\|A_n^{-1} P_n (T - T_n) x\| : x \in R(P), \|x\| = 1\} \\ &\leq 2 \|P_n\| \, \|(T - T_n)_{|R(P)}\|. \end{aligned}$$

If $\Lambda = \{\lambda\}$ and the ascent of λ is l, then since $(\lambda I_{|R(P)} - \hat{T})^l = 0$, we have for $j = 1, \ldots, m$

$$\begin{aligned} |\lambda - \lambda_{n,j}|^l &\leq \|(\lambda I_{|R(P)} - \hat{T}_n)^l\| = \|(\lambda I_{|R(P)} - \hat{T}_n)^l - (\lambda I_{|R(P)} - \hat{T})^l\| \\ &= \|\sum_{k=0}^{l-1} (\lambda I_{|R(P)} - \hat{T}_n)^{l-1-k} (\hat{T} - \hat{T}_n)(\lambda I_{|R(P)} - \hat{T})^k\| \\ &\leq c_n \|\hat{T} - \hat{T}_n\|. \end{aligned}$$

Similarly, defining $\hat{T}_n = T_{n|R(P_n)}$ and $\hat{T} = B_n^{-1} T B_n$, we obtain the other estimates. The proof of the estimates for $|\lambda - \lambda_{n,j}|^l$ is adapted from [4], p. 685. □

Let $\Lambda = \{\lambda\}$ and l be the ascent of λ. We state the following theorem from [10].

Theorem 2.4 (Osborn [10]). *Let λ_n be an eigenvalue of T_n such that $\lambda_n \to \lambda$ as $n \to \infty$. Suppose that $w_n \in N(T_n - \lambda_n I)$ with $\|w_n\| = 1$. Then there is some $u_n \in N(T - \lambda I)$ such that*

$$\|u_n - w_n\| \le c\left\{\|(T - T_n)_{|R(P)}\|\right\}^{1/l},$$

where c is a constant independent of n.

3. A framework for higher order approximation

Let q be a positive integer and $\mathbf{X}_q$ denote the set of all column vectors $\mathbf{x} = [x_1, \dots, x_q]^t$ with $x_1, \dots, x_q$ in X. Define

$$\|\mathbf{x}\|_\infty = \max\{\|x_j\| : j = 1, \dots, q\}.$$

Then $\mathbf{X}_q$ is a Banach space with respect to the norm $\| \quad \|_\infty$. We shall identify the adjoint space of $\mathbf{X}_q$ with the set of all column vectors $\mathbf{x}^* = [x_1^*, \dots, x_q^*]^t$ with $x_1^*, \dots, x_q^*$ in X^*. Define

$$\|\mathbf{x}^*\|_1 = \|x_1^*\| + \cdots + \|x_q^*\|.$$

If we let

$$\langle \mathbf{x}, \mathbf{x}^*\rangle = \langle x_1, x_1^*\rangle + \cdots + \langle x_q, x_q^*\rangle,$$

then it is clear that $|\langle \mathbf{x}, \mathbf{x}^*\rangle| \le \|\mathbf{x}\|_\infty \|\mathbf{x}^*\|_1$. We have $\mathbf{X}_1 = X$ and we let $\mathbf{T}_1 = T$. Now let $q \ge 2$. Consider the operator $\mathbf{T}_q : \mathbf{X}_q \to \mathbf{X}_q$ given by

$$\mathbf{T}_q[x_1, \dots, x_q]^t = [Tx_1, x_1, \dots, x_{q-1}]^t.$$

Then $\mathbf{T}_q$ can be written as the $q \times q$ matrix

$$\begin{bmatrix} T & 0 & \cdots & \cdots & 0 \\ I & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & I & 0 \end{bmatrix}.$$

We have $\|\mathbf{T}_q\|_\infty = \max\{1, \|T\|\}$. For nonzero $z \in \mathbb{C}$, it can be easily seen that $\mathbf{T}_q - z\mathbf{I}_q$ is invertible if and only if $T - zI$ is invertible, and then $(\mathbf{T}_q - z\mathbf{I}_q)^{-1}$ can be written as the $q \times q$ matrix

$$\begin{bmatrix} R(z) & 0 & \dots & 0 \\ \dfrac{R(z)}{z} & -\dfrac{I}{z} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{R(z)}{z^{q-1}} & -\dfrac{I}{z^{q-1}} & \dots & -\dfrac{I}{z} \end{bmatrix}.$$

Thus $\sigma(\mathbf{T}_q) \setminus \{0\} = \sigma(T) \setminus \{0\}$. In particular, $\Gamma \subset \rho(\mathbf{T}_q)$ and $\sigma(\mathbf{T}_q) \cap \mathrm{Int}\Gamma = \Lambda$. Let

$$\mathbf{P}_q = -\frac{1}{2\pi i}\int_\Gamma (\mathbf{T}_q - z\mathbf{I}_q)^{-1} dz$$

denote the spectral projection associated with $\mathbf{T}_q$ and Λ. Since 0 lies outside Γ, we have

$$\mathbf{P}_q[x_1, \dots, x_q]^t = [Px_1, S_1x_1, \dots, S_{q-1}x_1]^t,$$

where

$$S_j = -\frac{1}{2\pi i}\int_\Gamma \frac{R(z)}{z^j}dz, \quad j = 1,\dots,q-1.$$

Now $[x_1,\dots,x_q]^t \in R(\mathbf{P}_q)$ if and only if $x_1 \in R(P), x_2 = S_1x_1,\dots,x_q = S_{q-1}x_1$. Hence the operator $J_q : R(P) \to R(\mathbf{P}_q)$ given by

$$J_qx = [x, S_1x,\dots,S_{q-1}x]^t, \quad x \in R(P),$$

is a surjective isomorphism and

$$\text{rank } \mathbf{P}_q = \text{ rank } P = m.$$

Next, the spectral projection associated with $\mathbf{T}_q^* : \mathbf{X}_q^* \to \mathbf{X}_q^*$ and $\overline{\Lambda} = \{\overline{\lambda} : \lambda \in \Lambda\}$ is given by

$$\mathbf{P}_q^* \, [x_1^*,\dots,x_q^*]^t = [P^*x_1^* + S_1^*x_2^* + \cdots + S_{q-1}^*x_q^*, 0,\dots,0]^t.$$

Thus $[x_1^*,\dots,x_q^*]^t \in R(\mathbf{P}_q^*)$ if and only if $x_1^* \in R(P^*), x_2^* = \cdots = x_q^* = 0$. Hence the operator $K_q : R(P^*) \to R(\mathbf{P}_q^*)$ given by

$$K_qx^* = [x^*, 0,\dots,0]^t, \quad x^* \in R(P^*),$$

is a surjective isomorphism. Also,

$$\langle J_qx,\, K_qx^*\rangle = \langle x,\, x^*\rangle \quad \text{for all } x \in R(P) \text{ and } x^* \in R(P^*).$$

Next, consider $\lambda \in \Lambda$ and let P_λ (resp. $\mathbf{P}_\lambda$) denote the spectral projection associated with T and λ (resp., $\mathbf{T}_q$ and λ). Then rank $\mathbf{P}_\lambda$ = rank P_λ just as before, so that the algebraic multiplicity of λ as an eigenvalue of $\mathbf{T}_q$ is the same as the algebraic multiplicity of λ as an eigenvalue of T. Consider

$$D_\lambda = P_\lambda(T - \lambda I) \quad \text{and} \quad \mathbf{D}_\lambda = \mathbf{P}_\lambda(\mathbf{T}_q - \lambda\mathbf{I}_q).$$

Then

$$\begin{aligned}\mathbf{D}_\lambda[x_1,\dots,x_q]^t &= \mathbf{P}_\lambda[(T-\lambda I)x_1,\, x_1 - \lambda x_2,\dots,x_{q-1} - \lambda x_q]^t \\ &= [P_\lambda(T-\lambda I)x_1,\, S_{\lambda,1}(T-\lambda I)x_1,\dots,S_{\lambda,q-1}(T-\lambda I)x_1]^t,\end{aligned}$$

where

$$S_{\lambda,j} = -\frac{1}{2\pi i}\int_{\Gamma_\lambda} \frac{R(z)}{z^j}\,dz, \quad j = 1,\dots,q-1\,,$$

Γ_λ being a simple closed curve which isolates λ from the rest of $\sigma(T)$ and from 0. By the usual techniques of contour integration, it can be seen that $S_{\lambda,j}P_\lambda = S_{\lambda,j} = P_\lambda S_{\lambda,j}$ for $j = 1,\dots,q-1$. Hence

$$\mathbf{D}_\lambda[x_1,\dots,x_q]^t = [D_\lambda x_1,\, S_{\lambda,1}D_\lambda x_1,\dots,S_{\lambda,q-1}D_\lambda x_1]^t.$$

Similarly, for $k = 2, 3,\dots$, we have

$$\mathbf{D}_\lambda^k[x_1,\dots,x_q]^t = [D_\lambda^k x_1,\, S_{\lambda,1}D_\lambda^k x_1,\dots,S_{\lambda,q-1}D_\lambda^k x_1]^t.$$

Thus for any positive integer k, we have $\mathbf{D}_\lambda^k = 0$ if and only if $D_\lambda^k = 0$. This shows that the ascent of λ as an eigenvalue of $\mathbf{T}_q$ is the same as the ascent of λ as an eigenvalue of T. Thus we see that the eigenvalue problem

$$T\phi = \lambda\phi, \quad \phi \in X, \quad \phi \neq 0$$

is equivalent to the eigenvalue problem

$$\mathbf{T}_q\Phi_q = \lambda\Phi_q, \quad \Phi_q \in \mathbf{X}_q, \quad \Phi_q \neq 0$$

for each $q = 2, 3, \dots$.

Let (T_n) be a sequence in $BL(X)$. We have $\mathbf{X}_1 = X$ and we let $\mathbf{T}_{1,n} = T_n$. Let now $q \geq 2$. For $n = 1, 2, \ldots$, let $\Delta_n = T - T_n$ and consider the operator $\mathbf{T}_{q,n} : \mathbf{X}_q \to \mathbf{X}_q$ given by

$$\mathbf{T}_{q,n}[x_1, \ldots, x_q]^t = \left[\sum_{j=0}^{q-1} \Delta_n^j T_n x_{j+1},\, x_1, \ldots, x_{q-1}\right]^t.$$

Then $\mathbf{T}_{q,n}$ can be written as the $q \times q$ matrix

$$\begin{bmatrix} T_n & \Delta_n T_n & \cdots & \cdots & \Delta_n^{q-1} T_n \\ I & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}.$$

We consider the eigenvalue problem

$$\mathbf{T}_{q,n}\Phi_{q,n} = \lambda_{q,n}\Phi_{q,n}, \quad \Phi_{q,n} \in \mathbf{X}_q, \quad \Phi_{q,n} \neq 0.$$

Then it is easy to see that $\Phi_{q,n} = \left[\phi_{q,n}, \dfrac{\phi_{q,n}}{\lambda_{q,n}}, \ldots, \dfrac{\phi_{q,n}}{(\lambda_{q,n})^q}\right]^t$, where the first component $\phi_{q,n} \in X$ satisfies

$$\left((\lambda_{q,n})^q I - \sum_{j=0}^{q-1} (\lambda_{q,n})^{q-1-j} \Delta_n^j T_n\right) \phi_{q,n} = 0$$

(cf. (2.4) of [5]). The case $q = 1$ is considered in Section 2. For the rest of the paper we let $q \geq 2$ and assume that

(H1) $(\|T_n\|)$ is a bounded sequence,

(H2') $\|(T - T_n)^2\| \to 0$ as $n \to \infty$.

Note that the results of Section 2, where $q = 1$, do not hold under the hypotheses (H1) and (H2'). As a simple example, consider $X = \mathbb{C}^2$ and

$$T = \begin{bmatrix} a & ab \\ 1 & b \end{bmatrix}, \quad T_n = \begin{bmatrix} a_n & a_n b_n \\ 0 & b_n \end{bmatrix}, \quad n = 1, 2, \ldots,$$

where a, b, a_n, b_n are nonzero complex numbers with $b \neq -a$ and $a_n \to a, b_n \to b$ as $n \to \infty$. Then $\sigma(T) = \{0, a + b\}$, while $\sigma(T_n) = \{a_n, b_n\}$. Thus the nonzero simple eigenvalue $a + b$ of T is not approximated by the nonzero eigenvalues a_n and b_n of T_n.

We have

$$\begin{aligned} \|\mathbf{T}_{q,n}\|_\infty &\leq \max\{1, \|T_n\| + \|\Delta_n T_n\| + \cdots + \|\Delta_n^{q-1} T_n\|\}, \\ \|(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{T}_q\|_\infty &\leq \|\Delta_n^2\| + \|\Delta_n^2 T_n\| + \cdots + \|\Delta_n^{q-1} T_n\|, \\ \|(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{T}_{q,n}\|_\infty &= \|\Delta_n^q T_n\|. \end{aligned}$$

Since $\|\Delta_n^2\| \to 0$ as $n \to \infty$, there is an integer n_0 such that $\|\Delta_n^2\| \leq 1/2$ for all $n \geq n_0$. As

$$\|\Delta_n^j T_n\| \leq \begin{cases} \|T_n\|\, \|\Delta_n^2\|^{j/2}, & \text{if } j \text{ is even} \\ \|\Delta_n T_n\|\, \|\Delta_n^2\|^{(j-1)/2}, & \text{if } j \text{ is odd} \end{cases}$$

and $\sum_{k=0}^{\infty} \|\Delta_n^2\|^k \leq 2$ for $n \geq n_0$, we have

$$\begin{aligned} \|\mathbf{T}_{q,n}\|_\infty &\leq \max\{1, 2(\|T_n\| + \|\Delta_n T_n\|)\}, \\ \|(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{T}_q\|_\infty &\leq 2(\|T_n\| + \|\Delta_n^2 T_n\|)\|\Delta_n^2\|, \\ \|(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{T}_{q,n}\|_\infty &\leq \max\{\|T_n\|, \|\Delta_n T_n\|\}\|\Delta_n^2\|. \end{aligned}$$

As $\|T_n\|$ is bounded, this shows that $\|\mathbf{T}_{q,n}\|_\infty$ is bounded in q and n,

$$\|(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{T}_q\|_\infty \to 0 \quad \text{and} \quad \|(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{T}_{q,n}\|_\infty \to 0$$

as $n \to \infty$, uniformly for $q = 2, 3, \ldots$. Thus the sequence $(\mathbf{T}_{q,n})$ satisfies the hypotheses (H1) and (H2) of Section 2 uniformly in $q = 2, 3, \ldots$.

The following two identities will be useful. For nonzero z in $\mathbb{C}$,

$$(*) \qquad \frac{zI - \Delta_n}{z}\left(zI - \sum_{j=0}^{q-1} \frac{\Delta_n^j T_n}{z^j}\right) = zI - T + \frac{\Delta_n^q T_n}{z^q},$$

$$(**) \qquad zI - \sum_{j=0}^{q-1} \frac{\Delta_n^j T_n}{z^j} - \sum_{j=0}^{q} \frac{\Delta_n^j (zI - T)}{z^j} = \frac{\Delta_n^q T}{z^q}.$$

3.1. **Main results.** We prove an important estimate.

Proposition 3.1. *Let* $\min\{|\lambda| : \lambda \in \Lambda\} > \epsilon$. *If the curve* Γ *lies in* $\{ z \in \mathbb{C} : |z| \geq \epsilon \}$ *and* $c_1 = \max_{z \in \Gamma} \|R(z)\|$, *then*

$$\|(\mathbf{T}_q - \mathbf{T}_{q,n})_{|R(\mathbf{P}_q)}\|_\infty \leq \frac{\ell(\Gamma) c_1}{2\pi \epsilon^{q-1}} \|(T - T_n)^q{}_{|R(P)}\|$$

for all n *and* q.

Proof. For $\mathbf{x} = [x_1, \ldots, x_q]^t \in \mathbf{X}_q$, we have

$$(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{x} = \left[\Delta_n x_1 - \sum_{j=1}^{q-1} \Delta_n^j T_n x_{j+1}, 0, \ldots, 0\right]^t.$$

Since

$$\mathbf{P}_q \mathbf{x} = [P x_1, S_1 x_1, \ldots, S_{q-1} x_1]^t,$$

it follows that

$$(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{P}_q \mathbf{x} = \left[(\Delta_n P - \sum_{j=1}^{q-1} \Delta_n^j T_n S_j) x_1, 0, \ldots, 0\right]^t.$$

By the definitions of P and S_j, we have

$$\Delta_n P - \sum_{j=1}^{q-1} \Delta_n^j T_n S_j = -\frac{1}{2\pi i} \int_\Gamma \left[\Delta_n - \sum_{j=1}^{q-1} \frac{\Delta_n^j T_n}{z^j}\right] R(z) dz.$$

But the identity (**) shows that

$$\begin{aligned}\left(\Delta_n - \sum_{j=1}^{q-1} \frac{\Delta_n^j T_n}{z^j}\right) R(z) &= \left[T - zI + zI - \sum_{j=0}^{q-1} \frac{\Delta_n^j T_n}{z^j}\right] R(z) \\ &= \left[T - zI + \sum_{j=0}^{q} \frac{\Delta_n^j (zI - T)}{z^j} + \frac{\Delta_n^q T}{z^q}\right] R(z) \\ &= -\sum_{j=1}^{q-1} \frac{\Delta_n^j}{z^j} + \frac{\Delta_n^q R(z)}{z^{q-1}},\end{aligned}$$

since $TR(z) = I + zR(z)$. Since 0 lies outside Γ, $\int_\Gamma \frac{dz}{z^j} = 0$ for $j = 1, \dots, q$, so that

$$\|(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{P}_q \mathbf{x}\|_\infty = \|\Delta_n^q y_1\|,$$

where $y_1 = \left(-\frac{1}{2\pi i}\int_\Gamma \frac{R(z)}{z^{q-1}} dz\right) x_1$. Now if $\mathbf{x} \in R(\mathbf{P}_q)$, then $x_1 = Px_1 \in R(P)$ and since P commutes with $R(z)$, we see that $y_1 \in R(P)$. Also, $\|y_1\| \le \frac{\ell(\Gamma)c_1\|x_1\|}{2\pi\epsilon^{q-1}}$ and hence

$$\|(\mathbf{T}_q - \mathbf{T}_{q,n})_{|R(\mathbf{P}_q)}\|_\infty \le \frac{\ell(\Gamma)c_1}{2\pi\epsilon^{q-1}} \|(T - T_n)^q{}_{|R(P)}\|.$$

□

It follows that if one fixes an integer $q \ge 2$, then the results given in Section 2 become available for the operators $\mathbf{T}_q$ and $\mathbf{T}_{q,n}$, and accelerated analogues of Theorem 2.3 and Theorem 2.4 would follow immediately. However, the constants appearing in various error estimates will depend on q. In order to find the nature of this dependence on q, we proceed as follows. It may be mentioned that the use of the norm $\| \ \|_\infty$ on $\mathbf{X}_q$ (instead of the commonly used norm $\| \ \|_2$) allows us to achieve our aim.

First we consider the invertibility of $\mathbf{T}_{q,n} - z\mathbf{I}_q$.

Proposition 3.2. (**a**) *If $z \ne 0$ and $zI - \sum_{j=0}^{q-1} \frac{\Delta_n^j T_n}{z^j}$ is invertible in $BL(X)$, then $\mathbf{T}_{q,n} - z\mathbf{I}_q$ is invertible in $BL(\mathbf{X}_q)$.*

(**b**) *Let E be a closed subset of $\rho(T)$ and $0 \notin E$. Then there is a positive integer n_0 such that for all $n \ge n_0$ and $q = 2, 3, \dots$, we have $E \subset \rho(\mathbf{T}_{q,n})$.*

If, in fact, $\min\{|z| : z \in E\} > 1$, *then for all $n \ge n_0$ and $q = 2, 3, \dots$,*

$$\max_{z\in E} \|(\mathbf{T}_q - z\mathbf{I}_q)^{-1}\|_\infty \le C_1 \quad \text{and} \quad \max_{z\in E} \|(\mathbf{T}_{q,n} - z\mathbf{I}_q)^{-1}\|_\infty \le C_2$$

for some constants C_1 and C_2 independent of n and q.

Proof. (**a**) Let

$$A_{k,n}(z) = zI - \sum_{j=0}^{k-1} \frac{\Delta_n^j T_n}{z^j}, \quad k = 1, \dots, q.$$

For $z \neq 0$, let $B_{q,n}(z)$ denote the inverse of $A_{q,n}(z)$. Then it can be verified by direct multiplication that the inverse of $\mathbf{T}_{q,n} - z\mathbf{I}_q$ is given by the $q \times q$ matrix

$$\begin{bmatrix} -B_q & I - B_qA_1 & z(I - B_qA_2) & \cdots & \cdots & z^{q-2}(I - B_qA_{q-1}) \\ -\frac{B_q}{z} & \frac{-B_qA_1}{z} & I - B_qA_2 & \cdots & \cdots & z^{q-3}(I - B_qA_{q-1}) \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{-B_q}{z^{q-2}} & \frac{-B_qA_1}{z^{q-2}} & \frac{-B_qA_2}{z^{q-3}} & \cdots & \frac{-B_qA_{q-2}}{z} & I - B_qA_{q-1} \\ \frac{-B_q}{z^{q-1}} & \frac{-B_qA_1}{z^{q-1}} & \frac{-B_qA_2}{z^{q-2}} & \cdots & \frac{-B_qA_{q-2}}{z^2} & \frac{-B_qA_{q-1}}{z} \end{bmatrix},$$

where we have written B_q for $B_{q,n}(z)$ and A_k for $A_{k,n}(z)$.

(b) Since E is a closed subset of $\mathbb{C}$ and $0 \notin E, \min\{|z| : z \in E\} = \delta > 0$. Since $\|T_n\|$ is bounded and $\max_{z \in E} \|R(z)\| < \infty$, there is some $M \geq 1$ such that

$$\left(\max_{z\in E} \|R(z)\|\right) \max_{n=1,2,\dots} \left\{ \|T_n\|, \frac{\|\Delta_n T_n\|}{\delta} \right\} \leq M.$$

Since $\|\Delta_n^2\| \to 0$ as $n \to \infty$, there is a positive integer n_0 such that $\|\Delta_n^2\| < \dfrac{\delta^2}{M}$ for all $n \geq n_0$. Let $z \in E$ and $n \geq n_0$. As $M \geq 1$ and $\|\Delta_n^2\|^{1/2} < \delta$, we see that $zI - \Delta_n$ is invertible.

By the identity $(*)$, we have

$$A_{q,n}(z) = z(zI - \Delta_n)^{-1} \left(zI - T + \frac{\Delta_n^q T_n}{z^q} \right).$$

Again, since

$$\frac{\|\Delta_n^q T_n\|}{|z^q|} \leq \begin{cases} \|T_n\| \|\dfrac{\Delta_n^2}{\delta^2}\|^{q/2}, & \text{if } q \text{ is even} \\ \dfrac{\|\Delta_n T_n\|}{\delta} \|\dfrac{\Delta_n^2}{\delta^2}\|^{(q-1)/2}, & \text{if } q \text{ is odd,} \end{cases}$$

it follows that

$$\|R(z) \frac{\Delta_n^q T_n}{z^q}\| < 1.$$

Hence for all $z \in E, n \geq n_0$ and $q = 2, 3, \dots$, the operators $zI - T + \dfrac{\Delta_n^q T_n}{z^q}$ (and consequently) $A_{q,n}(z)$ are invertible in $BL(X)$. By (a) above, it follows that $E \subset \rho(\mathbf{T}_{q,n})$ for all $n \geq n_0$ and $q = 2, 3, \dots$.

Next, assume that $\delta > 1$. Then

$$\begin{aligned} \|(\mathbf{T}_q - z\mathbf{I}_q)^{-1}\|_\infty &\leq \max \left\{ \|R(z)\|, \frac{\|R(z)\|}{|z|^k} + \sum_{j=1}^{k} \frac{1}{|z|^j} : \quad k = 1, \dots, q-1 \right\} \\ &\leq \|R(z)\| + \frac{1}{\delta - 1}. \end{aligned}$$

Thus

$$\max_{z \in E} \|(\mathbf{T}_q - z\mathbf{I}_q)^{-1}\|_\infty \leq C_1,$$

where C_1 is a constant independent of q.

We have noted earlier that as $n \to \infty$, $\|(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{T}_q\|_\infty$ and $\|(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{T}_{q,n}\|_\infty$ tend to 0 uniformly in $q = 2, 3, \ldots$. Hence we can assume, without loss of generality, that for all $n \geq n_0$ and $q = 2, 3, \ldots$,

$$\|(\mathbf{T}_q - \mathbf{T}_{q,n})^2\|_\infty < \delta^2 \quad \text{and} \quad C_1\|(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{T}_{q,n}\|_\infty \leq \frac{\delta}{2}.$$

By Lemma 2.1 applied to the operators $\mathbf{T}_q$ and $\mathbf{T}_{q,n}$, we have

$$\|(\mathbf{T}_{q,n} - z\mathbf{I}_q)^{-1}\|_\infty \leq 2C_1\left[1 + \frac{\|\mathbf{T}_q - \mathbf{T}_{q,n}\|_\infty}{\delta}\right] \leq C_2, \text{ say.}$$

□

We remark that the condition $\min\{\,|z| : z \in E\} > 1$ cannot be dropped from part (b) of Proposition 3.2, that is, if $\min\{\,|z| : z \in E\} \leq 1$, then $\|(\mathbf{T}_q - z\mathbf{I}_q)^{-1}\|_\infty$ may tend to infinity as $q \to \infty$. The simplest example is obtained by letting $X = \mathbb{C}$ and for a fixed $c \in \mathbb{C}$,

$$Tx = cx, \quad x \in X.$$

Then for $z \in \mathbb{C}$ with $z \neq c$ and $z \neq 0$ and for $x \in X$, $(T - zI)^{-1}x = x/(c - z)$, so that

$$(\mathbf{T}_q - z\mathbf{I}_q)^{-1}[x_1, \ldots, x_q]^t = \left[\frac{x_1}{c - z}, \frac{x_1}{z(c - z)} - \frac{x_2}{z}, \ldots, \left(\frac{x_1}{z^{q-1}(c - z)} - \frac{x_2}{z^{q-1}} - \cdots - \frac{x_q}{z}\right)\right]^t.$$

Since

$$(\mathbf{T}_q - z\mathbf{I}_q)^{-1}[1, 0, \ldots, 0]^t = \left[\frac{1}{c - z}, \frac{1}{z(c - z)}, \ldots, \frac{1}{z^{q-1}(c - z)}\right]^t,$$

we have

$$\|(\mathbf{T}_q - z\mathbf{I}_q)^{-1}\|_\infty \geq \frac{1}{|z|^{q-1}|c - z|}.$$

Thus if $|z| < 1$, then $\|(\mathbf{T}_q - z\mathbf{I}_q)^{-1}\|_\infty$ tends to infinity as $q \to \infty$.

Taking $E = \Gamma$, we see that for all $n \geq n_0$ and $q = 2, 3, \ldots$, $\Gamma \subset \rho(\mathbf{T}_{q,n})$, so that

$$\mathbf{P}_{q,n} = -\frac{1}{2\pi i}\int_\Gamma (\mathbf{T}_{q,n} - z\mathbf{I}_q)^{-1}\, dz$$

defines the spectral projection associated with $\mathbf{T}_{q,n}$ and $\Lambda_{q,n} = \sigma(\mathbf{T}_{q,n}) \cap \operatorname{Int}\Gamma$.

Theorem 3.3. *If* $\min\{|\lambda| : \lambda \in \Lambda\} > 1$ *and the curve* Γ *lies in* $\{z \in \mathbb{C} : |z| \geq \delta\}$, *where* $\delta > 1$, *then for all large* n *and* $q = 2, 3, \ldots$, *we have*

$$\max_{z\in\Gamma} \|(\mathbf{T}_q - z\mathbf{I}_q)^{-1}\|_\infty \leq C_1 \quad \text{and} \quad \max_{z\in\Gamma} \|(\mathbf{T}_{q,n} - z\mathbf{I}_q)^{-1}\|_\infty \leq C_2$$

for some constants C_1 *and* C_2, *independent of* n *and* q. *Also,*

$$\begin{aligned}
\|(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{P}_q\|_\infty &\leq \frac{\ell(\Gamma)}{2\pi\delta}C_1\|(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{T}_q\|_\infty, \\
\|(\mathbf{P}_q - \mathbf{P}_{q,n})\mathbf{P}_q\|_\infty &\leq \frac{\ell(\Gamma)}{2\pi}C_1C_2\|(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{P}_q\|_\infty, \\
\|(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{P}_{q,n}\|_\infty &\leq \frac{\ell(\Gamma)}{2\pi\delta}C_2\|(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{T}_{q,n}\|_\infty, \\
\|(\mathbf{P}_q - \mathbf{P}_{q,n})\mathbf{P}_{q,n}\|_\infty &\leq \frac{\ell(\Gamma)}{2\pi}C_1C_2\|(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{P}_{q,n}\|_\infty.
\end{aligned}$$

In particular,

$$\|(\mathbf{P}_q - \mathbf{P}_{q,n})\mathbf{P}_q\|_\infty \to 0 \quad \textit{and} \quad \|(\mathbf{P}_q - \mathbf{P}_{q,n})\mathbf{P}_{q,n}\|_\infty \to 0$$

as $n \to \infty$, *uniformly in* $q = 2, 3, \ldots$.

Proof. By part (b) of Proposition 3.2, we have

$$\max_{z \in \Gamma} \|(\mathbf{T}_q - z\mathbf{I}_q)^{-1}\|_\infty \le C_1, \quad \max_{z \in \Gamma} \|(\mathbf{T}_{q,n} - z\mathbf{I}_q)^{-1}\|_\infty \le C_2$$

for all $n \ge n_0$ and $q = 2, 3, \ldots$. By using the resolvent identity it follows, again as in the proof of Theorem 3.1 of [9], that

$$\begin{aligned}(\mathbf{P}_q - \mathbf{P}_{q,n})\mathbf{P}_q &= -\frac{1}{2\pi i}\int_\Gamma (\mathbf{T}_{q,n} - z\mathbf{I}_q)^{-1}(\mathbf{T}_{q,n} - \mathbf{T}_q)\mathbf{P}_q(\mathbf{T}_q - z\mathbf{I}_q)^{-1}\,dz,\\ (\mathbf{P}_q - \mathbf{P}_{q,n})\mathbf{P}_{q,n} &= -\frac{1}{2\pi i}\int_\Gamma (\mathbf{T}_q - z\mathbf{I}_q)^{-1}(\mathbf{T}_{q,n} - \mathbf{T}_q)\mathbf{P}_{q,n}(\mathbf{T}_{q,n} - z\mathbf{I}_q)^{-1}\,dz.\end{aligned}$$

Further,

$$\begin{aligned}(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{P}_q &= -\frac{1}{2\pi i}\int_\Gamma (\mathbf{T}_q - \mathbf{T}_{q,n})(\mathbf{T}_q - z\mathbf{I}_q)^{-1}\,dz\\ &= -\frac{1}{2\pi i}\int_\Gamma \frac{1}{z}(\mathbf{T}_q - \mathbf{T}_{q,n})\left[\mathbf{T}_q(\mathbf{T}_q - z\mathbf{I}_q)^{-1} - \mathbf{I}_q\right]dz,\end{aligned}$$

and

$$\begin{aligned}(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{P}_{q,n} &= -\frac{1}{2\pi i}\int_\Gamma (\mathbf{T}_q - \mathbf{T}_{q,n})(\mathbf{T}_{q,n} - z\mathbf{I}_q)^{-1}\,dz\\ &= -\frac{1}{2\pi i}\int_\Gamma \frac{1}{z}(\mathbf{T}_q - \mathbf{T}_{q,n})\left[\mathbf{T}_{q,n}(\mathbf{T}_{q,n} - z\mathbf{I}_q)^{-1} - \mathbf{I}_q\right]dz.\end{aligned}$$

The desired results follow by noting that $-\frac{1}{2\pi i}\int_\Gamma \frac{dz}{z} = 0$, as Γ does not enclose 0, and $\|(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{T}_q\|_\infty \to 0$, $\|(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{T}_{q,n}\|_\infty \to 0$ as $n \to \infty$ uniformly in $q = 2, 3, \ldots$. □

In order to treat the case when $\min\{|\lambda| : \lambda \in \Lambda\} \le 1$, we consider a scaling of the operator T. Let α be a positive number. Then

$$\sigma(\alpha T) = \{\alpha\lambda : \lambda \in \sigma(T)\}.$$

Also, if λ is an isolated eigenvalue of T having finite algebraic multiplicity and P_λ is the corresponding spectral projection, then $\alpha\lambda$ is an eigenvalue of αT with the same spectral projection, since

$$-\frac{1}{2\pi i}\int_{\alpha\Gamma}(\alpha T - wI)^{-1}dw = -\frac{1}{2\pi i}\int_\Gamma (T - zI)^{-1}dz = P_\lambda.$$

Let $\mathbf{T}_q(\alpha)$ and $\mathbf{T}_{q,n}(\alpha)$ denote the operators obtained by replacing T and T_n by αT and αT_n in $\mathbf{T}_q$ and $\mathbf{T}_{q,n}$, respectively.

Lemma 3.4. *For* $\alpha > 0$, *let* $D_q(\alpha) : \mathbf{X}_q \to \mathbf{X}_q$ *be given by*

$$D_q(\alpha)[x_1, \ldots, x_q]^t = [x_1, \alpha x_2, \ldots, \alpha^{q-1}x_q]^t.$$

Then

(a) $\mathbf{T}_q(\alpha) = (D_q(\alpha))^{-1}(\alpha\mathbf{T}_q)D_q(\alpha)$, $\quad \mathbf{T}_{q,n}(\alpha) = (D_q(\alpha))^{-1}(\alpha\mathbf{T}_{q,n})D_q(\alpha)$,

$$\sigma(\mathbf{T}_q(\alpha)) = \{\alpha\lambda : \lambda \in \sigma(\mathbf{T}_q)\}, \quad \sigma(\mathbf{T}_{q,n}(\alpha)) = \{\alpha\lambda : \lambda \in \sigma(\mathbf{T}_{q,n})\}.$$

(**b**) *Let* $\mathbf{P}_q(\alpha)$ *and* $\mathbf{P}_{q,n}(\alpha)$, *respectively, denote the spectral projections associated with the operators* $\mathbf{T}_q(\alpha)$ *and* $\mathbf{T}_{q,n}(\alpha)$ *with respect to the curve* $\alpha\Gamma$. *Then*

(i) $\mathbf{P}_q(\alpha) = (D_q(\alpha))^{-1}\mathbf{P}_q D_q(\alpha), \mathbf{P}_{q,n}(\alpha) = (D_q(\alpha))^{-1}\mathbf{P}_{q,n} D_q(\alpha)$,

(ii) $[x_1, \dots, x_q]^t \in R(\mathbf{P}_q(\alpha))$ *(resp.,* $R(\mathbf{P}_{q,n}(\alpha))$*) if and only if* $[x_1, \alpha x_2, \dots, \alpha^{q-1}x_q]^t \in R(\mathbf{P}_q)$ *(resp.,* $R(\mathbf{P}_{q,n})$*),*

(iii) $\operatorname{rank}\mathbf{P}_q(\alpha) = \operatorname{rank}\mathbf{P}_q$, $\operatorname{rank}\mathbf{P}_{q,n}(\alpha) = \operatorname{rank}\mathbf{P}_{q,n}$.

Proof. (**a**) Considering the $q \times q$ matrix representing the operator $\mathbf{T}_q$ and the $q \times q$ diagonal matrix diag $(1, \alpha, \dots, \alpha^{q-1})$ representing the operator $D_q(\alpha)$, we obtain $\mathbf{T}_q(\alpha) = (D_q(\alpha))^{-1}(\alpha\mathbf{T}_q)D_q(\alpha)$ by direct multiplication. Since $\mathbf{T}_q(\alpha)$ and $\alpha\mathbf{T}_q$ are thus similar operators, their spectra are identical. The consideration for $\mathbf{T}_{q,n}(\alpha)$ is exactly the same.

(**b**) We have

$$\begin{aligned}
\mathbf{P}_q(\alpha) &= -\frac{1}{2\pi i}\int_{\alpha\Gamma}(\mathbf{T}_q(\alpha) - w\mathbf{I}_q)^{-1}dw \\
&= -\frac{1}{2\pi i}\int_{\alpha\Gamma}(D_q(\alpha))^{-1}(\alpha\mathbf{T}_q - w\mathbf{I}_q)^{-1}D_q(\alpha)dw \\
&= (D_q(\alpha))^{-1}\left(-\frac{1}{2\pi i}\int_{\Gamma}(\mathbf{T}_q - z\mathbf{I}_q)^{-1}dz\right)D_q(\alpha) = (D_q(\alpha))^{-1}\mathbf{P}_q D_q(\alpha).
\end{aligned}$$

Now $\mathbf{x}$ belongs to $R(\mathbf{P}_q(\alpha))$, that is, $\mathbf{P}_q(\alpha)\mathbf{x} = \mathbf{x}$ if and only if $\mathbf{P}_q D_q(\alpha)\mathbf{x} = D_q(\alpha)\mathbf{x}$, that is, $D_q(\alpha)\mathbf{x}$ belongs to $R(\mathbf{P}_q)$. This implies that rank $\mathbf{P}_q(\alpha) =$ rank $\mathbf{P}_q$. The consideration for $\mathbf{P}_{q,n}(\alpha)$ are exactly the same. □

Theorem 3.5. *For all large* n *and all* $q = 2, 3, \dots$, *let*

$$Y_{q,n} = \{\, x_1 \in X \,:\, [x_1, \dots, x_q]^t \in R(\mathbf{P}_{q,n}) \text{ for some } x_2, \dots, x_q \in X\}.$$

Then

(**a**) $\operatorname{rank}\mathbf{P}_{q,n} = \operatorname{rank}\mathbf{P}_q = \operatorname{rank} P = \dim Y_{q,n}$.

(**b**) *Let* $\min\{|\lambda| : \lambda \in \Lambda\} > \epsilon$. *Then for all large* n *and all* $q = 2, 3, \dots$,

$$\hat{\delta}(R(P), Y_{q,n}) \le \frac{C}{\epsilon^{q-1}}\min\left\{\|(T - T_n)^q{}_{|R(P)}\|,\ \|(T - T_n)^q T_n\|\right\}$$

for some constant C, *independent of* n *and* q.

Proof. First we consider a special case when $\min\{|\lambda| : \lambda \in \Lambda\} > 1$. In that case, we assume $\min\{|z| : z \in \Gamma\} = \delta > 1$.

(**a**) Since

$$\|(\mathbf{P}_q - \mathbf{P}_{q,n})^2\|_\infty \le \|(\mathbf{P}_q - \mathbf{P}_{q,n})\mathbf{P}_q\|_\infty + \|(\mathbf{P}_q - \mathbf{P}_{q,n})\mathbf{P}_{q,n}\|_\infty,$$

it follows from Theorem 3.3 that there is a positive integer n_0 such that for all $n \ge n_0$ and all $q = 2, 3, \dots$, we have $\|(\mathbf{P}_q - \mathbf{P}_{q,n})^2\|_\infty < 1$, so that rank $\mathbf{P}_{q,n} =$ rank $\mathbf{P}_q$. We have already noted that rank $\mathbf{P}_q =$ rank P for all q. Next, since $\delta > 1$, we note that for $j = 1, \dots, q-1$,

$$\begin{aligned}
\|S_j\| &= \|-\frac{1}{2\pi i}\int_\Gamma \frac{R(z)}{z^j}dz\| \le \frac{\ell(\Gamma)}{2\pi\delta^j}\max_{z\in\Gamma}\|R(z)\| \\
&\le \frac{\ell(\Gamma)}{2\pi}\max_{z\in\Gamma}\|R(z)\| = c, \text{ say.}
\end{aligned}$$

Thus for $x \in R(P)$, we have

$$\begin{aligned} \|J_q x\|_\infty &= \max\{\|x\|, \|S_1 x\|, \dots, \|S_{q-1}x\|\} \\ &\le \max\{1, \|S_1\|, \dots, \|S_{q-1}\|\}\|x\| \\ &\le \max\{1, c\}\|x\|. \end{aligned}$$

Let $\phi_1, \dots, \phi_m$ be a basis of $R(P)$ and $\phi_1^*, \dots, \phi_m^*$ be the corresponding adjoint basis of $R(P^*)$. For $j = 1, \dots, m$, let $\psi_{q,n,j}$ denote the first component of $\mathbf{P}_{q,n}J_q\phi_j$. For $k = 1, \dots, m$, we have $K_q\phi_k^* = [\phi_k^*, 0, \dots, 0]^t$. Hence

$$\langle \psi_{q,n,j}, \phi_k^* \rangle = \langle \mathbf{P}_{q,n}J_q\phi_j, K_q\phi_k^* \rangle$$

for $j, k, = 1, \dots, m$. Now fix $j, 1 \le j \le m$. Then

$$\begin{aligned} \|\mathbf{P}_{q,n}J_q\phi_j - J_q\phi_j\|_\infty &= \|(\mathbf{P}_{q,n} - \mathbf{P}_q)\mathbf{P}_q J_q\phi_j\|_\infty \\ &\le \|(\mathbf{P}_{q,n} - \mathbf{P}_q)\mathbf{P}_q\|_\infty \, \|J_q\phi_j\|_\infty \\ &\le \max\{1, c\}\|(\mathbf{P}_{q,n} - \mathbf{P}_q)\mathbf{P}_q\|_\infty \, \|\phi_j\|. \end{aligned}$$

Since $\|(\mathbf{P}_{q,n} - \mathbf{P}_q)\mathbf{P}_q\|_\infty \to 0$ as $n \to \infty$ uniformly in $q = 2, 3, \dots$, by Theorem 3.3, we see that for all $k = 1, \dots, m$

$$\langle \psi_{q,n,j}, \phi_k^* \rangle \to \langle J_q\phi_j, K_q\phi_k^* \rangle = \delta_{j,k}$$

as $n \to \infty$, uniformly in $q = 2, 3 \dots$. This shows that there is a positive integer n_0 such that for all $n \ge n_0$ and all $q = 2, 3, \dots$, the $m \times m$ matrix $[\langle \psi_{q,n,j}, \phi_k^* \rangle]$ is nonsingular, and hence, $\{\psi_{q,n,1}, \dots, \psi_{q,n,m}\}$ is a linearly independent subset of $Y_{q,n}$. Thus dim $Y_{q,n} \ge m$. On the other hand, dim $Y_{q,n} \le \dim R(\mathbf{P}_{q,n}) =$ rank $P = m$ for all large n and all $q = 2, 3 \dots$. Hence, dim $Y_{q,n} =$ rank $P = m$.

(b) We have

$$\delta(R(P), Y_{q,n}) = \sup\{\text{dist }(x, Y_{q,n}) : x \in R(P), \|x\| = 1\}.$$

Consider $x \in R(P)$ with $\|x\| = 1$ and $J_q x = [x, S_1 x, \dots, S_q x]^t$. If $y_1 \in Y_{q,n}$, then there is some $\mathbf{y} \in R(\mathbf{P}_{q,n})$ with $\mathbf{y} = [y_1, \dots, y_q]$. Since $\|x - y_1\| \le \|J_q x - \mathbf{y}\|_\infty$, we have

$$\begin{aligned} \text{dist}(x, Y_{q,n}) &= \inf\{\|x - y_1\| : y_1 \in Y_{q,n}\} \\ &\le \inf\{\|J_q x - \mathbf{y}\|_\infty : \mathbf{y} \in R(\mathbf{P}_{q,n})\}. \end{aligned}$$

Let $\mathbf{x} = J_q x / \|J_q x\|_\infty$, so that $\mathbf{x} \in R(\mathbf{P}_q)$ and $\|\mathbf{x}\|_\infty = 1$. Thus

$$\begin{aligned} \text{dist}(x, Y_{q,n}) &\le \|J_q x\|_\infty \inf\{\|\mathbf{x} - \mathbf{y}\|_\infty : \mathbf{y} \in R(\mathbf{P}_{q,n})\} \\ &\le \|J_q x\|_\infty \delta(R(\mathbf{P}_q), R(\mathbf{P}_{q,n})) \\ &\le \max\{1, c\}\delta(R(\mathbf{P}_q), R(\mathbf{P}_{q,n}))\|x\|, \end{aligned}$$

since $\|J_q x\|_\infty \le \max\{1, c\}\|x\|$, as we have just seen. This implies that

$$\delta(R(P), Y_{q,n}) \le \max\{1, c\}\delta(R(\mathbf{P}_q), R(\mathbf{P}_{q,n})).$$

But since $\min\{\, |\lambda| : \lambda \in \Lambda\} > 1$, it follows from Theorem 3.3 that

$$\delta(R(\mathbf{P}_q), R(\mathbf{P}_{q,n})) \le \|(\mathbf{P}_q - \mathbf{P}_{q,n})\mathbf{P}_q\|_\infty \to 0$$

as $n \to \infty$ uniformly in $q = 2, 3 \dots$. Hence, we can choose n_0 so large that $\delta(R(P), Y_{q,n}) \le 1/2$ for all $n \ge n_0$ and $q = 2, 3, \dots$. Since dim $Y_{q,n} =$ dim $R(P)$, we have

$$\delta(Y_{q,n}, R(P)) \le \frac{\delta(R(P), Y_{q,n})}{1 - \delta(R(P), Y_{q,n})} \le 2\delta(R(P), Y_{q,n})$$

(see [8], p. 264-269). Thus

$$\begin{aligned}\hat{\delta}(R(P), Y_{q,n}) &= \max\{\delta(R(P), Y_{q,n}), \delta(Y_{q,n}, R(P))\} \\ &\leq 2\delta(R(P), Y_{q,n}) \\ &\leq 2\max\{1, c\}\delta(R(\mathbf{P}_q), R(\mathbf{P}_{q,n})) \\ &\leq 2\max\{1, c\}\hat{\delta}(R(\mathbf{P}_q), R(\mathbf{P}_{q,n})).\end{aligned}$$

Since $\|\mathbf{T}_{q,n}\|_\infty$ is bounded in q and n, and

$$\|(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{T}_q\|_\infty \to 0, \qquad \|(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{T}_{q,n}\|_\infty \to 0$$

as $n \to \infty$ uniformly in q, Theorem 2.2 applied to the operators $\mathbf{T}_q$ and $\mathbf{T}_{q,n}$ shows that

$$\hat{\delta}(R(\mathbf{P}_q), R(\mathbf{P}_{q,n})) \leq \frac{\ell(\Gamma)}{\pi} C_1 C_2 \min\left\{\|(\mathbf{T}_q - \mathbf{T}_{q,n})_{|R(\mathbf{P}_q)}\|_\infty, \right. \\ \left. \|(\mathbf{T}_q - \mathbf{T}_{q,n})_{|R(\mathbf{P}_{q,n})}\|_\infty\right\},$$

where $\sup_{z\in\Gamma} \|(\mathbf{T}_q - z\mathbf{I}_q)^{-1}\|_\infty \leq C_1$ and $\sup_{z\in\Gamma} \|(\mathbf{T}_{q,n} - z\mathbf{I}_q)^{-1}\|_\infty \leq C_2$. But by Proposition 3.1 with $\epsilon = 1$, we have

$$\|(\mathbf{T}_q - \mathbf{T}_{q,n})_{|R(\mathbf{P}_q)}\|_\infty \leq \frac{\ell(\Gamma)c_1}{2\pi}\|(T - T_n)^q{}_{|R(P)}\|.$$

Also,

$$\begin{aligned}\|(\mathbf{T}_q - \mathbf{T}_{q,n})_{|R(\mathbf{P}_{q,n})})\|_\infty &\leq \|(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{P}_{q,n}\|_\infty \\ &\leq \frac{\ell(\Gamma)}{2\pi\delta} C_2 \|(\mathbf{T}_q - \mathbf{T}_{q,n})\mathbf{T}_{q,n}\|_\infty \\ &\leq \frac{\ell(\Gamma)}{2\pi\delta} C_2 \|\Delta_n^q T_n\|,\end{aligned}$$

as we have already seen. (Note that $\min\{|z| : z \in \Gamma\} = \delta > 1$.) Thus

$$\hat{\delta}(R(P), Y_{q,n}) \leq C \min\left\{\|(T - T_n)^q{}_{|R(P)}\|, \|(T - T_n)^q T_n\|\right\}$$

for all large n, $q = 2, 3, \ldots$ and some constant C, independent of n and q.

Finally, we consider the general case when $\min\{\,|\lambda| : \lambda \in \Lambda\,\} > \epsilon$. We choose $\alpha = 1/\epsilon$ and consider the scaled operators $\mathbf{T}_q(\alpha), \mathbf{P}_q(\alpha), \mathbf{T}_{q,n}(\alpha)$ and $\mathbf{P}_{q,n}(\alpha)$. By what we have just proved and by Lemma 3.4,

$$\text{rank } \mathbf{P}_{q,n} = \text{ rank } \mathbf{P}_{q,n}(\alpha) = \text{ rank } \mathbf{P}_q(\alpha) = \text{ rank } \mathbf{P}_q$$

for all large n and $q = 2, 3\ldots.$ Also, $P(\alpha) = P$ and the first components of the elements of $R(\mathbf{P}_{q,n}(\alpha))$ and $R(\mathbf{P}_{q,n})$ are the same, that is, $Y_{q,n}(\alpha) = Y_{q,n}$. Hence there is a constant $C(\alpha)$ independent of n and q such that

$$\begin{aligned}\hat{\delta}(R(P), Y_{q,n}) &= \hat{\delta}(R(P(\alpha)), Y_{q,n}(\alpha)) \\ &\leq C(\alpha) \min\left\{\|(\alpha T - \alpha T_n)^q{}_{|R(P(\alpha))}\|, \|(\alpha T - \alpha T_n)^q \alpha T_n\|\right\} \\ &= \frac{\max\{\alpha, \alpha^2\}C(\alpha)}{\epsilon^{q-1}} \min\left\{\|(T - T_n)^q{}_{|R(P)}\|, \|(T - T_n)^q T_n\|\right\}.\end{aligned}$$

Hence the result. □

Recall that $\Lambda = \{\lambda_1, \ldots, \lambda_m\}$, where each $\lambda_j, j = 1, \ldots, m$, is an eigenvalue of T counted according to its algebraic multiplicity and

$$\hat{\lambda} = \frac{\lambda_1 + \ldots + \lambda_m}{m}.$$

Since rank $\mathbf{P}_{q,n} = m$ for all large n and all $q = 2, 3, \dots$,

$$\sigma(\mathbf{T}_{q,n}) \cap \text{ Int } \Gamma = \{\lambda_{q,n,1}, \dots , \lambda_{q,n,m}\},$$

where each $\lambda_{q,n,j}, j = 1, \dots , m$, is an eigenvalue of $\mathbf{T}_{q,n}$ counted according to its algebraic multiplicity.

These are the eigenvalues obtained from qth spectral analysis of T. In the words of Dellwo and Friedman, they comprise the legitimate portion of the qth order approximate spectra associated with Λ.

Let

$$\hat{\lambda}_{q,n} = \frac{\lambda_{q,n,1} + \dots + \lambda_{q,n,m}}{m}.$$

Theorem 3.6. *Let* $\min\{|\lambda| : \lambda \in \Lambda\} > \epsilon$. *Then for all large* n *and all* $q = 2, 3, \dots$,

$$|\hat{\lambda} - \hat{\lambda}_{q,n}| \le \frac{C}{\epsilon^{q-1}} \min \left\{ \|(T - T_n)^q{}_{|R(P)}\|, \; \|(T - T_n)^q T_n\| \right\}$$

for some constant C*, independent of* n *and* q.

If $\Lambda = \{\lambda\}$ *and the ascent of* λ *as an eigenvalue of* T *is* l*, then for each* $i = 1, \dots , m$*, we have*

$$|\lambda - \lambda_{q,n,i}|^l \le \frac{C'}{\epsilon^{q-1}} \min \left\{ \|(T - T_n)^q{}_{|R(P)}\|, \; \|(T - T_n)^q T_n\| \right\}$$

for some constant C'*, independent of* n *and* q.

Proof. First we consider a special case when $\min\{ |\lambda| : \lambda \in \Lambda\} > 1$. By Theorem 3.3, there is some n_0 such that for all $n \ge n_0$ and all $q = 2, 3, \dots$, we have $\|(\mathbf{P}_q - \mathbf{P}_{q,n})\mathbf{P}_q\|_\infty \le 1/2$. It follows that for all such n and $q = 2, 3, \dots$, the map $\mathbf{A}_{q,n}$ from $R(\mathbf{P}_q)$ to $R(\mathbf{P}_{q,n})$ given by $\mathbf{x} \longmapsto \mathbf{P}_{q,n}\mathbf{x}$ is an isomorphism and if $\mathbf{A}_{q,n}^{-1}$ denotes the inverse map from $R(\mathbf{P}_{q,n})$ to $R(\mathbf{P}_q)$, then $\|\mathbf{A}_{q,n}^{-1}\|_\infty \le 2$. The same argument also shows that the map $\mathbf{B}_{q,n}$ from $R(\mathbf{P}_{q,n})$ to $R(\mathbf{P}_q)$ given by $\mathbf{x} \longmapsto \mathbf{P}_q\mathbf{x}$ is an isomorphism and $\|\mathbf{B}_{q,n}^{-1}\|_\infty \le 2$. We choose Γ so that $\min\{|z| : z \in \Gamma\} = \delta > 1$. Then by Proposition 3.2, we have

$$\|\mathbf{P}_q\|_\infty \le \frac{\ell(\Gamma)}{2\pi} C_1 \quad \text{and} \quad \|\mathbf{P}_{q,n}\|_\infty \le \frac{\ell(\Gamma)}{2\pi} C_2,$$

where C_1 and C_2 are independent of n and q. Noting that the algebraic multiplicity of each λ_i as an eigenvalue of $\mathbf{T}_q$ is equal to its algebraic multiplicity as an eigenvalue of T, and applying Theorem 2.3 to the operators $\mathbf{T}_q$ and $\mathbf{T}_{q,n}$, we have

$$\begin{aligned} |\hat{\lambda} - \hat{\lambda}_{q,n}| \le 2 \min \{ &\|\mathbf{P}_{q,n}\|_\infty \|(\mathbf{T}_q - \mathbf{T}_{q,n})_{|R(\mathbf{P}_q)}\|_\infty, \\ &\|\mathbf{P}_q\|_\infty \|(\mathbf{T}_q - \mathbf{T}_{q,n})_{|R(\mathbf{P}_{q,n})}\|_\infty \}. \end{aligned}$$

But by Proposition 3.1 with $\epsilon = 1$, we have

$$\begin{aligned} 2\|\mathbf{P}_{q,n}\|_\infty \|(\mathbf{T}_q - \mathbf{T}_{q,n})_{|R(\mathbf{P}_q)}\|_\infty &\le \frac{\ell(\Gamma)}{\pi} C_2 \|(\mathbf{T}_q - \mathbf{T}_{q,n})_{|R(\mathbf{P}_q)}\|_\infty \\ &\le \frac{\ell(\Gamma)}{\pi} C_2 \frac{\ell(\Gamma)}{2\pi} c_1 \|(T - T_n)^q{}_{|R(P)}\|, \end{aligned}$$

and

$$\begin{aligned} 2\|\mathbf{P}_q\|_\infty \|(\mathbf{T}_q - \mathbf{T}_{q,n})_{|R(\mathbf{P}_{q,n})}\|_\infty &\le \frac{\ell(\Gamma)}{\pi} C_1 \|(\mathbf{T}_q - \mathbf{T}_{q,n})_{|R(\mathbf{P}_{q,n})}\|_\infty \\ &\le \frac{\ell(\Gamma)}{\pi} C_1 \frac{\ell(\Gamma)}{2\pi\delta} C_2 \|(T - T_n)^q T_n\|, \end{aligned}$$

as in the proof of Theorem 3.5(b). Thus

$$|\hat{\lambda} - \hat{\lambda}_{q,n}| \le C \min\left\{\|(T-T_n)^q{}_{|R(P)}\|,\ \|(T-T_n)^q T_n\|\right\}$$

for all large $n, q = 2, 3, \ldots$ and some constant C, independent of n and q.

Now let $\Lambda = \{\lambda\}$, so that $\hat{\lambda} = \lambda$. We have noted that the ascent of λ as an eigenvalue of $\mathbf{T}_q$ is equal to its ascent as an eigenvalue of T, namely l. Again, applying Theorem 2.3 to the operators $\mathbf{T}_q$ and $\mathbf{T}_{q,n}$, we have for $j = 1, \ldots, m$,

$$\begin{aligned}|\lambda - \lambda_{q,n,j}|^l \le 2\min\{&C_{q,n}\|\mathbf{P}_{q,n}\|_\infty \|(\mathbf{T}_q - \mathbf{T}_{q,n})_{|R(\mathbf{P}_q)}\|_\infty,\\ &D_{q,n}\|\mathbf{P}_q\|_\infty \|(\mathbf{T}_q - \mathbf{T}_{q,n})_{|R(\mathbf{P}_{q,n})}\|_\infty\},\end{aligned}$$

where

$$\begin{aligned} C_{q,n} &= \sum_{k=0}^{l-1} \|\lambda \mathbf{I}_{q|R(\mathbf{P}_q)} - \mathbf{A}_{q,n}^{-1}\mathbf{T}_{q,n}\mathbf{A}_{q,n}\|_\infty^{l-1-k}\, \|\lambda\mathbf{I}_{q|R(\mathbf{P}_q)} - \mathbf{T}_{q|R(\mathbf{P}_q)}\|_\infty^k,\\ D_{q,n} &= \sum_{k=0}^{l-1} \|\lambda \mathbf{I}_{q|R(\mathbf{P}_{q,n})} - \mathbf{T}_{q,n|R(\mathbf{P}_{q,n})}\|_\infty^{l-1-k}\, \|\lambda\mathbf{I}_{q|R(\mathbf{P}_{q,n})} - \mathbf{B}_{q,n}^{-1}\mathbf{T}_q\mathbf{B}_{q,n}\|_\infty^k. \end{aligned}$$

Note that $\min\{|z| : z \in \Gamma\} = \delta > 1$. Therefore, for $j = 1, \ldots, m$ we have

$$|\lambda - \lambda_{q,n,j}|^l \le C' \min\left\{\|(T-T_n)^q{}_{|R(P)}\|,\ \|(T-T_n)^q T_n\|\right\}$$

for all large $n, q = 2, 3, \ldots$ and some constant C', independent of n and q.

Finally, we consider the general when case $\min\{\,|\lambda| : \lambda \in \Lambda\} > \epsilon$. We choose $\alpha = 1/\epsilon$ and consider the scaled operators $\mathbf{T}_q(\alpha)$ and $\mathbf{T}_{q,n}(\alpha)$. By Lemma 3.4, we find that $\alpha\lambda_1, \ldots, \alpha\lambda_m$ are the eigenvalues of $\mathbf{T}_q(\alpha)$ inside the curve $\alpha\Gamma$ and $\alpha\lambda_{q,n,1}, \ldots, \alpha\lambda_{q,n,m}$ are the eigenvalues of $\mathbf{T}_{q,n}(\alpha)$ inside the curve $\alpha\Gamma$, counted according to their algebraic multiplicities. By what we have just proved, there is some constant $C(\alpha)$, independent of n and q such that

$$|\alpha\hat{\lambda} - \alpha\hat{\lambda}_{q,n}| \le C(\alpha) \min\left\{\, \|(\alpha T - \alpha T_n)^q{}_{|R(P)}\|,\ \|(\alpha T - \alpha T_n)^q \alpha T_n\|\,\right\},$$

so that

$$|\hat{\lambda} - \hat{\lambda}_{q,n}| \le \frac{\max\{1, \alpha\}C(\alpha)}{\epsilon^{q-1}} \min\left\{\|(T-T_n)^q{}_{|R(P)}\|,\ \|(T-T_n)^q T_n\|\,\right\},$$

as desired. If $\Lambda = \{\lambda\}$, then the ascent of $\alpha\lambda$ as an eigenvalue of $\mathbf{T}_q(\alpha)$ equals l. Hence the estimates for $|\lambda - \lambda_{q,n,j}|^l, j = 1, \ldots, m$, follow similarly. □

If $\Lambda = \{\lambda\}$ with $|\lambda| > \epsilon$ and f is an analytic function in the neighborhood of λ, then by the functional calculus it is easy to see that for all large n and $q = 2, 3, \ldots$,

$$|f(\lambda) - \frac{1}{m}\sum_{j=1}^{m} f(\lambda_{q,n,j})| \le \frac{C}{\epsilon^{q-1}} \min\left\{\|(T-T_n)^q{}_{|R(P)}\|,\ \|(T-T_n)^q T_n\|\right\},$$

where C is a constant independent of n and q. The above-mentioned estimate is, in fact, an accelerated analogue of a result of Descloux, Nassif and Rappaz ([6], [7]).

Next, we consider approximation of an element of $R(P)$ by an eigenvector obtained from a higher order spectral analysis of T.

Theorem 3.7. *Let* $\mathbf{x} = [x_1, \dots, x_q]^t$ *be an eigenvector of* $\mathbf{T}_{q,n}$ *corresponding to an eigenvalue in* $\Lambda_{q,n}$ *such that* $\|x_1\| = 1$. *If* $\min\{ |\lambda| : \lambda \in \Lambda\} > \epsilon$, *then there is a constant* C *such that*

$$\|x_1 - Px_1\| \le \frac{C}{\epsilon^{q-1}} \|(T - T_n)^q T_n\|$$

for all large n *and* $q = 2, 3, \dots$.

Proof. Since $\mathbf{P}_q\mathbf{x} = [Px_1,\ S_1x_1, \dots, S_{q-1}x_1]^t$ and $\mathbf{P}_{q,n}\mathbf{x} = \mathbf{x}$, we have

$$\|x_1 - Px_1\| \le \|\mathbf{x} - \mathbf{P}_q\mathbf{x}\|_\infty = \|(\mathbf{P}_{q,n} - \mathbf{P}_q)\mathbf{P}_{q,n}\mathbf{x}\|_\infty.$$

If $\min\{ |\lambda| : \lambda \in \Lambda\} > 1$, then $\|\mathbf{x}\|_\infty = \|x_1\| = 1$ and hence Theorem 3.3 shows there is a constant C with

$$\|x_1 - Px_1\| \le \|(\mathbf{P}_{q,n} - \mathbf{P}_q)\mathbf{P}_{q,n}\|_\infty \le C\|(\mathbf{T}_{q,n} - \mathbf{T}_q)\mathbf{T}_{q,n}\|_\infty = C\|(T - T_n)^q T_n\|.$$

If $\min\{ |\lambda| : \lambda \in \Lambda\} > \epsilon$, then, as before, we choose $\alpha = 1/\epsilon$ and consider the scaled operators $\mathbf{T}_{q,n}(\alpha), \mathbf{P}_{q,n}(\alpha), \mathbf{T}_q(\alpha)$ and $\mathbf{P}_q(\alpha)$. Since $P(\alpha) = P$ and the first components of the elements of $R(\mathbf{P}_{q,n}(\alpha))$ and $R(\mathbf{P}_{q,n})$ are the same, there is a constant $C(\alpha)$ such that

$$\|x_1 - Px_1\| \le C(\alpha)\|(\alpha T - \alpha T_n)^q \alpha T_n\| = \frac{\alpha^2 C(\alpha)}{\epsilon^{q-1}} \|(T - T_n)^q T_n\|.$$

□

The result in Theorem 3.7 was also obtained by Dellwo and Friedman in [5].

Let $\Lambda = \{\lambda\}$ and l be the ascent of λ. Let $\lambda_{q,n} \in \Lambda_{q,n}$ such that $\lambda_{q,n} \to \lambda$ as $n \to \infty$ uniformly in $q = 2, 3, \dots$.

Theorem 3.8. *Let* $|\lambda| > \epsilon$. *Suppose that*

$$\mathbf{w}_{q,n} = \left[w_{q,n}, \frac{w_{q,n}}{\lambda_{q,n}}, \dots, \frac{w_{q,n}}{(\lambda_{q,n})^{q-1}} \right]^t \in N(\mathbf{T}_{q,n} - \lambda_{q,n}\mathbf{I}_q)$$

with $\|w_{q,n}\| = 1$. *Then there is some* $u_{q,n} \in N(T - \lambda I)$ *such that*

$$\|w_{q,n} - u_{q,n}\| \le C \left\{ \frac{1}{\epsilon^{q-1}} \|(T - T_n)^q{}_{|R(P)}\| \right\}^{1/l}$$

for all $q = 2, 3, \dots$, *where* C *is a constant independent of* q *and* n.

Proof. Consider the particular case when $|\lambda| > 1$. Then for all large n and $q = 2, 3, \dots$, $|\lambda_{q,n}| > 1$. Thus $\|\mathbf{w}_{q,n}\|_\infty = \|w_{q,n}\| = 1$. Applying Theorem 2.4 to the operators $\mathbf{T}_q$ and $\mathbf{T}_{q,n}$, we obtain some $\mathbf{u}_{q,n} \in N(\mathbf{T}_q - \lambda\mathbf{I}_q)$ such that

$$\|\mathbf{w}_{q,n} - \mathbf{u}_{q,n}\|_\infty \le C \left\{ \|(\mathbf{T}_q - \mathbf{T}_{q,n})_{|R(\mathbf{P}_q)}\|_\infty \right\}^{1/l}.$$

Since $\mathbf{u}_{q,n} = J_q u_{q,n}$ for some $u_{q,n} \in N(T - \lambda I)$, Proposition 3.1 (with $\epsilon = 1$) shows that

$$\|w_{q,n} - u_{q,n}\| \le \|\mathbf{w}_{q,n} - \mathbf{u}_{q,n}\|_\infty \le C' \left\{ \|(T - T_n)^q{}_{|R(P)}\| \right\}^{1/l},$$

for some constant C', independent of n and q.

Finally, to treat the general case, we choose $\alpha = 1/\epsilon$ and consider the scaled operators $\mathbf{T}_q(\alpha)$, $\mathbf{P}_q(\alpha)$, $\mathbf{T}_{q,n}(\alpha)$ and $\mathbf{P}_{q,n}(\alpha)$. Since $P(\alpha) = P$ and the first components of the elements of $R(\mathbf{P}_{q,n}(\alpha))$ and $R(\mathbf{P}_{q,n})$ are the same, there is a constant $C'(\alpha)$ such that

$$\begin{aligned}\|u_{q,n} - w_{q,n}\| &\leq C'(\alpha)\left\{\|(\alpha T - \alpha T_n)^q{}_{|R(P)}\|\right\}^{1/l}\\ &= C'(\alpha)\left\{\frac{1}{\epsilon^q}\|(T - T_n)^q{}_{|R(P)}\|\right\}^{1/l}. \quad \square\end{aligned}$$

Hence the result follows.

4. Numerical examples

Let λ be a nonzero defective eigenvalue of T of algebraic multiplicity m and ascent $l > 1$. We illustrate by numerical examples how the weighted arithmetic mean $\hat{\lambda}_{q,n}$ gives a better approximation than do the individual eigenvalues $\lambda_{q,n,1}, \ldots, \lambda_{q,n,m}$, provided by the qth order spectral analysis when $q = 2, 3, 4, 5, 6$.

Let $X = C([a, b])$ and T be an integral operator given by

$$Tx(s) = \int_a^b k(s,t)x(t)\,dt, \quad x \in X, \quad s \in [a,b],$$

where the kernel k is continuous on $[a,b] \times [a,b]$. In actual computations, T is replaced by its approximation $\tilde{T}$ given by

$$\tilde{T}x(s) = \sum_{j=1}^{M} w_j^{(M)} k(s, t_j^{(M)}) x(t_j^{(M)}), \quad x \in X, \quad s \in [a,b],$$

where M is very large. Here the nodes $t_1^{(M)}, \ldots, t_M^{(M)}$ in $[a,b]$ and the weights $w_1^{(M)}, \ldots, w_M^{(M)}$ in $\mathbb{C}$ give a convergent quadrature formula

$$Qx = \sum_{j=1}^{M} w_j^{(M)} x(t_j^{(M)}), \quad x \in X.$$

Consider a finite rank operator T_n given by

$$T_n x = \sum_{j=1}^{n} \langle x,\, x_j^* \rangle x_j, \quad x \in X,$$

where $x_1, \ldots, x_n$ are in X and $x_1^*, \ldots, x_n^*$ are in X^*. Then the eigenvalue problem for $\mathbf{T}_{q,n}$ can be reduced to an eigenvalue problem for the matrix $\mathbf{A}_{q,n}$ where

$$\mathbf{A}_{q,n} = \begin{bmatrix} A_n^{(0)} & A_n^{(1)} & \cdots & \cdots & A_n^{(q-1)} \\ I_n & 0 & \cdots & \therefore & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & I_n & 0 \end{bmatrix}.$$

Here $A_n^{(k)} = \left[\langle \Delta_n^k x_j,\, x_i^* \rangle\right]$ for $k = 0, 1, \ldots, q-1$ and I_n is the $n \times n$ identity matrix.

Note that this matrix eigenvalue problem is of size nq. If $\|T - T_n\| \to 0$, then both $\|(T-T_n)^q{}_{|R(P)}\|$ and $\|(T-T_n)^q T_n\|$ are less than or equal to a constant times $\|T-T_n\|^q$. Thus if we keep n fixed and increase the order q of the spectral analysis, then the size of the matrix eigenvalue problem increases arithmetically with respect to q while the accuracy of the spectral approximation increases geometrically.

Example. Let $a = 0, b = 1$ and

$$k(s,t) = \begin{cases} s - t/2, & \text{if} \quad 0 \le s \le t \le 1 \\ t/2, & \text{if} \quad 0 \le t < s \le 1. \end{cases}$$

Then each $\lambda_j = \dfrac{1}{(2j-1)^2\pi^2}$, $j = 1, 2, \dots$, is an eigenvalue of T of algebraic multiplicity $m = 2$ and ascent $l = 2$. We have chosen the nodes and weights as follows:

$$t_i^{(M)} = \begin{cases} \dfrac{i - 1/\sqrt{3}}{M}, & \text{if } i \text{ is odd,} \\ \dfrac{i - 1 + 1/\sqrt{3}}{M}, & \text{if } i \text{ is even,} \end{cases}$$

and

$$w_i^{(M)} = \frac{1}{M}, \quad i = 1, \dots, M.$$

These are obtained by the compound Gauss Two Point Rule on $[0, 1]$.

For $n \ll M$, let $w_1^{(n)}, \dots, w_n^{(n)}$ and $t_1^{(n)}, \dots, t_n^{(n)}$ be the weights and the nodes, respectively, associated with the compound Gauss Two Point Rule on $[0, 1]$. Let $e_1^{(n)}, \dots, e_n^{(n)}$ be the hat functions corresponding to the nodes $t_1^{(n)}, \dots, t_n^{(n)}$. We consider the following two approximations of the integral operator T.

i) Nyström Approximation:

$$T_n^N x(s) = \sum_{j=1}^{n} w_j^{(n)} k(s, t_j^{(n)}) x(t_j^{(n)}), \quad x \in C([0,1]), \quad s \in [0,1].$$

ii) Projection Approximation:

$$T_n^P = \pi_n T, \text{ where } \pi_n x = \sum_{j=1}^{n} x(t_j^{(n)}) e_j^{(n)}, \quad x \in C([0,1]).$$

Note that the sequence (T_n^N) satisfies the hypotheses (H1) and (H2), but $\|T - T_n^N\|$ does not tend to 0 as $n \to \infty$ (cf. [3], p. 197). On the other hand, $\|T - T_n^P\| \to 0$ as $n \to \infty$ (cf. Theorem 4.5 of [3]).

Let $\tilde{\lambda}$ denote the arithmetic mean of the two eigenvalues of $\tilde{T}$ which are close to the largest eigenvalue $\lambda = 1/\pi^2$ of T. Also, let $\lambda_{q,n,1}$ and $\lambda_{q,n,2}$ denote the eigenvalues provided by the qth order spectral analysis which are close to $\tilde{\lambda}$, and $\hat{\lambda}_{q,n} = \dfrac{\lambda_{q,n,1} + \lambda_{q,n,2}}{2}$. We have taken $M = 500$ and $n = 10, 20, 30, 40$. The following computations were performed on HP9000/700 model J200 in single precision with an accuracy of 7 digits and in double precision with an accuracy of 15 digits. These numerical results illustrate that, in general, the rate of convergence of the $\hat{\lambda}_{q,n}$ to $\tilde{\lambda}$ is faster than that of the individual eigenvalues $\lambda_{q,n,1}$ and $\lambda_{q,n,2}$.

Calculations in Single Precision

Table 4.1. $q = 2$, Nyström Approximation

n	$\lvert\tilde{\lambda} - \lambda_{q,n,1}\rvert$	$\lvert\tilde{\lambda} - \lambda_{q,n,2}\rvert$	$\lvert\tilde{\lambda} - \hat{\lambda}_{q,n}\rvert$
10	8.96×10^{-4}	9.50×10^{-4}	9.23×10^{-4}
20	2.15×10^{-4}	2.36×10^{-4}	2.25×10^{-4}
30	1.02×10^{-4}	1.02×10^{-4}	9.98×10^{-5}
40	1.44×10^{-5}	9.78×10^{-5}	5.61×10^{-5}

Table 4.2. $q = 2$, Projection Approximation

n	$\lvert\tilde{\lambda} - \lambda_{q,n,1}\rvert$	$\lvert\tilde{\lambda} - \lambda_{q,n,2}\rvert$	$\lvert\tilde{\lambda} - \hat{\lambda}_{q,n}\rvert$
10	5.29×10^{-5}	5.29×10^{-5}	9.79×10^{-6}
20	3.81×10^{-5}	3.81×10^{-5}	5.74×10^{-7}
30	2.00×10^{-5}	2.00×10^{-5}	7.45×10^{-8}
40	3.37×10^{-5}	3.35×10^{-5}	8.19×10^{-8}

Table 4.3. $q = 3$, Nyström Approximation

n	$\lvert\tilde{\lambda} - \lambda_{q,n,1}\rvert$	$\lvert\tilde{\lambda} - \lambda_{q,n,2}\rvert$	$\lvert\tilde{\lambda} - \hat{\lambda}_{q,n}\rvert$
10	6.69×10^{-5}	6.69×10^{-5}	1.37×10^{-5}
20	3.25×10^{-5}	3.25×10^{-5}	8.94×10^{-7}
30	3.93×10^{-5}	3.87×10^{-5}	3.05×10^{-7}
40	4.67×10^{-5}	4.67×10^{-5}	1.04×10^{-7}

Table 4.4. $q = 3$, Projection Approximation

n	$\lvert\tilde{\lambda} - \lambda_{q,n,1}\rvert$	$\lvert\tilde{\lambda} - \lambda_{q,n,2}\rvert$	$\lvert\tilde{\lambda} - \hat{\lambda}_{q,n}\rvert$
10	1.88×10^{-5}	1.88×10^{-5}	2.98×10^{-8}
20	6.91×10^{-5}	6.89×10^{-5}	7.45×10^{-8}
30	5.75×10^{-5}	5.75×10^{-5}	5.96×10^{-8}
40	$1..99\times10^{-5}$	1.98×10^{-5}	5.96×10^{-8}

Table 4.5. $q = 4$, Nyström Approximation

n	$\lvert\tilde{\lambda} - \lambda_{q,n,1}\rvert$	$\lvert\tilde{\lambda} - \lambda_{q,n,2}\rvert$	$\lvert\tilde{\lambda} - \hat{\lambda}_{q,n}\rvert$
10	2.65×10^{-5}	4.27×10^{-5}	8.08×10^{-6}
20	3.31×10^{-5}	3.31×10^{-5}	4.10×10^{-7}
30	3.95×10^{-5}	3.95×10^{-5}	2.98×10^{-8}
40	5.34×10^{-5}	5.34×10^{-5}	3.72×10^{-8}

Table 4.6. $q = 4$, Projection Approximation

n	$\lvert\tilde{\lambda} - \lambda_{q,n,1}\rvert$	$\lvert\tilde{\lambda} - \lambda_{q,n,2}\rvert$	$\lvert\tilde{\lambda} - \hat{\lambda}_{q,n}\rvert$
10	1.71×10^{-5}	1.71×10^{-5}	1.04×10^{-7}
20	6.47×10^{-5}	6.45×10^{-5}	1.04×10^{-7}
30	3.99×10^{-5}	3.99×10^{-5}	5.96×10^{-8}
40	1.07×10^{-5}	1.07×10^{-5}	8.94×10^{-8}

Calculations in Double Precision

Table 4.7. $q = 5$, Nyström Approximation

n	$\lvert\tilde{\lambda} - \lambda_{q,n,1}\rvert$	$\lvert\tilde{\lambda} - \lambda_{q,n,2}\rvert$	$\lvert\tilde{\lambda} - \hat{\lambda}_{q,n}\rvert$
10	1.72×10^{-7}	1.71×10^{-7}	1.72×10^{-7}
20	4.22×10^{-9}	1.03×10^{-9}	2.62×10^{-9}
30	2.11×10^{-9}	2.11×10^{-9}	2.30×10^{-10}
40	3.01×10^{-9}	3.01×10^{-9}	4.11×10^{-11}

Table 4.8. $q = 5$, Projection Approximation

n	$\lvert\tilde{\lambda} - \lambda_{q,n,1}\rvert$	$\lvert\tilde{\lambda} - \lambda_{q,n,2}\rvert$	$\lvert\tilde{\lambda} - \hat{\lambda}_{q,n}\rvert$
10	2.57×10^{-9}	2.57×10^{-9}	1.99×10^{-11}
20	1.30×10^{-9}	1.30×10^{-9}	1.93×10^{-14}
30	1.63×10^{-9}	1.63×10^{-9}	2.22×10^{-16}
40	2.24×10^{-9}	2.24×10^{-9}	5.97×10^{-16}

Table 4.9. $q = 6$, Nyström Approximation

n	$\lvert\tilde{\lambda} - \lambda_{q,n,1}\rvert$	$\lvert\tilde{\lambda} - \lambda_{q,n,2}\rvert$	$\lvert\tilde{\lambda} - \hat{\lambda}_{q,n}\rvert$
10	7.39×10^{-8}	7.39×10^{-8}	7.37×10^{-8}
20	4.64×10^{-10}	2.67×10^{-9}	1.10×10^{-9}
30	2.25×10^{-9}	2.25×10^{-9}	9.66×10^{-11}
40	8.74×10^{-10}	8.74×10^{-10}	1.73×10^{-11}

Table 4.10. $q = 6$, Projection Approximation

n	$\lvert\tilde{\lambda} - \lambda_{q,n,1}\rvert$	$\lvert\tilde{\lambda} - \lambda_{q,n,2}\rvert$	$\lvert\tilde{\lambda} - \hat{\lambda}_{q,n}\rvert$
10	7.80×10^{-10}	7.81×10^{-10}	2.61×10^{-13}
20	1.29×10^{-9}	1.29×10^{-9}	5.69×10^{-16}
30	1.69×10^{-9}	1.69×10^{-9}	5.27×10^{-16}
40	1.75×10^{-9}	1.75×10^{-9}	5.83×10^{-16}

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