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CONVERGENCE OF NEWTON'S METHOD AND INVERSE FUNCTION THEOREM IN BANACH SPACE

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ABSTRACT. Under the hypothesis that the derivative satisfies some kind of weak Lipschitz condition, a proper condition which makes Newton's method converge, and an exact estimate for the radius of the ball of the inverse function theorem are given in a Banach space. Also, the relevant results on premises of Kantorovich and Smale types are improved in this paper.

We continue to discuss the problem of convergence in the Newton method

$$x_{n+1} = x_n - f'(x_n)^{-1} f(x_n), \quad n = 0, 1, \cdots, \tag{0.1}$$

to solve an operator equation f which maps from some domain D in a real or complex Banach space $\mathbf{X}$ to another Banach space $\mathbf{Y}$,

$$f(x) = 0. \tag{0.2}$$

Now we come back to the problem which we bypassed in [1].

We always assume that $f'(x_0)^{-1}$ exists and $f'(x_0)^{-1}f'$ satisfies some kind of Lipschitz condition similar to that of [1] in some open ball $\mathrm{B}(x_0, r) \subset D$ with center x_0 and radius r (or some closed ball $\overline{\mathrm{B}(x_0, r)} \subset D$) in order to study the convergence of Newton's method and the domain of the local inverse function of f at x_0.

1. The domain of the inverse function

The inverse function theorem asserts that there is an inverse function $f_{x_0}^{-1}$ defined on some open ball $\mathrm{B}(f(x_0), \varepsilon) \subset \mathbf{Y}$ with the property that

$$f_{x_0}^{-1}(f(x_0)) = x_0,$$
$$f(f_{x_0}^{-1}(y)) = y, \qquad \forall y \in \mathrm{B}(f(x_0), \varepsilon),$$

and $f_{x_0}^{-1}$ is differentiable. Now we study the exact lower bound estimate of the radius of this ball.

For this reason, we assume that f has a continuous derivative in the ball $\mathrm{B}(x_0, r)$, $f'(x_0)^{-1}$ exists and $f'(x_0)^{-1}f'$ satisfies the center Lipschitz condition with the L average,

$$\left\| f'(x_0)^{-1} f'(x) - \mathrm{I} \right\| \le \int_0^{\rho(x)} L(u)\,du, \qquad \forall x \in \mathrm{B}(x_0, r), \tag{1.1}$$

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where $\rho(x) = \|x - x_0\|$ and L is a positive integrable function in the interval $(0, r)$. By Banach's theorem, when $r_0 \le r$, for all $x \in \mathrm{B}(x_0, r_0)$, $f'(x)^{-1}$ exists and

$$\|f'(x)^{-1}f'(x_0)\| \le \frac{1}{1 - \int_0^{\rho(x)} L(u)du}, \tag{1.2}$$

where r_0 satisfies

$$\int_0^{r_0} L(u)du = 1. \tag{1.3}$$

Theorem 1.1. *Suppose that $r \ge r_0$ and $b = \int_0^{r_0} L(u)udu$. Then under the hypothesis of condition* (1.1), *we have*

$$\mathrm{B}(f(x_0), b/\|f'(x_0)^{-1}\|) \subset f(\mathrm{B}(x_0, r_0)), \tag{1.4}$$

and in the open ball on the left, $f_{x_0}^{-1}$ exists and is differentiable. Moreover, the radius of the ball is the best possible.

Lemma 1.2. *Let*

$$h(t) = \beta - t + \int_0^t L(u)(t-u)du, \quad 0 \le t \le R, \tag{1.5}$$

where R satisfies

$$\frac{1}{R}\int_0^R L(u)(R-u)du = 1. \tag{1.6}$$

Then when $0 < \beta < b$, h is decreasing monotonically in $[0, r_0]$, while it is increasing monotonically in $[r_0, R]$ and

$$h(\beta) > 0, \quad h(r_0) = \beta - b < 0, \quad h(R) = \beta > 0.$$

Moreover, h has a unique zero in each interval, denoted by r_1 and r_2. They satisfy

$$\beta < r_1 < \frac{r_0}{b}\beta < r_0 < r_2 < R. \tag{1.7}$$

Proof. It is obvious by the sign of $h'(t) = -1 + \int_0^t L(u)du$ that $h(t)$ is piecewise monotone. By the positivity of L, we see that $\varphi(t) := \frac{1}{t}\int_0^t L(u)(t-u)du$ is increasing monotonically with respect to t. In fact, for $0 < t_1 < t_2$,

$$\begin{aligned}
\varphi(t_2) - \varphi(t_1) &= \int_{t_1}^{t_2} L(u)du - \left(\frac{1}{t_2}\int_{t_1}^{t_2} + \left(\frac{1}{t_2} - \frac{1}{t_1}\right)\int_0^{t_1}\right) L(u)udu \\
&\ge \int_{t_1}^{t_2} L(u)du - \int_{t_1}^{t_2} L(u)du - \left(\frac{1}{t_2} - \frac{1}{t_1}\right)\int_0^{t_1} L(u)udu \\
&= \left(\frac{1}{t_1} - \frac{1}{t_2}\right)\int_0^{t_1} L(u)udu > 0.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\beta < r_1 &= h(r_1) + r_1 = \beta + \varphi(r_1)r_1 < \beta + \varphi(r_0)r_1 \\
&= \beta + \frac{1}{r_0}\int_0^{r_0} L(u)(r_0 - u)du \cdot r_1 = \beta + r_1 - \frac{b}{r_0}r_1. \quad \square
\end{aligned}$$

By this lemma, Theorem 1.1 implies a more precise proposition, as follows. For this purpose, we assume the inequality (1.1) can be extended to the boundary, i.e.

$$\|f'(x_0)^{-1}f'(x) - \mathrm{I}\| \leq \int_0^{\rho(x)} L(u)du, \qquad \forall x \in \overline{\mathrm{B}(x_0, r)}. \tag{1.1'}$$

Proposition 1.3. *Suppose that $r \geq r_1$ and $0 < \beta < b = \int_0^{r_0} L(u)udu$, where r_1 is determined by Lemma 1.2 . Then, under the hypothesis of the condition (1.1'),*

$$\overline{\mathrm{B}(f(x_0), \beta/\|f'(x_0)^{-1}\|)} \subset f(\overline{\mathrm{B}(x_0, r_1)}), \tag{1.8}$$

and in the closed ball on the left, $f_{x_0}^{-1}$ exists, is differentiable, and its derivative $(f_{x_0}^{-1})'(y) = f'(x)^{-1}$ at $y = f(x)$ satisfies (1.2). Moreover, as a closed ball of the image, the radius r_1 is as small as possible.

Proof. Arbitrarily choosing

$$y \in \overline{\mathrm{B}(f(x_0), \beta/\|f'(x_0)^{-1}\|)}, \tag{1.9}$$

we consider two sequences $\{x_n\} \subset \mathbf{X}$ and $\{t_n\} \subset \mathbf{R}$, respectively given by

$$x_{n+1} = x_n - f'(x_0)^{-1}(f(x_n) - y), \quad n = 0, 1, \cdots, \tag{1.10}$$

and

$$t_{n+1} = t_n + h(t_n), \quad t_0 = 0, \quad n = 0, 1, \cdots. \tag{1.11}$$

First, by the fact that $h(t) + t$ increases monotonically with respect to t and $t_0 = 0 < t_1 = \beta < r_1$, we inductively find that $\{t_n\}$ increases monotonically and is less than r_1. Thus $\{t_n\}$ converges to r_1.

Then, by induction, for all n we will prove that

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n. \tag{1.12}$$

By (1.9) and (1.7),

$$\|x_1 - x_0\| \leq \|f'(x_0)^{-1}\| \cdot \|f(x_0) - y\| \leq \beta = t_1 - t_0.$$

This means (1.12) is true for $n = 0$. Suppose that (1.12) is valid until some $n - 1$. For $0 \leq \tau \leq 1$, let

$$\begin{aligned} x_{n-1+\tau} &= x_{n-1} + \tau(x_n - x_{n-1}), \\ t_{n-1+\tau} &= t_{n-1} + \tau(t_n - t_{n-1}). \end{aligned} \tag{1.13}$$

We have

$$\begin{aligned} \|x_{n-1+\tau} - x_0\| &\leq \|x_1 - x_0\| + \cdots + \|x_{n-1} - x_{n-2}\| + \tau\|x_n - x_{n-1}\| \\ &\leq (t_1 - t_0) + \cdots + (t_{n-1} - t_{n-2}) + \tau(t_n - t_{n-1}) \\ &= t_{n-1+\tau} < r_1 \leq r. \end{aligned}$$

Thus, by virtue of the equality

$$\begin{aligned} x_{n+1} - x_n &= -f'(x_0)^{-1}(f(x_n) - f(x_{n-1}) - f'(x_0)(x_n - x_{n-1})) \\ &= -\int_0^1 \left(f'(x_0)^{-1}f'(x_{n-1+\tau}) - \mathrm{I}\right)(x_n - x_{n-1})d\tau \end{aligned}$$

and (1.1), we obtain

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \int_0^1 \|f'(x_0)^{-1}f'(x_{n-1+\tau})-\mathrm{I}\|\,\|x_n-x_{n-1}\|d\tau\\ &\le \int_0^1\int_0^{\rho(x_{n-1+\tau})} L(u)du\|x_n-x_{n-1}\|d\tau\\ &\le \int_0^1\int_0^{t_{n-1+\tau}} L(u)du(t_n-t_{n-1})d\tau\\ &= \int_0^{t_n} L(u)(t_n-u)du-\int_0^{t_{n-1}} L(u)(t_{n-1}-u)du\\ &= t_{n+1}-t_n.\end{aligned}$$

This indicates that (1.12) is valid for all n.

The inequality (1.12) above shows that the sequence $\{x_n\}$ is self-convergent and so is convergent. Taking the limit on both sides in (1.10), we see that $x=\lim x_n$ satisfies

$$f(x)=y. \tag{1.14}$$

Also, since $\|x_n-x_0\|\le r_1$, we have

$$x=f_{x_0}^{-1}(y)\in\overline{\mathrm{B}(x_0,r_1)}. \tag{1.15}$$

For this reason we have to prove x satifying (1.14) is unique in the closed ball. This will be given togather with the proof of the next proposition. Finally, the differentiablity of the inverse function follows by (1.2). □

Remark. Except for the differentiablity of the inverse function, the proposition is also true for $\beta=b$.

Besides Proposition 1.3, we have the following proprosition, which is called the branch separation theorem

Proposition 1.4. *Suppose that $r_1\le r<r_2$ and $0<\beta<b$, where r_1,r_2 and b are determined by Lemma 1.2 and Theorem 1.1. Then, under the condition* (1.1′),

$$\overline{\mathrm{B}(f(x_0),\beta/\|f'(x_0)^{-1}\|)}\bigcap f(\mathrm{B}(x_0,r)\setminus\overline{\mathrm{B}(x_0,r_1)})=\emptyset. \tag{1.16}$$

Proof. Arbitrarily choose

$$y\in\mathrm{B}(f(x_0),\beta/\|f'(x_0)^{-1}\|),\qquad x_0'\in\mathrm{B}(x_0,r). \tag{1.17}$$

Let

$$x_{n+1}'=x_n'-f'(x_0)^{-1}(f(x_n')-y),\ \ n=0,1,\cdots, \tag{1.18}$$

$$t_{n+1}'=t_n'+h(t_n'),\ \ t_0'=\|x_0'-x_0\|,\ \ n=0,1,\cdots. \tag{1.19}$$

Since

(1.20)

$$x_{n+1}'-x_{n+1}=-\int_0^1\left(f'(x_0)^{-1}f'(x_n+\tau(x_n'-x_n))-\mathrm{I}\right)(x_n'-x_n)d\tau,$$

we can prove that

$$\|x_n'-x_n\|\le t_n'-t_n,\ \ n=0,1,\cdots. \tag{1.21}$$

Hence, $\{x_n'\}$ is also convergent to $x=\lim x_n$. Therefore, there is only one $x\in\overline{\mathrm{B}(x_0,r_1)}$ in the open ball $\mathrm{B}(x_0,r)$ that satisfies (1.14). □

Remark. The proof of Propositions 1.3 and 1.4 may be viewed as the proof of the existence and uniqueness theorems about the solution of the equation $f(x) = y$, and the premise (1.9) and (1.17) can be replaced by

$$\|f'(x_0)^{-1}(f(x_0) - y)\| \leq \beta. \tag{1.22}$$

Hence, setting $y = 0$ and $\beta = \|f'(x_0)^{-1}f(x_0)\|$, we have

Theorem 1.5. *Let $\beta = \|f'(x_0)^{-1}f(x_0)\| \leq b$. Assume that $r_1 \leq r < r_2$ if $\beta < b$, or $r = r_1$ if $\beta = b$, where r_1, r_2 and b are determined by Lemma 1.2 and Theorem 1.1. Then, under the conditions* (1.1′), *the equation* (0.2) *has a unique solution*

$$x^* \in \overline{B(x_0 - f'(x_0)^{-1}f(x_0), r_1 - \beta)} \subset \overline{B(x_0, r_1)} \tag{1.23}$$

in the closed ball $\overline{\mathrm{B}(x_0, r)}$.

2. Further discussion of Lipschitz conditions

In the ball $\mathrm{B}(x_0, r)$, the Lipschitz condition with the constant L is

$$\|f(x) - f(x')\| \leq L\|x - x'\|, \tag{2.1}$$

where $x, x' \in \mathrm{B}(x_0, r)$. If (2.1) is only true for all $x \in \mathrm{B}(x_0, r)$ and $x' = x_0$, then it is called the center Lipschitz condition in [1]; if (2.1) is valid for all $x' \in \mathrm{B}(x_0, r)$ and for all $x = x_0 + \tau(x' - x_0)$ $(0 \leq \tau \leq 1)$, then it is called the radius Lipschitz condition. Now, if (2.1) is valid for all $x \in \mathrm{B}(x_0, r)$ and for all $x' \in \mathrm{B}(x, r - \rho(x))$, then we call it the center Lipschitz condition in the inscribed sphere. For a constant or positive integrable function L, among all Lipschitz conditions with the constant L or the average of L, there is an implication relation $\succ$ as follows:

Lipschitz condition in a ball
$\succ$ the center Lipschitz condition in the inscribed sphere
$\succ$ the radius Lipschitz condition
$\succ$ the center Lipschitz condition.

Custom is the only reason why we give the names of different Lipschitz conditions. It is not necessary in essence; see Theorems 6.3 and 6.4 in [1]. Sometimes, however, we have to pay attention to such accustomed thinking because it determines the development of the literature.

3. Convergence of Newton's method

Suppose that f has a continuous derivative in the closed ball $\overline{\mathrm{B}(x_0, r)}$, $f'(x_0)^{-1}$ exists and $f'(x_0)^{-1}f'$ satisfies the center Lipschitz condition in the inscribed sphere with the average of L,

$$\begin{gathered} \left\|f'(x_0)^{-1}(f'(x) - f'(x'))\right\| \leq \int_{\rho(x)}^{\rho(\overline{xx'})} L(u)du, \\ \forall x \in \mathrm{B}(x_0, r), \ \ \forall x' \in \overline{\rho m B}(x, r - \rho(x)), \end{gathered} \tag{3.1}$$

where $\rho(x) = ||x - x_0||, \rho(\overline{xx'}) = \rho(x) + \|x' - x\| \leq r$, and L is a positive nondecreasing function in $[0, r]$. Under this hypothesis, the conditions (1.1) and (1.1′) are of course satisfied, and thus Theorems 1.1 and 1.5, Propositions 1.3 and 1.4 hold.

Theorem 3.1. *Assume that* $\beta = ||f'(x_0)^{-1}f(x_0)|| \le b$ *and* $r \ge r_1$, *where* b *and* r_1 *are determined by Theorem 1.1 and Lemma 1.2. Then, under the condition* (3.1), *Newton's method* (0.1) *is defined for all* n *and converges to a solution* x^* *of equation* (0.2),

$$x^* \in \overline{\mathrm{B}(x_1, r_1 - \beta)} \subset \overline{\mathrm{B}(x_0, r_1)}. \tag{3.2}$$

Moreover, for all $n \ge n_0 \ge 0$, *the best possible error bounds*

$$\|x^* - x_n\| \le (r_1 - t_n)\left(\frac{\|x^* - x_{n_0}\|}{r_1 - t_{n_0}}\right)^{2^{n-n_0}} \tag{3.3}$$

and

(3.4)

$$\frac{2\|x_{n+1} - x_n\|}{1 + \sqrt{1 + 4\dfrac{r_1 - t_{n+1}}{(r_1 - t_n)^2}(t_{n+1} - t_n)}} \le \|x^* - x_n\| \le (r_1 - t_n)\left(\frac{\|x_{n_0+1} - x_{n_0}\|}{t_{n_0+1} - t_{n_0}}\right)^{2^{n-n_0}}$$

are valid with

$$t_{n+1} = t_n - \frac{h(t_n)}{h'(t_n)}, \quad t_0 = 0, \quad n = 0, 1, \cdots. \tag{3.5}$$

In order to prove Theorem 3.1, we need

Proposition 3.2. *Under the assumptions of Theorem 3.1, for any natural number* $n \ge 1$, *we have*

$$||x_n - x_{n-1}|| \le t_n - t_{n-1}, \tag{3.6}$$

$$||f'(x_0)^{-1}f(x_n)|| \le h(t_n)\left(\frac{||x_n - x_{n-1}||}{t_n - t_{n-1}}\right)^2, \tag{3.7}$$

$$\frac{||f'(x_0)^{-1}f(x_n)||}{||f'(x_0)^{-1}f(x_{n-1})||} \le \frac{h(t_n)}{h(t_{n-1})} \cdot \frac{||x_n - x_{n-1}||}{t_n - t_{n-1}}, \tag{3.8}$$

and

$$||x_{n+1} - x_n|| \le (t_{n+1} - t_n)\left(\frac{||x_n - x_{n-1}||}{t_n - t_{n-1}}\right)^2. \tag{3.9}$$

Proof. By the hypotheses, (3.6) is true for $n = 1$. Now assume that it holds for some $n \ge 1$. Then

$$x_n \in \mathrm{B}(x^*, t_n) \subset \mathrm{B}(x^*, r_1). \tag{3.10}$$

Since

$$\begin{aligned} f(x_n) &= f(x_n) - f(x_{n-1}) - f'(x_{n-1})(x_n - x_{n-1}) \\ &= \int_0^1 (f'(x_{n-1+\tau}) - f'(x_{n-1}))(x_n - x_{n-1})d\tau, \end{aligned}$$

where

$$x_{n-1+\tau} = x_{n-1} + \tau(x_n - x_{n-1}), \quad 0 \le \tau \le 1,$$

we obtain

$$||f'(x_0)^{-1}f(x_n)|| \le \int_0^1 ||f'(x_0)^{-1}(f'(x_{n-1+\tau}) - f'(x_{n-1}))|| \cdot ||(x_n - x_{n-1})||d\tau.$$

By the hypothesis (3.1), we have

$$\begin{aligned}||f'(x_0)^{-1}f(x_n)|| &\le \int_0^1 \int_{\rho(x_{n-1})}^{\rho(\overline{x_{n-1}x_{n-1+\tau}})} L(u)du||x_n - x_{n-1}||d\tau \\ &= \int_0^{||x_n - x_{n-1}||} L(||x_{n-1} - x_0|| + u)(||x_n - x_{n-1}|| - u)du.\end{aligned}$$

Since L is a nondecreasing function, $\varphi(t) := \frac{1}{t^2}\int_0^t L(\rho+u)(t-u)du$ is nondecreasing with respect to t in $[0, r-\rho]$. In fact, when $0 < t_1 < t_2 \le r - \rho$, we have

$$\begin{aligned}\varphi(t_2) - \varphi(t_1) = &\left(\frac{1}{t_1^2} - \frac{1}{t_2^2}\right)\int_0^{\frac{t_1}{2}} \left(L\left(\rho + \frac{t_1}{2} + u\right) - L\left(\rho + \frac{t_1}{2} - u\right)\right) udu \\ &+ \frac{(t_2 - t_1)^2}{2t_2^2 t_1}\int_0^{t_1} (L(\rho + t_1) - L(\rho + u))du \\ &+ \frac{1}{t_2^2}\int_{t_1}^{t_2} (L(\rho + u) - L(\rho + t_1))(t_2 - u)du \ge 0.\end{aligned}$$

Hence

$$\begin{aligned}&||f'(x_0)^{-1}f(x_n)|| \\ &\le \frac{1}{\|x_n - x_{n-1}\|^2}\int_0^{\|x_n - x_{n-1}\|} L(\|x_{n-1} - x_0\| + u)(\|x_n - x_{n-1}\| - u)du \\ &\qquad \cdot \|x_n - x_{n-1}\|^2 \\ &\le \frac{1}{(t_n - t_{n-1})^2}\int_0^{t_n - t_{n-1}} L(t_{n-1} + u)(t_n - t_{n-1} - u)du\|x_n - x_{n-1}\|^2 \\ &= h(t_n)\left(\frac{\|x_n - x_{n-1}\|}{t_n - t_{n-1}}\right)^2,\end{aligned}$$

where we have used the inductive hypothesis (3.6). Therefore, (3.7) holds for all n, which makes (3.6) hold.

Since

$$(3.11)\qquad ||x_n - x_{n-1}|| \le ||f'(x_{n-1})^{-1}f'(x_0)|| \cdot ||f'(x_0)^{-1}f(x_{n-1})||,$$

we obtain

(3.12)

$$\frac{||f'(x_0)^{-1}f(x_n)||}{||f'(x_0)^{-1}f(x_{n-1})||} \le h(t_n)\frac{||x_n - x_{n-1}||}{(t_n - t_{n-1})^2}||f'(x_{n-1})^{-1}f'(x_0)||.$$

By (1.2) and (1.5) we have

(3.13)

$$||f'(x_{n-1})^{-1}f'(x_0)|| \le \frac{1}{1 - \displaystyle\int_0^{||x_{n-1}-x_0||} L(u)du} \le -\frac{1}{h'(t_{n-1})}.$$

Combining (3.12) and (3.13) and using (3.5), we get that (3.8) is also true if (3.6) is true for some n.

Increasing n to $n+1$ in (3.11) and (3.13) and applying (3.7) and (3.13) to (3.11), we get (3.9).

So (3.6) can be continued, and (3.6)-(3.9) hold for all $n \ge 1$. □

Proof of Theorem 3.1. Obviously, $\{t_n\}$ is convergent to r_1 monotonically. Therefore, the sequence $\{x_n\} \subset \mathrm{B}(x_0, r_1)$ converges. Also by (1.1), $\|f'(x_n)\|$ is bounded uniformly. So from

$$f(x_n) + f'(x_n)(x_{n+1} - x_n) = 0 \tag{3.14}$$

we get $\lim x_n = x^*$.

Finally, by (3.1) and

$$\begin{aligned} e_n &:= f'(x_0)^{-1}(f(x^*) - f(x_n) - f'(x_n)(x^* - x_n)) \\ &= \int_0^1 f'(x_0)^{-1}(f'(z_\tau) - f'(x_n))(x^* - x_n)d\tau, \\ z_\tau &:= x_n + \tau(x^* - x_n) \end{aligned}$$

we obtain

$$\begin{aligned} \|e_n\| &\le \int_0^1 \int_{\rho(x_n)}^{\rho(\overline{x_n z_\tau})} L(u)du\|x^* - x_n\|d\tau \\ &= \int_0^{\|x^* - x_n\|} L(\|x_n - x_0\| + u)(\|x^* - x_n\| - u)du. \end{aligned}$$

Since $\frac{1}{t^2}\int_0^t L(\rho + u)(t - u)du$ is nondecreasing with respect to t, we have

$$\begin{aligned} \|e_n\| &\le \frac{1}{\|x^* - x_n\|^2} \int_0^{\|x^* - x_n\|} L(\|x_n - x_0\| + u)(\|x^* - x_n\| - u)du \\ &\qquad \cdot \|x^* - x_n\|^2 \\ &\le \frac{1}{(r_1 - t_n)^2} \int_0^{r_1 - t_n} L(t_n + u)(r_1 - t_n - u)du\|x^* - x_n\|^2 \\ &= \int_{t_n}^{r_1} L(u)(r_1 - u)du \left(\frac{\|x^* - x_n\|}{r_1 - t_n}\right)^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \|x^* - x_{n+1}\| &\le \|f'(x_n)^{-1}f'(x_0)\| \cdot \|e_n\| \\ &\le \frac{\displaystyle\int_{t_n}^{r_1} L(u)(r_1 - u)du}{1 - \displaystyle\int_0^{t_n} L(u)du} \left(\frac{\|x^* - x_n\|}{r_1 - t_n}\right)^2. \end{aligned}$$

By the induction method, (3.3) follows.

By (3.9), for all $i \ge 0$ and $n \ge n_0 \ge 0$, we have

$$\|x_{n+i+1} - x_{n+i}\| \le (t_{n+i+1} - t_{n+i}) \left(\frac{\|x_{n_0+1} - x_{n_0}\|}{t_{n_0+1} - t_{n_0}}\right)^{2^{n-n_0}}.$$

Summing for all $i \ge 0$ results in the upper bound (3.4). It follows from (3.3) that

$$\|x_{n+1} - x_n\| \le \|x^* - x_n\| + \|x^* - x_{n+1}\| \le \|x^* - x_n\| + \frac{r_1 - t_{n+1}}{(r_1 - t_n)^2}\|x^* - x_n\|^2.$$

Then, using Gragg and Tapia [3], we obtain the proof of the lower bound (3.4). □

4. Under the premise of a Kantorovich type

About the convergence of Newton's method, the main point of Kantorovich [2] type premise is to make

$$h(t) = \beta - t + \frac{1}{2}Lt^2, \quad 0 \le t \le R, \tag{4.1}$$

become a majorizing function. For this reason, as $||x - x_0|| + ||x' - x|| \le r$, it is sufficient to assume that

$$\|f'(x_0)^{-1}(f'(x) - f'(x'))\| \le L\|x - x'\|, \tag{4.2}$$

for a positive constant L. As

$$\lambda = L\beta \le \frac{1}{2}, \tag{4.3}$$

corresponding to (1.7), the zeros of h

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = \frac{1 \mp \sqrt{1 - 2\lambda}}{L} \tag{4.4}$$

satisfy

$$\beta \le r_1 \le 2\beta \le \frac{1}{L} \le r_2 \le \frac{2}{L}, \tag{4.5}$$

because $r_0 = 1/L, R = 2/L, b = 1/(2L)$ in this case. So Theorems 1.1 and 1.5, Propositions 1.3 and 1.4 all have concrete forms. The concretization of Theorem 3.1 requires that the solution of the sequence (3.4) has a closed form

$$t_n = \frac{1 - q^{2^n - 1}}{1 - q^{2^n}} r_1, \quad n = 0, 1, \cdots, \tag{4.6}$$

where

$$q = \frac{1 - \sqrt{1 - 2\lambda}}{1 + \sqrt{1 - 2\lambda}}. \tag{4.7}$$

(4.6) is independently given by [3]–[5].

For instance, the concrete forms of Theorems 1.1, 1.5 and 3.1 are, respectively,

Theorem 4.1. *Let L be a positive constant. Assume that f satisfies the condition*

$$||f'(x_0)^{-1}f'(x) - \mathrm{I}|| \le L||x - x_0||, \qquad \forall x \in \mathrm{B}(x_0, 1/L). \tag{4.8}$$

Then $f_{x_0}^{-1}$ exists and is differentiable in the open ball

$$\mathrm{B}(f(x_0), 1/(2L||f'(x_0)^{-1}||)) \subset f(\mathrm{B}(x_0, 1/L)). \tag{4.9}$$

Moreover, the radius of this ball (the left in (4.9)) is the best possible.

Theorem 4.2. *Let L be a positive constant, $\beta = ||f'(x_0)^{-1}f(x_0)||$ and $\lambda = L\beta \le \frac{1}{2}$. Assume that f satisfies the condition*

$$||f'(x_0)^{-1}f'(x) - I\,|| \le L||x - x_0||, \qquad \forall x \in \overline{\mathrm{B}(x_0, r)}, \tag{4.10}$$

where $r_1 \le r < r_2$ if $\lambda < \frac{1}{2}$, or $r = r_1$ if $\lambda = \frac{1}{2}$, while r_1 and r_2 are determined by (4.4). Then the equation (0.2) has a unique solution x^* satisfying (1.23) in the closed ball $\overline{\mathrm{B}(x_0, r)}$.

Theorem 4.3. *Let L be a positive constant, $\beta = \|f'(x_0)^{-1}f(x_0)\|$ and $\lambda = L\beta \leq \frac{1}{2}$. Assume that f satisfies the condition* (4.2). *Then Newton's method* (0.1) *is well defined for all n and converges to the solution x^* satisfying* (3.2) *of the equation* (0.2). *Moreover, for all $n \geq 0$, the best possible error bounds*

$$\|x^* - x_n\| \leq \frac{q^{2^n-1}}{\sum_{i=0}^{2^n-1} q^i}\|x^* - x_0\| \leq \frac{q^{2^n-1}}{\sum_{i=0}^{2^{n-1}-1} q^{2i}}\|x_1 - x_0\| \tag{4.11}$$

and

(4.12)

$$\frac{2\|x_{n+1} - x_n\|}{1 + \sqrt{1 + 4q^{2^n}/(1+q^{2^n})^2}} \leq \|x^* - x_n\| \leq \frac{q}{\beta}\sum_{i=0}^{2^{n-1}-1} q^{2i}\|x_n - x_{n-1}\|^2$$
$$\leq q^{2^{n-1}}\|x_n - x_{n-1}\|$$

are valid with (4.7).

Remark. It is a posterior estimation to use $\|x_n - x_{n-1}\|$ to estimate $\|x^* - x_n\|$. The posterior estimation in (4.12) can be obtained by setting $n_0 = n - 1$ in (3.4). In the hypothesis of Kantorovich's type, more precise posterior estimations were studied by Potra [6] and Potra & Ptak [7].

5. Under a premise of Smale type

Under the hypotheses that f is analytic and satisfies

$$\left\|f'(x_0)^{-1}f^{(n)}(x_0)\right\| \leq n!\gamma^{n-1}, \quad n \geq 2, \tag{5.1}$$

Smale [8] studied the convergence and error estimation of Newton's iteration. Wang and Han [9] (also see [10], [11]) completely improved Smale's results by introducing a majorizing function

$$h(t) = \beta - t + \frac{\gamma t^2}{1 - \gamma t}\ , \quad 0 \leq t \leq R. \tag{5.2}$$

When $\gamma\|x - x_0\| < 1$, it is easy to derive from (5.1) that

$$\left\|f'(x_0)^{-1}f''(x)\right\| \leq h''(\|x - x_0\|) = \frac{2\gamma}{(1 - \gamma\|x - x_0\|)^3}$$

(see Lemma 3 in [12] or Lemma 3.5 in [13]). Hence, conditions (1.1) and (3.1) are satisfied for the function L defined by

$$L(u) = \frac{2\gamma}{(1 - \gamma u)^3}. \tag{5.3}$$

Furthermore, for this L, the function h given in (1.5) coincides with the one in (5.2).

As $\alpha = \gamma\beta \leq 3 - 2\sqrt{2}$, corresponding to (1.7), the zeros of h

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = \frac{1 + \alpha \mp \sqrt{(1+\alpha)^2 - 8\alpha}}{4\gamma} \tag{5.4}$$

satisfy

$$\beta \leq r_1 \leq (1 + \frac{1}{\sqrt{2}})\beta \leq (1 - \frac{1}{\sqrt{2}})\frac{1}{\gamma} \leq r_2 \leq \frac{1}{2\gamma}, \tag{5.5}$$

because $r_0 = (1 - \frac{1}{\sqrt{2}})\frac{1}{\gamma}, R = \frac{1}{2\gamma}, b = (3 - 2\sqrt{2})\frac{1}{\gamma}$ in this case. So Theorems 1.1 and 1.5, Propositions 1.3 and 1.4 all have concrete forms. The concretization of Theorem 3.1 requires that the solution of the sequence (3.4) has a closed form

$$t_n = \frac{1 - q^{2^n - 1}}{1 - q^{2^n - 1}\eta} r_1, \quad n = 0, 1, \cdots, \tag{5.6}$$

where

$$q = \frac{1 - \alpha - \sqrt{(1+\alpha)^2 - 8\alpha}}{1 - \alpha + \sqrt{(1+\alpha)^2 - 8\alpha}}, \quad \eta = \frac{1 + \alpha - \sqrt{(1+\alpha)^2 - 8\alpha}}{1 + \alpha + \sqrt{(1+\alpha)^2 - 8\alpha}}. \tag{5.7}$$

For instance, the concrete forms of Theorems 1.1, 1.5 and 3.1 are, respectively,

Theorem 5.1. *Let γ be a positive constant. Assume that f satisfies the condition*

$$\begin{gathered} \|f'(x_0)^{-1}f'(x) - I\| \le \frac{1}{(1 - \gamma\|x - x_0\|)^2} - 1, \\ \forall x \in \mathrm{B}(x_0, (1 - \tfrac{1}{\sqrt{2}})/\gamma). \end{gathered} \tag{5.8}$$

Then $f_{x_0}^{-1}$ exists and is differentiable in the open ball

$$\mathrm{B}(f(x_0), (3 - 2\sqrt{2})/(\gamma\|f'(x_0)^{-1}\|)) \subset f(\mathrm{B}(x_0, (1 - \frac{1}{\sqrt{2}})/\gamma)). \tag{5.9}$$

Moreover, the radius of this ball (the left in (5.9)) is the best possible.

Theorem 5.2. *Let γ be a positive constant, $\beta = \|f'(x_0)^{-1}f(x_0)\|$ and $\alpha = \beta\gamma \le 3 - 2\sqrt{2}$. Assume that f satisfies the condition*

$$\|f'(x_0)^{-1}f'(x) - \mathrm{I}\| \le \frac{1}{(1 - \gamma\|x - x_0\|)^2} - 1, \qquad \forall x \in \overline{\mathrm{B}(x_0, r)}, \tag{5.10}$$

where $r_1 \le r < r_2$ if $\alpha < 3 - 2\sqrt{2}$, or $r = r_1$ if $\alpha = 3 - 2\sqrt{2}$, while r_1 and r_2 are determined by (5.4). *Then the equation* (0.2) *has a unique solution x^* satisfying* (1.23) *in the closed ball $\overline{\mathrm{B}(x_0, r)}$.*

Theorem 5.3. *Let γ be a positive constant, $\beta = \|f'(x_0)^{-1}f(x_0)\|$ and $\alpha = \beta\gamma \le 3 - 2\sqrt{2}$. Assume that f satisfies the condition*

$$\begin{gathered} \|f'(x_0)^{-1}(f'(x) - f'(x'))\| \le \frac{1}{(1 - \gamma\|x - x_0\| - \gamma\|x' - x\|)^2} - \frac{1}{(1 - \gamma\|x - x_0\|)^2}, \\ \|x - x_0\| + \|x' - x\| \le r. \end{gathered} \tag{5.11}$$

Then Newton's method (0.1) *is well defined for all n and converges to the solution x^* satisfying* (3.2) *of the equation* (0.2). *Moreover, for all $n \ge 0$, the best possible error bounds*

$$\begin{aligned} \|x^* - x_n\| &\le \frac{1 - \eta}{1 - q^{2^n - 1}\eta} q^{2^n - 1}\|x^* - x_0\| \\ &\le \frac{1 - \eta^2}{(1 + \alpha)(1 - q^{2^n - 1}\eta)} q^{2^n - 1}\|x_1 - x_0\| \end{aligned} \tag{5.12}$$

and

(5.13)

$$\frac{2\|x_{n+1}-x_n\|}{1+\sqrt{1+4q^{2^n}\frac{(1-q^{2^n})(1-q^{2^n-1}\eta)}{(1-q^{2^{n+1}-1}\eta)^2}}}$$
$$\leq \|x^*-x_n\| \leq \frac{q(1-q^{2^n-1}\eta)}{r_1(1-\eta)}\left(\frac{1-q^{2^{n-1}-1}\eta}{1-q^{2^{n-1}}}\right)^2\|x_n-x_{n-1}\|^2$$
$$\leq q^{2^{n-1}}\frac{1-q^{2^{n-1}-1}\eta}{1-q^{2^{n-1}}}\|x_n-x_{n-1}\|$$

are valid with (5.7).

The results above can be made more general by replacing (5.3). Now we take

$$L(u)=\frac{2c\gamma}{(1-\gamma u)^3}, \tag{5.3$'$}$$

where c is a positive number. In this case the majorizing function is

$$h(t)=\beta-t+\frac{c\gamma t^2}{1-\gamma t}, \tag{5.2$'$}$$

and its zeros are

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = \frac{1+\alpha\mp\sqrt{(1+\alpha)^2-4(1+c)\alpha}}{2(1+c)\gamma}. \tag{5.4$'$}$$

They satisfy

$$\beta\leq r_1\leq (1+\sqrt{\frac{c}{c+1}})\beta\leq(1-\sqrt{\frac{c}{c+1}})\frac{1}{\gamma}\leq r_2\leq\frac{1}{(c+1)\gamma}, \tag{5.5$'$}$$

because $r_0=(1-\sqrt{\frac{c}{c+1}})\frac{1}{\gamma}$, $R=\frac{1}{(c+1)\gamma}$, $b=(1+2c-2\sqrt{c(c+1)})\frac{1}{\gamma}$. Hence, we have

Theorem 5.3$'$. *Let γ and c be positive constants, $\beta=||f'(x_0)^{-1}f(x_0)||$ and $\alpha=\beta\gamma\leq 1+2c-2\sqrt{c(c+1)}$. Assume that f satisfies the condition*

(5.11$'$)

$$\|f'(x_0)^{-1}(f'(x)-f'(x'))\|\leq\frac{c}{(1-\gamma\|x-x_0\|-\gamma\|x'-x\|)^2}-\frac{c}{(1-\gamma\|x-x_0\|)^2},$$
$$||x-x_0||+||x'-x||\leq r.$$

Then Newton's method (0.1) *is well defined for all n and converges to the solution x^* satisfying* (3.2) *of the equation* (0.2). *Moreover, for all $n\geq 0$, the best possible error bounds* (5.12) *and* (5.13) *are valid with*

(5.7$'$)

$$q=\frac{1-\alpha-\sqrt{(1+\alpha)^2-4(1+c)\alpha}}{1-\alpha+\sqrt{(1+\alpha)^2-4(1+c)\alpha}},\quad \eta=\frac{1+\alpha-\sqrt{(1+\alpha)^2-4(1+c)\alpha}}{1+\alpha+\sqrt{(1+\alpha)^2-4(1+c)\alpha}}.$$

6. Under the premise of analyticity

We come back to the analytic premise about f, to see what stronger conclusion can be obtained. When f is assumed to be analytic in the ball $\mathrm{B}(x_0, r)$, f can be expanded to a convergent power series

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n. \tag{6.1}$$

If we suppose

$$\left\| f'(x_0)^{-1} f^{(n)}(x_0) \right\| \leq \gamma_n, \quad n \geq 2, \tag{6.2}$$

and write

$$g(t) = \sum_{n=2}^{\infty} \frac{\gamma_n}{n!} t^n, \tag{6.3}$$

where the sequence γ_n satisfies

$$\limsup \sqrt[n]{\frac{\gamma_n}{n!}} \leq \frac{1}{r}, \tag{6.4}$$

then $f'(x_0)^{-1} f'$ satisfies the Lipschitz condition about g'' in $\mathrm{B}(x_0, r)$. Thus, Theorem 1.1 asserts that $f_{x_0}^{-1}$ exists in $\mathrm{B}(f(x_0), b/\|f'(x_0)^{-1}\|)$ and is analytic, where

$$b = \int_0^{r_0} g''(u) u du = r_0 - g(r_0), \tag{6.5}$$

and r_0 satisfies

$$\int_0^{r_0} g''(u) du = g'(r_0) = 1. \tag{6.6}$$

So, we have

Theorem 6.1. *Assume that f is analytic in the ball $\mathrm{B}(x_0, r)$ and $r \geq r_0$. If*

$$\|y - f(x_0)\| < \frac{b}{\|f'(x_0)^{-1}\|}, \tag{6.7}$$

then the Euler series

$$x = x_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{d}{dy} \right)^n f_{x_0}^{-1}(y)_{y=f(x_0)} (y - f(x_0))^n \tag{6.8}$$

converges, and the constant b in the right of (6.7), which is determined by (6.5), is the best possible.

When $\mathbf{X} = \mathbf{Y} = \mathbf{C}$, we have

Theorem 6.2. *Assume that f and F are analytic in the open ball $\mathrm{B}(x_0, r) \subset \mathbf{C}$ and $r \geq r_0$. Then the convergence radius, $R(F \circ f_{x_0}^{-1})$, of the Lagrange series*

(6.9)

$$F(f_{x_0}^{-1}(y)) = F(x_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{d}{dx} \right)^{n-1} \left(F'(x) \left(\frac{x - x_0}{f(x) - f(x_0)} \right)^n \right)_{x=x_0} (y - f(x_0))^n$$

has an exact lower bound

$$R(F \circ f_{x_0}^{-1}) \geq b|f'(x_0)|. \tag{6.10}$$

It is a very technical thing to choose the sequence $\{\gamma_n\}$ or the function g such that it can give a bound of the different Taylor coefficients of f and be convenient to give the values of the parameters b and r_0 in Theorems 6.1 and 6.2. In this paper we propose to choose a different generating function G of the unit sequence $\{1, 1, \cdots\}$ with the positive constants γ and c, and then the function g can be obtained by

$$g(t) = \frac{c}{G'(0)\gamma}\left\{G(\gamma t) - G'(0)\gamma t - G(0)\right\}.$$

Example 1 (Exponential type). Taking $G(t) = e^t$ as the exponential generating function of the unit sequence, we have

$$g(t) = \frac{c}{\gamma}(e^{\gamma t} - \gamma t - 1).$$

Under the condition

$$||f'(x_0)^{-1}f^{(n)}(x_0)|| \leq c\gamma^{n-1}, \quad n \geq 2,$$

we obtain that

$$\gamma r_0 = \ln\frac{c+1}{c},$$
$$\gamma b = (c+1)\ln\frac{c+1}{c} - 1.$$

Especially, as $c = 1$, we have

$$\gamma r_0 = \ln 2 = 0.69314\cdots,$$
$$\gamma b = \ln 4 - 1 = 0.38629\cdots.$$

Theorem 6.1 with the values above has been obtained in [14] by the method of taking the exponential generating function of the number of Schröder system as the majorizing sequence.

Example 2 (Binomial type). Taking $G(t) = 1 + \mathrm{sign}(m)\{(1-t)^{-m} - 1\}$ as the binomial generating function of the unit sequence, where $m > -1$ and $m \neq 0$ is a real number, we have

$$g(t) = \frac{c}{m\gamma}\left\{(1-\gamma t)^{-m} - m\gamma t - 1\right\}.$$

Under the condition

$$||f'(x_0)^{-1}f^{(n)}(x_0)|| \leq c(m+1)(m+2)\cdots(m+n-1)\gamma^{n-1}, \quad n \geq 2,$$

we obtain that

$$\gamma r_0 = 1 - \left(\frac{c}{c+1}\right)^{\frac{1}{m+1}},$$
$$\gamma b = 1 + c\frac{m+1}{m}\left(1 - \left(\frac{c+1}{c}\right)^{\frac{m}{m+1}}\right).$$

Especially, as $m = 1$ and $c = 1$, we have

$$\gamma r_0 = 1 - \frac{1}{\sqrt{2}} = 0.29289\cdots,$$
$$\gamma b = 3 - 2\sqrt{2} = 0.17157\cdots.$$

Theorem 6.1 with the values above has been obtained in [13] by the method of taking the normal generating function of the number of blankets added to n letters as the majorizing sequence.

Also, as $m = \frac{1}{2}, c = 1$, we have

$$\gamma r_0 = 1 - \sqrt[3]{\frac{1}{4}} = 0.37003\cdots,$$
$$\gamma b = 4 - 3\sqrt[3]{2} = 0.22023\cdots.$$

As $m = -\frac{1}{2}, c = 1$, we have

$$\gamma r_0 = \frac{3}{4},$$
$$\gamma b = \tfrac{1}{2}.$$

The required condition of these simple numbers is not complicated, i.e.

$$||f'(x_0)^{-1}f^{(n)}(x_0)|| \leq \frac{(2n-3)!!}{2^{n-1}}\gamma^{n-1}, \quad n \geq 2,$$

Example 3 (The first logarithmic type). Taking $G(t) = 1 - \ln(1-t)$ as the first logarithmic generating function of the unit sequence, we have

$$g(t) = \frac{c}{\gamma}\ln\frac{1}{1-\gamma t} - ct.$$

Under the condition

$$||f'(x_0)^{-1}f^{(n)}(x_0)|| \leq c(n-1)!\gamma^{n-1}, \quad n \geq 2,$$

we obtain that

$$\gamma r_0 = \frac{1}{c+1},$$
$$\gamma b = 1 - c\ln\frac{c+1}{c}.$$

Especially, as $c = 1$, we have

$$\gamma r_0 = \frac{1}{2},$$
$$\gamma b = 1 - \ln 2 = 0.30685\cdots.$$

Example 4 (The second logarithmic type). Taking $G(t) = 1+2t+(1-t)\ln(1-t)$ as the second logarithmic generating function of the unit sequence, we have

$$g(t) = \frac{c}{\gamma}(1-\gamma t)\ln(1-\gamma t) + c\,t.$$

Under the condition

$$||f'(x_0)^{-1}f^{(n)}(x_0)|| \leq c(n-2)!\gamma^{n-1}, \quad n \geq 2,$$

we obtain that

$$\gamma r_0 = 1 - e^{-\frac{1}{c}},$$
$$\gamma b = 1 - c + ce^{-\frac{1}{c}}.$$

Especially, as $c = 1$, we have

$$\gamma r_0 = 1 - \frac{1}{e} = 0.63212\cdots,$$
$$\gamma b = \frac{1}{e} = 0.36787\cdots.$$

7. Applications to Smale's α-theory

We continue the discussion of Chapter 7 in [1]. It is well known that Smale [8] first used the criterion

$$\alpha(f, x_0) = \gamma||f'(x_0)^{-1}f(x_0)|| \tag{7.1}$$

to judge x_0 is an approximate zero of Newton's iteration of f, where

$$\gamma = \sup_{n\geq 2}\left|\left|\frac{1}{n!}f'(x_0)^{-1}f^{(n)}(x_0)\right|\right|^{\frac{1}{n-1}}. \tag{7.2}$$

Definition 7.1. Suppose $x_0 \in D$ is such that Newton's iteration (0.1) is well defined for $f : D \subset X \to Y$ and satisfies

$$e(x_n) \leq \left(\frac{1}{2}\right)^{2^{n-1}} e(x_{n-1}),$$

for all positive integers n, where $e(x_n)$ denotes some measurement of the approximation degree between x_n and x^*. Then x_0 is said to be an approximate zero of f in the sense of $e(x_n)$.

The approximate zero defined in [8] was introduced in the sense of $||x_{n+1} - x_n||$, while the second kind of approximate zero is defined in the sense of $||x^* - x_n||$. Now a more reasonable definition for the second kind was introduced in [15]. We find that it is not necessary to introduce the definition of an approximate zero in the sense of $||f'(x_0)^{-1}f(x_n)||$.

In fact, similarly to Theorem 7.2 in [1], by Theorem 5.3′ we have

Theorem 7.2. *Let γ, c and q be positive numbers, $0 < q < 1$. Assume that f satisfies the condition*

$$\begin{aligned} &||f'(x_0)^{-1}(f'(x) - f'(x'))|| \\ &\leq \frac{c}{(1-\gamma||x-x_0|| - \gamma||x'-x||)^2} - \frac{c}{(1-\gamma||x-x_0||)^2}, \\ &\gamma||x-x_0|| + \gamma||x'-x|| \leq 1 - \sqrt{\frac{c}{c+1}}. \end{aligned} \tag{7.3}$$

Then, as

$$\alpha \leq \frac{2q + c(1+q)^2 - (1+q)\sqrt{c^2(1+q)^2 + 4cq}}{2q} \tag{7.4}$$

for all natural numbers $n \geq 1$, it follows that

$$||x_n - x^*|| \leq q^{2^{n-1}}||x_{n-1} - x^*||, \tag{7.5}$$

$$||x_{n+1} - x_n|| \leq q^{2^{n-1}}||x_n - x_{n-1}||, \tag{7.6}$$

and

$$||f'(x_0)^{-1}f(x_n)|| \leq q^{2^{n-1}}||f'(x_0)^{-1}f(x_{n-1})||, \tag{7.7}$$

where x^* satisfies $f(x^*) = 0$.

Especially, as

$$\alpha(f, x_0) \leq \frac{4 + 9c - 3\sqrt{c(9c+8)}}{4}, \tag{7.4a}$$

x_0 is an approximate zero of f in any sense of $\|x^* - x_n\|, \|x_{n+1} - x_n\|$ or $\|f'(x_0)^{-1}f(x_n)\|$.

Proof. The representation in the inequality (7.4) at the right side can be obtained from (5.7′) by representing q by α. Hence, under the hyposethis of (7.4), by Theorem 5.3′ and Proposition 3.2, we have

$$||x_n - x^*|| \le \frac{r_1 - t_n}{r_1 - t_{n-1}}||x_{n-1} - x^*||,$$

$$||x_{n+1} - x_n|| \le \frac{t_{n+1} - t_n}{t_n - t_{n-1}}||x_n - x_{n-1}||$$

and

$$||f'(x_0)^{-1}f(x_n)|| \le \frac{h(t_n)}{h(t_{n-1})}||f'(x_0)^{-1}f(x_{n-1})||.$$

Thus, Theorem 7.2 follows from the following lemma. □

Lemma 7.3. *For (5.2′), (5.4′) and (5.7′), if $\alpha = \beta\gamma \le 1 = 2c - 2\sqrt{c(c+1)}$, then*

(7.8) $$\frac{r_1 - t_n}{r_1 - t_{n-1}} = \frac{1 - q^{2^{n-1}-1}\eta}{1 - q^{2^n-1}\eta}q^{2^{n-1}} \le q^{2^{n-1}},$$

(7.9) $$\frac{t_{n+1} - t_n}{t_n - t_{n-1}} = \frac{1 - q^{2^n}}{1 - q^{2^{n-1}}} \cdot \frac{1 - q^{2^{n-1}-1}\eta}{1 - q^{2^{n+1}-1}\eta}q^{2^{n-1}} \le q^{2^{n-1}}$$

and

(7.10) $$\frac{h(t_n)}{h(t_{n-1})} = \frac{1 - c\left(\frac{1}{(1-\gamma t_n)^2} - 1\right)}{1 - c\left(\frac{1}{(1-\gamma t_{n-1})^2} - 1\right)} \cdot \frac{t_{n+1} - t_n}{t_n - t_{n-1}} \le q^{2^{n-1}}.$$

Proof. As (5.2) becomes (5.2′), the representation (5.6) about $r_1 - t_n$ remains true provided that r_1 and r_2, q and η are determined by (5.4′) and (5.7′). Hence, (7.8)-(7.10) follow immediately. □

Finally, similarly to Colloray 7.3 in [1], we have

Corollary 7.4. *Let γ be a posiitve number. Assume that $f'(x_0)^{-1}$ exists, f is analytic in $\mathrm{B}(x_0, 1/\gamma)$, and for some $q \in (0,1)$*

(7.11) $$\left\|\frac{1}{n!}f'(x_0)^{-1}f^{(n)}(x_0)\right\| \le \left(\frac{1-q}{1+q}\right)^2 \gamma^{n-1}, \quad n \ge 2.$$

Then, as

(7.12) $$\alpha(f, x_0) \le q,$$

(7.5) holds.

Especially, as

(7.11a) $$\left\|\frac{1}{n!}f'(x_0)^{-1}f^{(n)}(x_0)\right\| \le \frac{\gamma^{n-1}}{9}, \quad n \ge 2,$$

and

(7.12a) $$\alpha(f, x_0) \le \frac{1}{2}$$

x_0 is an approximate zero of Newton's iteration of f.

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