MATHEMATICS OF COMPUTATION
Volume 67, Number 224, October 1998, Pages 1697–1705
S 0025-5718(98)01006-0

POLYNOMIALS WITH SMALL MAHLER MEASURE

MICHAEL J. MOSSINGHOFF

ABSTRACT. We describe several searches for polynomials with integer coefficients and small Mahler measure. We describe the algorithm used to test Mahler measures. We determine all polynomials with degree at most 24 and Mahler measure less than 1.3, test all reciprocal and antireciprocal polynomials with height 1 and degree at most 40, and check certain sparse polynomials with height 1 and degree as large as 181. We find a new limit point of Mahler measures near 1.309, four new Salem numbers less than 1.3, and many new polynomials with small Mahler measure. None has measure smaller than that of Lehmer's degree 10 polynomial.

1. INTRODUCTION

The Mahler measure of a polynomial

$$f(x) = \sum_{k=0}^{d} a_k x^k = a_d \prod_{k=1}^{d} (x - \alpha_k)$$

is

$$M(f) = |a_d| \prod_{k=1}^{d} \max\{1, |\alpha_k|\} = \exp\left(\int_0^1 \log|f(e(t))|\, dt\right),$$

where $e(t) = e^{2\pi i t}$. The Mahler measure is clearly multiplicative, and satisfies

$$M(f(x)) = M(f(-x)) = M(f(x^k)) = M(f^*(x))$$

for every $k \geq 1$, where $f^*(x) = x^d f(1/x)$. We restrict our attention to polynomials with integer coefficients. Thus $M(f) \geq 1$, and a classical theorem of Kronecker implies that $M(f) = 1$ if and only if $f(x)$ is a product of cyclotomic polynomials and the monomial x. In [9], D. H. Lehmer asks if there exist polynomials with Mahler measure between 1 and $1 + \epsilon$ for arbitrary $\epsilon > 0$, and notes that the polynomial

$$\ell(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

has $M(\ell) = 1.1762808\ldots$. Several extensive searches [4, 6, 11, 12] have failed to find a polynomial with smaller measure.

The best general lower bound on $M(f)$ (up to the constant c) is due to Dobrowolski [7]: if f is a noncyclotomic, irreducible polynomial of degree $d > 2$, then

$$M(f) > 1 + c\left(\frac{\log\log d}{\log d}\right)^3.$$

Received by the editor May 20, 1997.
1991 *Mathematics Subject Classification.* Primary 12–04; Secondary 11Y40.
Key words and phrases. Lehmer's problem, Mahler measure.

A polynomial f is reciprocal if $f = f^*$ and antireciprocal if $f = -f^*$. Smyth [13] proves that if f is a nonreciprocal, irreducible polynomial and $f(x) \neq x - 1$ or x, then $M(f) \geq 1.324717\ldots$, the real root of $x^3 - x - 1$ and the smallest Pisot-Vijayaraghavan number.

The height and length of f are defined respectively by

$$H(f) = \max_{0 \leq k \leq d} |a_k|, \qquad L(f) = \sum_{k=0}^{d} |a_k|.$$

Let $\Phi_n(x)$ denote the nth cyclotomic polynomial. We say a polynomial f is primitive if $f(x)$ cannot be written as $g(x^k)$ with $k > 1$ for some polynomial g.

This article describes some recent extensive searches for polynomials with small Mahler measure: an exhaustive search through degree 24, a search of reciprocal and antireciprocal polynomials with height 1 through degree 40, and a search of certain sparse polynomials up to degree 181. Section 2 describes the algorithm used to detect polynomials with small measure. Section 3 describes each search and summarizes the polynomials found by each. Section 4 notes a new limit point of Mahler measures near 1.309 and lists four new Salem numbers less than 1.3. Three tables in the Supplement summarize the polynomials found by these searches.

2. The polynomial processor

We describe an algorithm for testing whether a given polynomial has Mahler measure less than a prescribed bound. This algorithm is based on that used in [4] and uses the root-squaring procedure of Graeffe to detect polynomials with large Mahler measure quickly. We review the Graeffe algorithm. Given a polynomial f, let g and h be polynomials so that

$$f(x) = g(x^2) + xh(x^2)$$

and define

$$f_1(x) = g(x)^2 - xh(x)^2.$$

Then the roots of f_1 are precisely the squares of the roots of f, and $M(f_1) = M(f)^2$. Let f_m denote the polynomial obtained from f by iterating the Graeffe procedure m times.

We note the following properties of the Graeffe algorithm.

Lemma 2.1. *If $f(x) = g(x^2) + xh(x^2)$ and $L(g)^2 + L(h)^2 \leq Y$, then f_1 can be computed exactly using integers no larger than Y in absolute value.*

Proof. Immediate from the fact that $L(f_1) \leq L(g)^2 + L(h)^2$. □

Lemma 2.2. *If $f(x)$ is a polynomial, then $f_1(1) = f(1)f(-1)$. If $f(x)$ is a reciprocal polynomial of even degree, then $f_1(-1)$ is a perfect square.*

Proof. Let $f(x) = g(x^2) + xh(x^2)$, then

$$f_1(1) = (g(1) + h(1))(g(1) - h(1)) = f(1)f(-1).$$

For the second statement, suppose $\deg f = 2N$ and N is odd. Then

$$\begin{aligned} g(-1) &= \sum_{k=0}^{\lfloor N/2 \rfloor} a_{2k}(-1)^k + \sum_{k=\lceil N/2 \rceil}^{N} a_{2k}(-1)^k \\ &= \sum_{k=0}^{\lfloor N/2 \rfloor} a_{2k}(-1)^k + \sum_{k=0}^{\lfloor N/2 \rfloor} a_{2N-2k}(-1)^{N-k} \\ &= 0 \end{aligned}$$

since $a_{2N-2k} = a_{2k}$. Thus $f_1(-1) = h(-1)^2$. Similarly, if N is even, we find that $f_1(-1) = g(-1)^2$. □

Lemma 2.3. *If $f(x)$ is a product of cyclotomic polynomials and $m > \log_2 \deg(f)$, then $f_m(x) = f_{m+1}(x)$.*

Proof. Suppose $f(x) = \Phi_n(x)$ and $n = 2^r q$ with q odd. Then

$$f_m(x) = \begin{cases} \left(\Phi_{2^{r-m}q}(x)\right)^{2^m} & \text{if } m < r, \\ \left(\Phi_q(x)\right)^{2^{r-1}} & \text{if } m \geq r. \end{cases}$$

Thus $f_m(x) = f_{m+1}(x)$ if $m \geq r$. Since $\log_2 \deg(f) \geq r - 1$, the statement follows. □

We describe the algorithm. Since reciprocal polynomials of odd degree are divisible by $x + 1$, we assume the given polynomial has even degree.

Algorithm 2.4. *Test Mahler Measure.*

Input: f, a monic, reciprocal polynomial of even degree d having integer coefficients, and M, a real number satisfying $1 < M \leq 1.4$.
Output: If $1 < M(f) < M$, return $M(f)$ and the noncyclotomic part of f.

Step 1. Root-squaring. Let $a_{n,m}$ denote the coefficient of x^n in $f_m(x)$. If $M(f) \leq M$, we see from [4] that

$$|a_{n,m}| \leq \binom{d}{n} + \binom{d-2}{n-1}\left(M^{2^m} + M^{-2^m} - 2\right) \tag{2.1}$$

for all m, and if in addition $a_{1,m} \geq d - 4$ and $m \geq 1$, then

$$\begin{aligned} |a_{n,m}| \leq \binom{d}{n} &+ \binom{d-4}{n-2}\left(M^{2^m} + M^{-2^m} - 2\right) \\ &+ 2\left(M^{2^{m-1}} + M^{-2^{m-1}} - 2\right)\left(\binom{d-4}{n-3} + \binom{d-4}{n-1}\right). \end{aligned} \tag{2.2}$$

In [4], this latter inequality requires that f_m have no negative real roots, but the proof requires only that any negative real roots have multiplicity greater than 1. This is assured by the condition $m \geq 1$.

We perform the root-squaring procedure at most m_0 times, rejecting the polynomial if at any stage the appropriate inequality (2.1) or (2.2) is not satisfied. The parameter m_0 is selected to minimize the total computation time. If m_0 is too small, Step 1 passes too many polynomials with $M(f) > M$, invoking Steps 2 and 3 much more often. On the other hand, the $a_{n,m}$ are computed using exact arithmetic, and in general $a_{n,m+1}$ has about twice as many digits as $a_{n,m}$. Thus, selecting m_0 too large greatly increases the time required in Step 1. We set $m_0 = 10$ in the exhaustive search and the height 1 search, and $m_0 = 12$ in the sparse polynomial search.

For small values of m, we store the $a_{n,m}$ as double-precision, floating-point numbers. This allows us to use the fast arithmetic of the hardware to compute several of the f_m. Once the $a_{n,m}$ require more than 53 bits of precision, we switch to a big integer representation implemented in software. Because the algorithm typically rejects many polynomials after only a few iterations of the root-squaring procedure, this strategy saves a considerable amount of time. We determine when to switch to the big integer representation using the criterion of Lemma 2.1 with $Y = 2^{53} - 1$.

Let m_1 denote the number of root-squaring operations performed on f using the hardware's arithmetic. For $m \leq m_1$, if $f_m(-1) = 0$, we remove all factors of $x+1$ from $f_m(x)$. Assume then that $f_m(-1) \neq 0$ for $m \leq m_1$. Set $s_0 = f(1)$ and $t_0 = f(-1)$, and for $1 \leq m \leq m_1$, let $s_m = s_{m-1}t_{m-1}$ and $t_m = (f_m(-1))^{1/2}$. By Lemma 2.2, a prime p divides $f_m(1)$ if and only if p divides s_m. The integer s_{m_1} is used in Step 2.

Finally, we reject f if we detect that $f_m = f_{m-1}$ for some m. We assume that $m_0 > 1 + \log_2 d$, so by Lemma 2.3 we reject all products of cyclotomic polynomials.

Step 2. Remove cyclotomic factors. The smallest Mahler measure among polynomials of degree at most 6 is $M_6 = M(x^6 - x^4 + x^3 - x^2 + 1) \approx 1.401268$. Because f has a noncyclotomic factor and $M < M_6$, we need only test f for cyclotomic factors of degree at most $d-8$. The following two observations speed this test. Both make use of $f_{m_1}(x)$, the last polynomial computed using the hardware's arithmetic in Step 1.

First, a cyclotomic factor $\Phi_n(x)$ of $f(x)$, where $n = 2^r q$ with q odd, stabilizes as a factor $\Phi_q(x)$ with multiplicity 2^{r-1} of $f_m(x)$ when $m \geq r$. Thus, for each odd integer q with $\varphi(q) \leq d - 8$, we test whether $\Phi_q(x)$ divides $f_{m_1}(x)$. If it does not, we conclude that $\Phi_{2^r q}(x)$ with $r \leq m_1$ does not divide $f(x)$.

Second, we avoid this trial division whenever $\Phi_q(1)$ does not divide $f_{m_1}(1)$. Since

$$\Phi_n(1) = \begin{cases} 0 & \text{if } n = 1, \\ p & \text{if } n = p^r,\ p \text{ a prime}, \\ 1 & \text{otherwise}, \end{cases}$$

it suffices to check if $\Phi_q(1)$ divides s_{m_1}.

After the cyclotomic factors of f are removed, we check if f is among the known polynomials with $M(f) \leq M$. These polynomials are stored in a binary tree to facilitate this check. If f is new, we continue with Step 3.

Step 3. Compute Mahler measure. We first compute an approximation to $M(f)$ using Bairstow's method [14] for finding roots of polynomials. We implement this procedure using hardware arithmetic and exploit the fact that f is reciprocal, so this test is quite fast. If the estimated value of $M(f)$ is less than $M + \delta$, with δ a specified positive tolerance, we pass f to a more accurate procedure for computing $M(f)$. The software packages PARI and Maple are used to compute $M(f)$ in this second stage.

We omit the preliminary estimate of $M(f)$ when testing polynomials of large degree in the sparse polynomial search.

3. Searches

Three sets of polynomials were tested with Algorithm 2.4: a set containing all polynomials of a given degree with measure less than M, reciprocal and antireciprocal polynomials with height 1, and polynomials with height 1 and a fixed number of nonzero coefficients. A fourth set of polynomials checked using Algorithm 2.4 is described in [11].

3.1. **Exhaustive Search.** In [4, p. 1369], Boyd describes a method for finding all reciprocal polynomials of a given even degree d with Mahler measure less than a given bound. In [4] and [6], Boyd uses this procedure to find all polynomials of degree $d \leq 20$ having Mahler measure less than 1.3. We use this same method to extend the exhaustive search through degree 24. We find 48 primitive, irreducible, noncyclotomic polynomials with Mahler measure less than 1.3 of degree 22 and 46 such polynomials of degree 24. These are precisely the polynomials of degrees 22 and 24 in [6] that were found in Boyd's height 1 search.

Extending this search to degree 24 involved testing about 9.8 billion polynomials and required approximately 15 days of computer time on an Intel Pentium 120.

3.2. **Height 1 Search.** Suppose f is an irreducible polynomial of degree d. Corollary 2 of [1] shows that for any positive integer L there exists a polynomial g such that $\deg g \leq L$ and

$$L \log H(fg) \leq (d+L-1)\log M(f) + \frac{d}{2}\log\left(\frac{d+L}{4}\right) + \frac{3d}{4}.$$

Suppose $M(f) \leq M < 2$, and choose L so large that

$$(d+L-1)\log M(f) + \frac{d}{2}\log\left(\frac{d+L}{4}\right) + \frac{3d}{4} < L\log 2. \tag{3.1}$$

Then there exists a polynomial g with $\deg g \leq L$ and $H(fg) = 1$. We call such a g a mollifier of f.

Note that (3.1) implies a mollifier of f exists with degree $O(d \log d / \log(2/M))$, and we may compute an explicit bound on the degree in specific cases. For example, if there exists a polynomial of degree 26 with Mahler measure less than 1.1762808, then it is a factor of a polynomial with height 1 and degree at most 161.

In [4], Boyd remarks that if f is a polynomial with small Mahler measure, a mollifier g of f seems to exist with $M(g) = 1$ and degree fairly small relative to the degree of f. He therefore proposes searching polynomials with height 1, and in [6] reports the results of testing reciprocal polynomials of even degree with height 1 through degree 32. We extend this search by testing all reciprocal and antireciprocal polynomials with height 1 (the odd as well as the even degrees) through degree 40.

Note that if $f(x)$ is a reciprocal polynomial of odd degree, then $f(x)/(x+1)$ is a reciprocal polynomial of even degree, so we invoke Algorithm 2.4 on $f(x)/(x+1)$. Likewise, if $f(x)$ is an antireciprocal polynomial of odd degree, we pass $f(x)/(x-1)$ to Algorithm 2.4, and if $f(x)$ is an antireciprocal polynomial of even degree, we test $f(x)/(x^2-1)$.

This search finds many new polynomials with Mahler measure less than 1.3, including a number of polynomials with degree at most 32 and height greater than 1. These polynomials are listed in Table 1. None have degree less than 30.

The search tested approximately 5 billion polynomials and required about 5.5 weeks of computation on a Pentium 120.

TABLE 1. Polynomials with height > 1 and $d \le 32$ not in [6] (ν is the number of roots outside the unit disk)

d	Measure	ν	Half of coefficients
30	1.285530553671	4	1 0 1 1 1 1 1 2 0 1 1 0 0 0 0 −1
30	1.288113357594	2	1 1 2 1 1 −1 −1 −2 −1 −1 0 0 1 0 0 −1
30	1.292745216074	4	1 0 1 0 1 1 0 2 0 2 0 1 1 0 2 −1
30	1.295830812559	3	1 1 1 1 1 1 1 2 2 2 2 2 2 2 2 3
30	1.296432383243	6	1 0 1 1 1 2 1 3 2 3 3 3 4 3 4 3
30	1.296533365392	3	1 1 1 0 0 −1 −1 −1 0 0 1 0 0 −1 −1 −2
30	1.296872723796	3	1 0 −1 0 −1 1 1 0 0 −2 0 1 1 1 −1 −1
30	1.297599482921	3	1 1 0 −1 −2 −2 −1 0 1 2 2 1 0 −1 −1 −1
30	1.299672830907	3	1 0 −1 1 0 −1 1 0 −2 1 1 −1 0 0 0 1
32	1.236083368052	4	1 1 1 1 0 −1 −1 −2 −1 0 0 1 1 0 0 0 −1
32	1.249688298465	4	1 2 2 1 0 −1 −1 −1 0 1 2 2 2 1 0 −1 −1
32	1.268321917905	2	1 1 1 1 0 0 0 0 1 1 1 1 0 −1 −2 −3 −3
32	1.268867282818	4	1 1 1 1 0 −1 −2 −2 −2 −2 −1 −1 0 1 2 3 3
32	1.270932746058	4	1 1 1 0 0 −1 −1 −1 0 1 2 2 2 1 0 −1 −1
32	1.279387162064	2	1 2 2 2 1 −1 −2 −3 −3 −2 −1 0 1 1 1 1 1
32	1.286650909902	6	1 0 −1 1 0 −2 1 1 −2 1 2 −2 0 2 −2 −1 3
32	1.287530573906	3	1 1 0 −1 −1 −1 −1 0 1 1 0 −1 −1 −1 0 1 2
32	1.289386554481	4	1 1 1 0 −1 −2 −1 1 2 2 0 −2 −3 −2 0 2 3
32	1.291024122419	4	1 1 1 0 −1 −1 −1 −1 −1 0 1 2 1 0 −1 −1 −1
32	1.294553682172	4	1 1 0 −1 −1 −1 0 1 2 2 0 −2 −3 −2 0 2 3
32	1.294774730521	4	1 1 −1 −2 0 2 1 −1 −1 0 0 0 1 1 0 −1 −1
32	1.298256684864	2	1 2 3 4 4 3 2 1 0 −1 −2 −3 −3 −3 −3 −3 −3
32	1.298335890166	6	1 1 1 1 0 −1 −1 −1 −1 −1 −1 −1 −1 0 1 2 3
32	1.299312144051	4	1 1 0 0 1 1 1 1 1 1 2 2 1 1 2 2 1

3.3. **Sparse Polynomial Search.** The Mahler measure of a polynomial $f(x, y)$ in two variables is defined by

$$M(f(x,y)) = \exp\left(\int_0^1 \int_0^1 \log|f(e(s), e(t))|\, ds\, dt\right). \tag{3.2}$$

Boyd [5] proves that the Mahler measure of a polynomial in two variables is the limit of the Mahler measures of certain associated polynomials in one variable:

$$M(f(x,y)) = \lim_{n\to\infty} M(f(x, x^n)).$$

The smallest known limit points of Mahler measures arise from two-variable polynomials with height 1 having at most six terms:

$$M(x^2(y^2-1) + x(y^3-1) + y(y^2-1)) = 1.255433\ldots,$$
$$M(x^2 + x(y^2+y+1) + y^2) = 1.285734\ldots,$$
$$M(x^2(y^3-1) + x(y^5-1) + y^2(y^3-1)) = 1.315692\ldots,$$
$$M(x(y^3-y-1) - (y^3+y^2+1)) = 1.324717\ldots.$$

This suggests testing the Mahler measure of sparse polynomials with height 1. We use Algorithm 2.4 to test all reciprocal and antireciprocal polynomials with a fixed number n of ± 1 coefficients up to some maximal degree. The following table shows the maximum degree d tested for each $n \geq 5$ (reciprocal and antireciprocal polynomials with $n \leq 4$ have Mahler measure 1 or $(1+\sqrt{5})/2$).

n	d
5, 6, 7	181
8, 9	131
10, 11	101
12, 13	75
14, 15	55
16, 17	47
18, 19	43

About 800 million polynomials were tested, requiring about three months of computation on Pentium and DEC Alpha computers.

This search found every previously known polynomial with Mahler measure less than 1.3, including all the polynomials found in the two previously described searches, as well as those found in [11]. It also found several polynomials with measure less than 1.3 that are not obviously associated with any of the known limit points of measures (including the new limit point described in the next section). The largest degree among these sporadic polynomials is 106; the smallest measure is about 1.239861, the 64th smallest Mahler measure greater than 1 known. (This polynomial appears in the list of smallest known Mahler measures in the Supplement.)

All but three of the polynomials with Mahler measure less than 1.3 were found in searches with $n \leq 12$. Using lattice reduction to search for sparse reciprocal or antireciprocal multiples of these three exceptional polynomials, we find that one is a factor of the polynomial

$$x^{78} - x^{76} + x^{72} - x^{55} - x^{50} - x^{43} + x^{35} + x^{28} + x^{23} - x^{6} + x^{2} - 1$$

with $n = 12$. All the auxiliary factors of this polynomial are cyclotomic. The best cyclotomic multiples that were found of the other two have $n = 14$ and $n = 13$:

$$x^{45} - x^{42} + x^{36} - x^{34} - x^{32} + x^{31} - x^{24} + x^{21} - x^{14} + x^{13} + x^{11} - x^{9} + x^{3} - 1,$$
$$x^{48} + x^{46} + x^{44} - x^{41} - x^{32} - x^{31} + x^{24} - x^{17} - x^{16} - x^{7} + x^{4} + x^{2} + 1.$$

We find that the noncyclotomic part of the latter polynomial divides

$$x^{56} + x^{51} - x^{50} + x^{47} - x^{39} - x^{28} - x^{17} + x^{9} - x^{6} + x^{5} + 1$$

with $n = 11$, but one of the auxiliary factors is not cyclotomic (it is $\ell(-x)$). We note that the method of [8] could be used to determine if other sparse multiples of these polynomials exist.

4. Results

4.1. **A New Limit Point.** Many of the polynomials produced by these searches have the form

$$(x^{3n-1} + 1)(x^{n+1} + 1) + x^{2n-1}(x^2 - x + 1).$$

Replacing x^n with y and multiplying by x yields

$$f(x, y) = x^2 y(y + 1) + x(y^4 - y^2 + 1) + y^2(y + 1).$$

Let $x = \alpha(t)$ and $x = \beta(t)$ be the two roots of $f(x, e(t))$ for any t, taking

$$\alpha(t) = \frac{1 - 2\cos(4\pi t) + \sqrt{(1 - 2\cos(4\pi t))^2 - 16\cos^2(\pi t)}}{4\cos(\pi t)} e(t/2).$$

Since $|\alpha(t)| = 1/|\beta(t)|$ and $|\alpha(t)| = |\alpha(1-t)|$, we have by (3.2) and Jensen's formula

$$\log M(f) = 2\int_0^{1/2} \big|\log|\alpha(t)|\big|\, dt.$$

Now $|\alpha(t)| = 1$ on $[0, t_1]$ and $[1/3, t_2]$, where $t_1 = .23454\ldots$ and $t_2 = .45028\ldots$ satisfy $(1 - 2\cos(4\pi t))^2 = 16\cos^2(\pi t)$. Computing the integral over the remaining intervals yields $M(f) = 1.309098380652328\ldots$.

4.2. **Small Salem Numbers.** Table 2 lists four new Salem numbers less than 1.3. Each is listed with its minimal polynomial and its rank among the 47 known Salem numbers less than 1.3. The other 43 known small Salem numbers can be found in [2] and [3].

TABLE 2. New small Salem numbers

Rank	Salem number	Minimal polynomial
39	1.292418657582	$x^{40} + x^{37} + x^{35} + x^{33} + x^{31} + x^{29} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{11} + x^9 + x^7 + x^5 + x^3 + 1$
40	1.292900721780	$x^{46} - x^{42} + x^{41} - x^{40} + x^{39} - x^{25} + x^{24} - x^{23} + x^{22} - x^{21} + x^7 - x^6 + x^5 - x^4 + 1$
43	1.296210659593	$x^{34} + x^{33} + x^{31} - x^{29} - x^{27} - 2x^{26} - x^{23} + x^{22} + x^{21} - x^{20} + x^{19} + x^{18} - x^{17} + x^{16} + x^{15} - x^{14} + x^{13} + x^{12} - x^{11} - 2x^8 - x^7 - x^5 + x^3 + x + 1$
46	1.298429835475	$x^{36} + x^{35} - x^{33} - 2x^{32} - 2x^{31} - x^{30} + x^{28} + x^{27} - x^{25} - x^{24} + x^{22} + x^{21} - x^{19} - x^{18} - x^{17} + x^{15} + x^{14} - x^{12} - x^{11} + x^9 + x^8 - x^6 - 2x^5 - 2x^4 - x^3 + x + 1$

4.3. **Records and Summaries.** Three tables in the Supplement at the end of this issue summarize the results of the searches. The first lists the 64 smallest known Mahler measures greater than 1, together with half of their coefficients and the number of roots ν of each that lie outside the unit disk. These are all the known Mahler measures greater than 1 and less than 1.24. The second shows the primitive, irreducible, noncyclotomic polynomial with the smallest known Mahler measure of a given even degree d, for $8 \le d \le 100$. The third summarizes the primitive, irreducible polynomials with measure less than 1.3 found in the searches, classifying them by degree and the number of roots outside the unit disk.

More extensive summaries and lists of all the polynomials found with measure less than 1.3 can be found at the author's World Wide Web site, accessible from the number theory web.

ACKNOWLEDGMENTS

I would like to thank David Boyd and Jeffrey Vaaler for several helpful suggestions.

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Department of Mathematical Sciences, Appalachian State University, Boone, North Carolina 28608

E-mail address: `mjm@math.appstate.edu`