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NON-PRIMITIVE NUMBER FIELDS OF DEGREE EIGHT AND OF SIGNATURE $(2,3)$, $(4,2)$ AND $(6,1)$ WITH SMALL DISCRIMINANT

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ABSTRACT. We give the lists of all non-primitive number fields of degree eight having two, four and six real places of discriminant less than 6688609, 24363884 and 92810082, respectively, in absolute value. For each field in the lists, we give its discriminant, the discriminant of its subfields, a relative polynomial generating the field over one of its subfields and its discriminant, the corresponding polynomial over $\mathbf{Q}$, and the Galois group of its Galois closure.

1. INTRODUCTION

It is well known that, in degree eight, the minima for discriminants are only known for signatures $(0,4)$ [7] and $(8,0)$ [16]. For the other cases only partial tables of euclidean fields for the signatures $(2,3)$ and $(4,2)$ [11] are available.

In this work, we give the table of non-primitive number fields of degree 8, of signature $(2,3)$ (resp. $(4,2)$, $(6,1)$) and of discriminant majorized, in absolute value, by 6688609 (resp. 24363884, 92810082).

To establish these lists, we have explicitly constructed all non-primitive number fields of degree 8, of desired signatures and of discriminant within the previously chosen bounds, each field being defined by a polynomial with coefficients chosen in a convenient subfield. We have followed the method of explicit construction of relative extensions described in [12].

This paper is organized into several sections. Section 2 provides the notations and mathematical basis for the expression of relative extensions. In Section 3, we justify the choice of the bounds, which choice is related to lower bounds for discriminants with Odlyzko-Poitou-Serre local corrections [18]. The consequences of these lower bounds are gathered in a lemma and bring important simplifications in the computations. Section 4 is devoted to the description of computations which allow us to find by the number-geometric method all the non-primitive extensions of degree 8, of signatures $(2,3)$, $(4,2)$ and $(6,1)$ and of absolute discriminant smaller than the previously chosen bounds. We prove the existence of two non-isomorphic fields of discriminant -5365963 and two non-isomorphic fields of discriminant -6647387. These are the only fields in the limits of the given tables which are not characterized by their discriminant. Finally, we study the Galois group of the Galois closure of each field in the tables.

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2. Notations

If K is a number field of degree n and of signature (r, s), we denote by d_K its discriminant, by $\mathbf{Z}_K$ its ring of integers and by $J(K)$ the set of distinct $\mathbf{Q}$-isomorphisms of K into $\mathbf{C}$. For every ξ in K, we denote by $\xi^{(1)}, \cdots, \xi^{(r)}$ its real conjugates, and by $\xi^{(r+1)}, \xi^{(r+2)} = \overline{\xi^{(r+1)}}, \ldots, \xi^{(n-1)}, \xi^{(n)} = \overline{\xi^{(n-1)}}$ its complex conjugates, and we set $T_j(\xi) = \sum_{i=1}^{n} |\xi^{(i)}|^j$.

Let K be a number field of degree n, an extension of degree m of a subfield F of degree n'. For $\sigma \in J(F)$ we set

$$J_\sigma(K) = \{\tau \in J(K) \colon \tau_{/F} = \sigma\}.$$

Clearly,

$$J(K) = \bigcup_{\sigma \in J(F)} J_\sigma(K).$$

For θ an integer of K we define

$$Tr_{\sigma,K/F}(\theta) = \sum_{\tau \in J_\sigma(K)} \tau(\theta).$$

If we assume $K = F(\theta)$, then θ is a root of a polynomial $P(x) \in \mathbf{Z}_F[x]$,

$$P(x) = x^m + a_1 x^{m-1} + \cdots + a_m.$$

If we denote by $P_\sigma(x)$, $\sigma \in J(F)$, the polynomial

$$P_\sigma(x) = x^m + \sigma(a_1) x^{m-1} + \cdots + \sigma(a_m),$$

then the polynomial $f(x) = \prod_{\sigma \in J(F)} P_\sigma(x)$ has integer coefficients and is either irreducible or a power of an irreducible polynomial. Let $\theta_1, \ldots, \theta_n$ be the roots of $f(x)$ ordered so that $\theta_1, \ldots, \theta_m$ are the roots of $P(x)$. For each natural number j we consider the power sums

$$s_j = s_j(\theta) = \sum_{i=1}^{m} \theta_i^j.$$

Clearly,

$$\sum_{i=1}^{n'} |s_j^{(i)}| \leq \sum_{i=1}^{n} |\theta_i|^j \qquad (2 \leq j \leq m).$$

Let δ be the relative discriminant of K over F, and N the absolute norm in the extension $F/\mathbf{Q}$. The discriminants of K and F are then related by

$$|\mathrm{d}_K| = |\mathrm{d}_F|^m N(\delta), \tag{1}$$

and if η denotes the number of complex places of K whose restriction to a place of F is real, a result of J. Martinet [13] asserts that

$$N(\delta) \equiv 0 \text{ or } (-1)^\eta \pmod 4. \tag{2}$$

We recall that a relative discriminant is the product of an integer ideal δ_0 of F by the set of infinite places ramified in K/F: $\delta = \delta_0 \infty_1 \cdots \infty_\eta$.

3. Lower bounds for discriminants

The tables of lower bounds for discriminants [5] indicate that the absolute value of the discriminant of a number field of degree 8 and of signature $(2,3)$ (resp. $(4,2)$, $(6,1)$) is larger than 3404641 (resp. 11666965, 42098660). This lower bound, which is obtained in the absence of all hypotheses on the decomposition of prime ideals in the extension $K/\mathbf{Q}$, can be improved by taking into account the local corrections corresponding to small prime numbers [18]. Assuming that the prime number $\mathfrak{p}$ is divisible by a prime ideal of K of norm $\mathfrak{p}^{\mathrm{f}}$, we find the following lower bounds for each fixed signature.

Lower bounds

$\mathfrak{p}$	f	(2,3)	(4,2)	(6,1)
2	1	11725962	42765015	162569966
2	2	6688609	24363884	92810082
3	1	8336752	30393069	115852707
3	2	4160401	14972957	52529001
5	1	5726300	20829049	79259702
7	1	4682933	16957023	64309248

In order to find all the number fields K of degree 8, of signature $(2,3)$ (resp. $(4,2)$, $(6,1)$) and of absolute discriminant d_K such that $|\mathrm{d}_K| \leq \mathrm{M}$, we choose $\mathrm{M} = 6688609$ (resp. 24363884, 92810082). This choice is justified by the fact that we can apply the following lemma to these fields.

Lemma 1. *Let K be a number field of degree 8 over $\mathbf{Q}$, of signature $(2,3)$ (resp. $(4,2)$, $(6,1)$) with $|\mathrm{d}_K| \leq \mathrm{M}$. Let θ be an integer of K of absolute norm a. If $a = 2^x 3^y c$, c prime with 2 and with 3, then $x = 0$ or $x \geq 3$, and $y = 0$ or $y \geq 2$.*

This lemma is an immediate consequence of the lower bounds given above.

4. Description of the computations

From now on, K denotes a non-primitive number field of degree 8, of signature $(r,s) = (2,3)$ (resp. $(4,2)$, $(6,1)$) and of discriminant d_K such that $|\mathrm{d}_K| \leq \mathrm{M}$. The field K being non-primitive, it contains either a quartic subfield or a quadratic subfield. As we are concerned with the construction of lists of the non-primitive fields with $|\mathrm{d}_K| \leq \mathrm{M}$, we must consider all the quartic subfields F (resp. quadratic subfields $\mathfrak{L}$) whose signature (r', s') is compatible with that of K, that is to say, $s \geq 2s'$ (resp. $s \geq 4s'$) and whose discriminant satisfies $|\mathrm{d}_F| \leq \mathrm{M}^{1/2}$ (resp. $|\mathrm{d}_{\mathfrak{L}}| \leq \mathrm{M}^{1/4}$).

a) Quadratic extensions of quartic subfields. Each relative quadratic extension of a quartic field may be defined by a polynomial of second degree with coefficients in the subfield. In this section we develop a method of computation which allows us to construct explicitly all the relative polynomials

$$P(x) = x^2 + bx + c \in \mathbf{Z}_F[x]$$

of which one of the roots θ defines an octic field K of signature (r,s) such that $K = F(\theta)$. The basic tool of this method is a generalization of the Hunter-Pohst theorem given by J. Martinet in [12].

Theorem 1. *There exists an integer $\theta \in K$, $\theta \notin F$, such that $K = F(\theta)$ and*

$$\sum_{i=1}^{8} |\theta^{(i)}|^2 \leq \frac{1}{2} \sum_{\sigma \in J(F)} |Tr_{\sigma, K/F}(\theta)|^2 + B, \tag{3}$$

where $B = (\mathrm{M}/4|\mathrm{d}_F|)^{1/4}$. This inequality is also valid for all elements of K of the form $\theta + \gamma$ or $-\theta$, where γ is any integer of F.

To construct all polynomials $P(x)$ of which one root θ generates one of the desired fields K over F, we will work in the field F. We assume that the discriminant d_F and an integral basis $\mathcal{B} = \{w_1 = 1, w_2, w_3, w_4\}$ of F are known.

The knowledge of the ordinary and the strict class numbers of F as well as the use of certain simplifications and techniques gathered in the following lemma allow us to exclude several quartic fields.

Lemma 2. *For $|\mathrm{d}_K| < \mathrm{M}$ and $N(\delta_0) > 1$ we have*

(i) $\mathrm{Max}\{7, 3404641|\mathrm{d}_F|^{-2}\} \leq N(\delta_0) \leq 6688609|\mathrm{d}_F|^{-2}$ (*resp.* $\mathrm{Max}\{5, 11666965|\mathrm{d}_F|^{-2}\} \leq N(\delta_0) \leq 24363884|\mathrm{d}_F|^{-2}$, $\mathrm{Max}\{7, 42098660|\mathrm{d}_F|^{-2}\} \leq N(\delta_0) \leq 92810082|\mathrm{d}_F|^{-2}$).
(ii) *If $N(\delta_0) = 5u$ with $(u, 5) = 1$, then $|\mathrm{d}_K| \geq 5726300$ (resp. 20829049, 79259702).*
(iii) *If $N(\delta_0) = 7u$ with $(u, 7) = 1$, then $|\mathrm{d}_K| \geq 4682933$ (resp. 16957023, 64309248).*
(iv) *$N(\delta_0) \neq 3u$ with $(u, 3) = 1$.*
(v) *If $\delta_0 = \prod_{i=1}^{t} \wp_i^{e_i}$, then $e_i = 1$ for $N(\wp_i) \equiv 1 \pmod 2$.*
(vi) *$N(\delta_0)$ is odd for the signatures $(2, 3)$ and $(6, 1)$.*

Proof. The assertions (i)–(iv) come from lower bounds for discriminants with local corrections and from formulas (1) and (2) for (i). The extension K/F is of relative degree 2; the ramification of a prime ideal $\wp$ in the extension K/F is either wild or tame according to whether $N(\wp)$ is even or odd. If $N(\wp) \equiv 1 \pmod 2$, then the ramification is tame; hence $e_i = 1$ and assertion (v) holds. To show assertion (vi), we notice that if $N(\wp) = 2^i$ $(1 \leq i \leq 3)$ then the extension $K/\mathbf{Q}$ contains either an ideal of norm 2 or an ideal of norm 4; this case is excluded according to the lower bounds for discriminants with local corrections. Finally, if 2 remains inert in $F/\mathbf{Q}$, since the ramification in K/F is wild, we should have $N(\delta_0) = 2^8 l$, which leads to $|\mathrm{d}_F| \leq 161$ (resp. $|\mathrm{d}_F| \leq 602$). However, there exist no fields of signature $(4, 0)$ or (2.1) (resp. $(4, 0)$) with discriminant $|\mathrm{d}_F|$ smaller than 275 (resp. 725) [9, 10]. □

Construction of relative polynomials. The second part of Theorem 1 shows that the coefficient b of $P(x)$ may be chosen of the form

$$b = \mathfrak{x}_1 w_1 + \cdots + \mathfrak{x}_4 w_4 \quad \text{with } \mathfrak{x}_i \in \{0, 1\} \text{ for } i = 1, \ldots, 4.$$

We notice that for a fixed value of b, the value of $T_2(\theta)$ is majorized by a real constant which only depends on the chosen value of b. On the other hand, if the root θ of $P(x)$ is a generator of the extension K/F, $\theta + \gamma$ is also a generator of K/F for all γ in $\mathbf{Z}_F$; additionally (3) remains valid if we replace θ by $\theta + \gamma$, since $Tr_{K/F}(\theta + \gamma) = -b + 2\gamma := -\beta$. If we represent β by means of the basis $\mathcal{B}$ of F in the form $\beta = \sum_{i=1}^{4} \beta_i w_i$, then $T_2(\beta)$ becomes a positive definite quadratic form $q(\nu) = \nu \mathrm{A} \nu^t$ in the coefficients $\beta_1, \ldots, \beta_4$ $(\nu := (\beta_1, \ldots, \beta_4))$, where $\mathrm{A} = (m_{ij})$ and

$$m_{ij} = \sum_{k=1}^{4} w_i^{(k)} \overline{w}_j^{(k)} \qquad (1 \leq i, j \leq 4).$$

There exists at least one choice of γ such that $T_2(\beta)$ will be minimum. We obtain the possible β, first using the algorithm A [17] to decompose the matrix A into a sum of squares by the Cholesky method, then using the algorithm B [17] to compute all the solutions β subject to $q(\nu) \leq E$ $(E := T_2(b))$. Among the β values, we only keep those which verify $\beta \equiv b \pmod{2\mathbf{Z}_F}$. We will later denote by b the value of β for which $T_2(\beta)$ is minimum, and we set $\mathcal{C} = \frac{1}{2}T_2(b) + B$.

Once a convenient value of b is determined, we determine the possible values of c from the second relative symmetric function $s_2 = \sum_{i=1}^{4} y_i w_i$ and from the inequalities

$$\sum_{i=1}^{4} |s_2^{(i)}|^2 \leq T_2(\theta)^2 \leq \mathcal{C}^2.$$

The possible values of the integer s_2 are obtained by letting $y_1, \ldots, y_4$ run through the integer values for which $q(y_1, \ldots, y_4) \leq \mathcal{C}^2$ and such that $s_2 \equiv b^2 \pmod{2\mathbf{Z}_F}$. For each value of s_2 obtained in this way, we evaluate in $\mathbf{Z}_F$ the integer $b^2 - s_2$. If all the coordinates of the latter are even, we then obtain a value of c; namely, $c = (b^2 - s_2)/2$.

We start by verifying whether $P(x)$ can define a field of the desired signature. This question is solved by simply examining the sign of the discriminant $\Delta = b^2 - 4c$ of each real conjugate of $P(x)$. We considerably reduced the number of polynomials $P(x)$ to be considered by using the inequality

$$\sum_{i=1}^{4} |\Delta^{(i)}| \leq 2B,$$

which follows from inequality (3) and from the equality

$$|\theta_1 + \theta_2|^2 + |\theta_1 - \theta_2|^2 = 2(|\theta_1|^2 + |\theta_2|^2),$$

where θ_1, θ_2 are complex numbers.

We compute $L = |N(\Delta)|$; if L is squarefree, the relative discriminant δ has a norm L and the computation of L permits the elimination of all polynomials with $L > \mathrm{M}/\mathrm{d}_F^2$. The computation of the roots of the four conjugate polynomials is only necessary for $r = 4$ and $r' = 2$ to test the irreducibility of the polynomial P. For $r = 4$ and $r' = 4$, the polynomials which are squares of irreducible fourth degree polynomials are eliminated.

For the computation of the discriminant of K, we have first determined the relative discriminant δ, by using a theorem on ramification in Kummer extensions, then deduced the value $\mathrm{d}_K = (-1)^s \mathrm{d}_F N(\delta)$. Let us now mention two examples of computation of δ and d_K.

- Let $K = F(\theta)$ and $F = \mathbf{Q}(\rho)$, where θ is a root of

  $$P(x) = x^2 + (-1 + \rho^2 - \rho^3)x + (-\rho + \rho^2)$$

  and ρ is a root of

  $$g(x) = x^4 - 2x^3 + 2x^2 - x - 1.$$

  We have $\mathrm{d}_F = \mathrm{d}_g = -475$, $\mathrm{d}_P \equiv 3\rho - 3\rho^2 + \rho^3 \pmod{g(\rho)}$ and $N(\mathrm{d}_P) = -19$. Then $g(x) \equiv (x+12)(x+6)(x+9)^2 \pmod{19}$ and $\mathrm{d}_P = \rho(\rho+6)(\rho+10) \pmod{19}$. It follows that the relative discriminant δ is equal to $\wp\infty_1$, where $\wp$ is the prime ideal $(19, \rho+6)$ of F over 19 and $\mathrm{d}_K = -(-475)^2 19 = -4286875$.

- Let $K = F(\theta)$ and $F = \mathbf{Q}(\rho)$ where θ is a root of $P(x)$ and ρ a root of $g(x)$:
$$P(x) = x^2 + (-\rho - \rho^2 + \rho^3) \quad \text{and} \quad g(x) = x^4 - x^3 - 1.$$
We have $\mathrm{d}_F = \mathrm{d}_g = -283$, $\mathrm{d}_P = 4(\rho + \rho^2 - \rho^3)$ and $N(\mathrm{d}_P) = 256$. Then $2\mathbf{Z}_F = \wp$, with $N(\wp) = 2^4$ and $\nu_\wp(\mathrm{d}_P) = 2$. Notice also that
$$P(x) \equiv (x + \rho^3 + 1)^2 \pmod{\wp}.$$
We can obtain an $(x + \rho^3 + 1)$-development of the polynomial $P(x)$ as
$$P(x) = (x + \rho^3 + 1)^2 + (-2\rho^3 - 2)(x + \rho^3 + 1) + (4\rho^3 + 2).$$
All the coefficients are divisible by $\wp$ and have a valuation equal to 1, proving that the polynomial is an Eisenstein polynomial and $\wp$ is fully ramified in K/F. Then $\delta = \wp^2$ and $\mathrm{d}_K = (-283)^2 256 = 20502784$.

To decide if two equal discriminants correspond to the same field up to an isomorphism, we determine whether the relative ideal discriminants are conjugates. We obtain the following results.

Proposition 1. *Within the limits of Table 1, there exist, for the signature* $(2, 3)$, *two nonisomorphic fields of discriminant* -5365963 *and two nonisomorphic fields of discriminant* -6647387. *All other fields in Table 1 are characterized by their discriminant.*

A second proof of this proposition can be given by decomposing a suitable prime number in each of the two fields with the same discriminant.

Denoting by K_1, K_2 the two fields with the same discriminant in the order in which they appear in the table, and by f_1, f_2 the respective polynomials defining these fields, we obtain the following decompositions, where d_K is fixed:

- $\mathrm{d}_K = -5365963$:
$$f_1(x) = x^8 + 3x^7 + x^6 - x^5 + 4x^4 + 4x^3 + x^2 + x - 1, \qquad 13\mathbf{Z}_{K_1} = \wp_1 \wp_1' \wp_6,$$
$$f_2(x) = x^8 - 4x^7 + 8x^6 - 13x^5 + 15x^4 - 13x^3 + 8x^2 - 4x + 1, \quad 13\mathbf{Z}_{K_2} = \wp_2 \wp_3 \wp_3';$$
- $\mathrm{d}_K = -6647387$:
$$f_1(x) = x^8 - 4x^7 + 5x^6 - 4x^5 + 4x^4 - x^3 - x^2 - 1, \qquad 11\mathbf{Z}_{K_2} = \wp_1 \wp_1' \wp_3 \wp_3',$$
$$f_2(x) = x^8 - 2x^6 + x^5 + 5x^4 - 7x^3 + 3x^2 + x - 1, \qquad 11\mathbf{Z}_{K_1} = \wp_2 \wp_6.$$

We indicate, in the list below, the polynomials defining the quartic fields used in the computations. These polynomials are obtained by applying the POLRED algorithm [4] to the polynomials given in the tables of Godwin [9, 10]. We note that the computational time when using these polynomials is very low as compared with the computational time when using polynomials taken directly from Godwin's tables.

$x^4 - x^3 - 1$	-283	$x^4 - x^3 + 2x - 1$	-275	$x^4 - x^3 - 3x^2 + x + 1$	725
$x^4 - x^3 + x^2 + x - 1$	-331	$x^4 - x^2 - 1$	-400	$x^4 - x^3 - 4x^2 + 4x + 1$	1125
$x^4 - x^3 + x^2 - x - 1$	-563	$x^4 - 2x^3 + x^2 + 2x - 1$	-448	$x^4 - 6x^2 + 4$	1600
$x^4 - x^3 - 2x + 1$	-643	$x^4 - 2x^3 + 2x^2 - x - 1$	-475	$x^4 - 7x^2 + 11$	4400
$x^4 - 2x^3 + 3x^2 - 1$	-976				

Among the k conjugate extensions found we only give one conjugate field for each subfield.

TABLE 1

d_K	d_F	k	$P(x)$	d_P	$N(d_P)$	δ
			$(r,s)=(2,3)$			
-4286875	-475	2	$x^2+(-1+\rho^2-\rho^3)x+(-\rho+\rho^2)$	$3\rho-3\rho^2+\rho^3$	-19	$(19,\rho+6)\infty_1$
-4461875	-275	2	$x^2+(1+\rho^3)x+(1-\rho^2+\rho^3)$	$-2-\rho+3\rho^2-3\rho^3$	-59	$(59,\rho+56)\infty_1$
-4616192	-448	2	$x^2+(-2+\rho^2-\rho^3)x+(1+\rho-2\rho^2+\rho^3)$	$-4\rho+5\rho^2-2\rho^3$	-23	$(23,\rho+15)\infty_1$
-4711123	-331	1	$x^2+\rho^2x-\rho$	$1+3\rho-\rho^2+\rho^3$	-43	$(43,\rho+35)\infty_1$
-4725251	-283	1	$x^2+(\rho^2-\rho^3)x+(-1-\rho+\rho^2)$	$4+3\rho-3\rho^2$	-59	$(59,\rho+35)\infty_1$
-4960000	-400	2	$x^2+(-1+\rho-\rho^3)x-\rho$	$2\rho+\rho^2+2\rho^3$	-31	$(31,\rho+9)\infty_1$
-5149367	-331	1	$x^2+(-\rho+\rho^2-\rho^3)x+1$	$-3-2\rho+2\rho^2-\rho^3$	-47	$(47,\rho+31)\infty_1$
-5365963	-283	1	$x^2+(1-\rho^3)x+\rho$	$2-3\rho+\rho^2-\rho^3$	-67	$(67,\rho+51)\infty_1$
-5365963	-283	1	$x^2+(-1+\rho-\rho^2)x+1$	$-2-2\rho+3\rho^2-\rho^3$	-67	$(67,\rho+42)\infty_1$
-5369375	-275	2	$x^2+(-1+\rho^2-\rho^3)x+(1-\rho+\rho^3)$	$-3+3\rho+\rho^2-4\rho^3$	-71	$(71,\rho+42)\infty_1$
-5781875	725	2	$x^2+(-1+3\rho+\rho^2-\rho^3)x+1$	$-2\rho-2\rho^2+\rho^3$	-11	$(11,\rho+5)\infty_1\infty_2\infty_3$
-6022411	-563	1	$x^2+(1-2\rho+2\rho^2-\rho^3)x(1-\rho+\rho^2-\rho^3)$	$1+\rho-2\rho^2+2\rho^3$	-19	$(19,\rho+5)\infty_1$
-6464099	-331	1	$x^2+\rho^2x+(\rho-\rho^2+\rho^3)$	$1-5\rho+3\rho^2-3\rho^3$	-59	$(59,\rho+40)\infty_1$
-6647387	-283	1	$x^2+(\rho-\rho^3)x+(-1+\rho^2)$	$3+\rho-2\rho^2-\rho^3$	-83	$(83,\rho+55)\infty_1$
-6647387	-283	1	$x^2+(-1+\rho^2-\rho^3)x-\rho$	$1+3\rho-\rho^2+2\rho^3$	-83	$(83,\rho+77)\infty_1$
-6668032	-976	1	$x^2+(-1-2\rho+\rho^2-\rho^3)x+(3\rho-2\rho^2+\rho^3)$	$3-8\rho+5\rho^2-2\rho^3$	-7	$(7,\rho+3)\infty_1$
			$(r,s)=(4,2)$			
15243125	725	1	$x^2+(-2+2\rho+2\rho^2-\rho^3)x+(1-2\rho-\rho^2+\rho^3)$	$3\rho+2\rho^2-2\rho^3$	29	$(29,\rho+17)\infty_1\infty_2$
15297613	-643	1	$x^2+(1+\rho^2-\rho^3)x+(\rho-\rho^2)$	$1-3\rho+3\rho^2$	37	$(37,\rho+33)$
16324589	-331	1	$x^2+(-1+\rho-\rho^2)x+(-1-\rho+\rho^2-\rho^3)$	$6+\rho-2\rho^2+3\rho^3$	149	$(149,\rho+120)$
17318125	-275	2	$x^2+(-2+\rho+\rho^2-\rho^3)x+\rho^3$	$2-\rho-2\rho^3$	229	$(229,\rho+173)$
18340381	-283	1	$x^2+(\rho^2-\rho^3)x-1$	$4-\rho+\rho^2$	229	$(229,\rho+168)$
18660737	-283	1	$x^2+(\rho-\rho^3)x-1$	$3+\rho+2\rho^2-\rho^3$	233	$(233,\rho+37)$
19360000	-275	1	$x^2+(-1+\rho)$	$4-4\rho$	256	$\wp^2: 2Z_F=\wp$
19360000	-400	1	$x^2+(\rho-\rho^3)x-1$	$3+\rho^2$	121	$(11,\rho^2+3)$
19360000	4400	1	$x^2+\rho x+1$	$-4+\rho^2$	1	$\infty_1\infty_2$
20262517	-283	1	$x^2+(-1+\rho^2-\rho^3)x+(-\rho-\rho^2+\rho^3)$	$1+3\rho+3\rho^2-2\rho^3$	253	$(11,\rho+4)(23,\rho+2)$
20502784	-283	1	$x^2+(-\rho-\rho^2+\rho^3)$	$4\rho+4\rho^2-4\rho^3$	256	$\wp^2: 2Z_F=\wp$
21543941	-283	1	$x^2+\rho^3x+(-1+\rho^2)$	$5+\rho-3\rho^2+\rho^3$	269	$(269,\rho+180)$
21550625	725	2	$x^2+(-1+3\rho+\rho^2-\rho^3)x+(2+\rho-\rho^2)$	$-4-6\rho+2\rho^2+\rho^3$	41	$(41,\rho+19)\infty_1\infty_2$
23040000	1600	2	$x^2+(1+2w_2-w_3-w_4)x+(1+w_2)$	$-2-2w_2+w_3$	9	$(3,\rho^2+2\rho+2)\infty_1\infty_2$
24212981	-331	1	$x^2+\rho^2x-\rho^2$	$1-\rho+3\rho^2+\rho^3$	221	$(13,\rho+10)(17,\rho+9)$
			$(r,s)=(6,1)$			
-68856875	725	2	$x^2+(1+\rho-\rho^2)x+(-1+2\rho+\rho^2-\rho^3)$	$4-7\rho-2\rho^2+3\rho^3$	-131	$(131,\rho+79)\infty_1$
-73061875	725	2	$x^2+x+(-1-\rho)$	$5+4\rho$	-139	$(139,\rho+36)\infty_1$
-74671875	1125	4	$x^2+(2-\rho^2)x+(-3\rho+\rho^3)$	$3+8\rho-3\rho^3$	59	$(59,\rho+13)\infty_1$

b) Quartic extensions of quadratic fields. Let K be an octic number field, an extension of degree 4 of a quadratic subfield $\mathfrak{L} = Q(\sqrt{\partial})$. Theorem 2.8 of [12] asserts that there exists an integer element $\theta \in K$, $\theta \notin \mathfrak{L}$, such that

$$\sum_{i=1}^{8} |\theta^{(i)}|^2 \leq (\alpha_1^2 + \beta_1^2 \mathfrak{d}_{\mathfrak{L}})/8 + B, \tag{4}$$

where $B = (4\mathrm{M}/3\mathrm{d}_{\mathfrak{L}})^{1/6}$ and $Tr_{\sigma,K/\mathfrak{L}}(\theta) = (\alpha_1 + \beta_1\sqrt{\mathfrak{d}_{\mathfrak{L}}})/2$ is an integer of $\mathfrak{L}$.

The field $\mathfrak{L}(\theta)$ is a non-trivial extension of $\mathfrak{L}$ and thus an intermediate field between $\mathfrak{L}$ and K. Since the study of the fields K containing a subfield of degree 4 over $\mathbf{Q}$ has already been made in Section 4.a, we can stipulate without loss of generality that θ is a primitive element of $K/\mathbf{Q}$. Let $f(x) \in \mathbf{Z}[x]$ be the minimal polynomial of the integer θ over $\mathbf{Q}$. Then $f(x)$ decomposes in $\mathbf{Z}_{\mathfrak{L}}[x]$ into a product of two conjugate irreducible polynomials

$$P(x) = x^4 + a_1x^3 + \cdots + a_4 \quad \text{and} \quad P'(x) = x^4 + \sigma(a_1)x^3 + \cdots + \sigma(a_4).$$

Notice that each integer γ in $\mathfrak{L}$ can be written as $\gamma = (\alpha + \beta\sqrt{\mathfrak{d}_{\mathfrak{L}}})/2$ with $\alpha \equiv \beta \pmod 2$ for $\mathfrak{d} \equiv 1 \pmod 4$ and α even for $\mathfrak{d} \equiv 2$ or $3 \pmod 4$. We denote this integer by the couple (α, β).

The fact that the inequality (4) remains valid if we replace θ by $-\theta$ or $\theta + \lambda$ for an arbitrary $\lambda \in \mathbf{Z}_{\mathfrak{L}}$ and the fact that P and P' define the same field (up to conjugacy) allow us to choose (α_1, β_1) from the set

$$\{(0,0),(1,1),(2,0),(2,2),(3,1),(4,0),(4,2)\} \quad \text{for } \mathfrak{d} \equiv 1 \pmod 4$$

and from

$$\{(0,0),(0,1),(0,2),(2,0),(2,1),(2,2),(4,0),(4,1),(4,2)\} \quad \text{for } \mathfrak{d} \equiv 2 \text{ or } 3 \pmod 4.$$

For each of these pairs, we start by computing an upper bound for $T_2(\theta)$ by (4). We then compute the upper bounds for $T_j(\theta)$ with $j = -1, 3, 4$ using Theorem 4 of [15]. Finally, we evaluate the other coefficients by induction with the help of Newton's formulas. The values of $s_j = (\alpha_j + \beta_j\sqrt{\mathfrak{d}_{\mathfrak{L}}})/2$ must satisfy not only the inequalities

$$|s_j| + |\sigma(s_j)| = \max\{|\alpha_j|, |\beta_j|\sqrt{\mathfrak{d}_{\mathfrak{L}}}\} \leq T_j(\theta) \qquad (2 \leq j \leq 4)$$

but also the congruences

$$s_j \equiv -\sum_{i=1}^{j-1} a_i s_{j-i} \pmod{j\mathbf{Z}_{\mathfrak{L}}} \qquad (2 \leq j \leq 4).$$

In spite of the fact that we have obtained all the polynomials of interest to us, we can further shorten the list of these polynomials. Indeed, the inequality

$$|N(a_4)| \leq (T_2(\theta)/8)^4,$$

which follows from the inequality between arithmetic and geometric means, and the fact that the values taken by $N(a_4)$ must be compatible with the conclusions of Lemma 1, makes it possible to reduce considerably the number of polynomials. The inequality

$$|e| + |\sigma(e)| \leq 4B,$$

where $e = 3a_1^2 - 8a_2$, which follows from the inequality (4) and from

$$\left|(k-1)\left(\sum_{i=1}^{k}\theta_i\right)^2 - 2k\sum_{1\le i\le j\le k}\theta_i\theta_j\right| + \left|\sum_{i=1}^{k}\theta_i\right|^2 = k\sum_{i=1}^{k}|\theta_i|^2,$$

where $\theta_1,\dots,\theta_k$ are complex numbers, allows the elimination of several values of a_2. Furthermore, for the signature $(6,1)$, since one of the relative polynomials must have only real roots, its coefficients must satisfy Newton's inequalities. So either we have

$$3a_1^2 - 8a_2 \ge 0, \qquad 4a_2^2 - 9a_1a_3 \ge 0, \qquad 3a_3^2 - 8a_2a_4 \ge 0,$$

or the conjugates in $\mathfrak{L}$ of these numbers are all positive or null.

Once all the coefficients of $P(x)$ are obtained, we start by ensuring that we have

$$|\sigma(a_3)a_4 + a_3\sigma(a_4)| \le N(a_4)T_{-1}(\theta).$$

We verify that P has only integer coefficients when $\beta_1 = 0$. We compute the relative discriminant d_P of the relative polynomial from the formula

$$\begin{aligned}27\mathrm{d}_P &= 4(a_2^2 - 3a_1a_3 + 12a_4)^3\\ &\quad - (2a_2^3 - 72a_2a_4 + 27a_1^2a_4 - 9a_1a_2a_3 + 27a_3^2)^2.\end{aligned}$$

We then make sure that the sign of $N(\mathrm{d}_P)$ coincides with that of $(-1)^\eta$. If $N(\mathrm{d}_P)$ is squarefree, then we must have $|N(\mathrm{d}_P)| \le \mathrm{M}/\mathrm{d}_{\mathfrak{L}}^4$. We notice that for $r = 6$, denoting by P the relative polynomial of positive relative discriminant, we have

$$\sum_{i=1}^{4}|1+\theta_i|^2 = 4 \pm 2s_1 + s_2.$$

The use of the inequality between arithmetic and geometric means yields

$$|P(\mp 1)| = [(4 \pm 2s_1 + s_2)/4]^2,$$

giving an important reduction of the number of polynomials to consider.

Some simplifications can also be made for $a_1 = 0$; for β_1 or for $a_1 = 0$ and $a_3 = 0$. To solve the question of signature of the field we have used Sturm's theorem only for the relative polynomials with positive discriminants. We have later computed the roots, on the one hand to estimate $\sum_{i=1}^{8}|\theta_i|^2$ (which we compare with the bound on $T_2(\theta)$ given by geometric methods, allowing the elimination of a great number of polynomials) and on the other hand to test the irreducibility of the polynomial $P(x)$ in $\mathbf{Z}_{\mathfrak{L}}[x]$. The test consists of determining whether there exist divisors of degree 1 or 2 of $P(x)$ in $\mathbf{Z}_{\mathfrak{L}}[x]$. For $r = 6$ (resp. 2) we have 8 (resp. 0) possibilities for the factors of degree 1 and 6 (resp. 2) possibilities for the factors of degree 2. For $r = 4$ we have 4 (resp. 2) tests if $\mathrm{d}_P < 0$ and no (resp. 6) tests if $\mathrm{d}_P > 0$ for the factors of degree 1 (resp. 2). If P is irreducible in $\mathbf{Z}_{\mathfrak{L}}[x]$, then the same is true for $f(x)$ in $\mathbf{Z}[x]$, and we have $K = \mathbf{Q}(\theta)$ and $[K\colon \mathfrak{L}] = 4$. We only need to compute the discriminant d_K to see if K lies within the limits of the search. When $N(\mathrm{d}_P)$ is squarefree, we immediately get $\mathrm{d}_K = \mathrm{d}_{\mathfrak{L}}^4 N(\mathrm{d}_P)$; otherwise we compute the discriminant of the field K by means of the Zassenhaus "ROUND 2" algorithm in a version due to D. Ford, implemented in GP PARI [2]. As we obtained, in general, more than one polynomial for a given discriminant, we have used the POLRED algorithm [4] to decide whether or not the polynomials define the same field up to isomorphism.

TABLE 2

d_K	$\mathfrak{d}_{\mathfrak{L}}$	a_1	a_2	a_3	a_4	d_P	$N(\mathrm{d}_P)$
$(r,s)=(2,3)$							
-4286875	5	$(-1,-1)$	$(-3,1)$	$(2,0)$	$(3,-1)$	$(-3097,1387)$	-6859
-4461875	5	$(-1,-1)$	$(2,0)$	$(3,-1)$	$(1,-1)$	$(343,-171)$	-7139
-4616192	8	$(-2,0)$	$(2,1)$	$(-2,0)$	$(2,0)$	$(-162,62)$	-1127
-4960000	5	$(-2,-2)$	$(4,2)$	$(-2,-2)$	$(1,1)$	$(-224,128)$	-7936
-5369375	5	$(2,0)$	$(-3,1)$	$(-1,1)$	$(3,-1)$	$(194,-120)$	-8591
-5756875	5	$(-1,-1)$	$(-1,1)$	$(0,0)$	$(1,-1)$	$(91,-95)$	-9211
-5781875	5	$(-1,-1)$	$(2,0)$	$(-1,-1)$	$(2,0)$	$(79,-93)$	-9251
-5856875	5	$(-1,-1)$	$(-3,1)$	$(-1,1)$	$(2,0)$	$(19,87)$	-9371
$(r,s)=(4,2)$							
15243125	5	$(-1,-1)$	$(-2,0)$	$(2,0)$	$(3,-1)$	$(1999,-883)$	24389
16643125	5	$(-1,-1)$	$(-1,1)$	$(-1,1)$	$(-2,0)$	$(-369,77)$	26629
17238125	5	$(-1,-1)$	$(-2,0)$	$(1,1)$	$(-1,1)$	$(-1393,605)$	27581
17318125	5	$(-2,-2)$	$(-2,0)$	$(5,3)$	$(-1,-1)$	$(7219,3225)$	27709
19268125	5	$(-4,0)$	$(-1,1)$	$(1,-1)$	$(3,-1)$	$(-1089,461)$	30829
19360000	5	$(0,0)$	$(-3,1)$	$(0,0)$	$(2,0)$	$(368,48)$	30976
20268125	5	$(-2,-2)$	$(-1,1)$	$(1,1)$	$(-1,-1)$	$(-3029,-1345)$	32429
20493125	5	$(-1,-1)$	$(1,1)$	$(1,-1)$	$(1,-1)$	$(-409,85)$	32789
20993125	5	$(0,0)$	$(-3,1)$	$(1,1)$	$(1,-1)$	$(-369,19)$	33589
21550625	5	$(-1,-1)$	$(0,0)$	$(-2,2)$	$(-3,1)$	$(-4862,2168)$	34481
22974464	8	$(-4,-1)$	$(4,1)$	$(-2,-1)$	$(2,1)$	$(-262,-76)$	5609
23040000	5	$(-4,0)$	$(2,0)$	(6.2)	$(1,1)$	$(-9024,-4032)$	36864
23040000	8	$(-2,-2)$	$(6,1)$	$(-2,2)$	$(2,0)$	$(-450,-150)$	5625
23643125	5	$(-3,-1)$	$(0,0)$	$(3,1)$	$(-3,-1)$	$(-3521,-1565)$	37829
$(r,s)=(6,1)$							
-68856875	5	$(-2,0)$	$(-5,-1)$	$(1,1)$	$(3,1)$	$(771,455)$	-110171
-73061875	5	$(-4,0)$	$(-3,-1)$	$(5,1)$	$(1,1)$	$(1747,839)$	-116899
-74671875	5	$(-3,-1)$	$(-1,1)$	$(-2,2)$	$(-3,1)$	$(-5085,2295)$	-119475
-74906875	5	$(-1,-1)$	$(-2,2)$	$(-2,0)$	$(1,-1)$	$(379,-353)$	-119851
-84356875	5	$(0,0)$	$(-3,-1)$	$(1,-1)$	$(-1,1)$	$(-961,541)$	-134971
-86606875	5	$(-1,-1)$	$(-1,-1)$	$(1,1)$	$(-4,2)$	$(-21469,9607)$	-138571

The number fields of degree 8, of signature $(2,3)$ (resp. $(4,2)$, $(6,1)$) and of discriminant $|\mathrm{d}_K|$ smaller than 6688609 (resp. 24363884, 92810082) containing a quadratic subfield found in this way are listed in Table 2.

CONCLUSION

Theorem 2. *There exist up to isomorphism exactly* 18 (*resp.* 21, 6) *non-primitive number fields of degree* 8 *and of signature* $(2,3)$ (*resp.* $(4,2)$, (6.1)) *and of discriminant smaller than* 6688609 (*resp.* 24363884, 92810082) *in absolute value. Except for the fields given in Proposition 1, all the other fields in the tables are characterized by their discriminant.*

Computation of Galois groups. To compute the Galois group of each polynomial represented in Table 3, we have used the method proposed in [1]. Through this method we can obtain the Galois group as a strong generating set, whose elements are permutations on all roots of the given polynomial. The notation for the group names is similar to that of Butler and McKay [3]. Groups preceded by "+" are groups of even permutations.

TABLE 3

		$(r,s)=(2,3)$			
$-4286875=-5^4\,19^3$	-475	$x^8-3x^7+4x^6-3x^5+3x^4-6x^3+6x^2-4x+1$	49	T_6	
$-4461875=-5^4\,11^2\,59$	-275	$x^8-x^7-2x^6+2x^4+2x^3-x^2-x-1$	1	T_{30}	*
$-4616192=-2^{12}\,7^2\,23$	-448	$x^8-2x^7-x^6+4x^5-2x^4-2x^3+2x^2-1$	1	T_{35}	
$-4711123=-43.331^2$	-331	$x^8-x^7-x^5+2x^3+x-1$	1	T_{23}	*
$-4725251=-59.283^2$	-283	$x^8-4x^6-2x^5+7x^4+5x^3-3x^2-4x-1$	1	T_{23}	
$-4960000=-2^8\,5^4\,31$	-400	$x^8-4x^7+7x^6-10x^5+8x^4-4x^3+2x-1$	1	T_{35}	
$-5149367=-47.331^2$	-331	$x^8+3x^7+6x^6+9x^5+9x^4+9x^3+6x^2+3x+1$	1	T_{23}	
$-5365963=-67.283^2$	-283	$x^8+3x^7+x^6-x^5+4x^4+4x^3+x^2+x-1$	1	T_{23}	*
$-5365963=-67.283^2$	-283	$x^8-4x^7+8x^6-13x^5+15x^4-13x^3+8x^2-4x+1$	1	T_{23}	*
$-5369375=-5^4\,11^2\,71$	-275	$x^8+2x^7-2x^6-4x^5+3x^4+2x^3-3x^2+x+1$	1	T_{30}	*
$-5756875=-5^4\,61.151$	5	$x^8-x^7-2x^6+3x^5-3x^3+2x^2-1$	1	T_{47}	*
$-5781875=-5^4\,11.29^2$	725	$x^8-x^7+x^6-2x^5+x^4-2x^3+x^2-x+1$	1	T_{35}	
$-5856875=-5^4\,9371$	5	$x^8-x^7-4x^6+3x^5+6x^4-2x^3-4x^2-x+1$	1	T_{47}	
$-6022411=-19.563^2$	-563	$x^8-x^7-5x^6+3x^5+9x^4-x^3-4x^2-2x-1$	121	T_{23}	
$-6464099=-59.331^2$	-331	$x^8-x^7-2x^6+4x^5-2x^4+3x^3-3x^2-1$	1	T_{23}	
$-6647387=-83.283^2$	-283	$x^8-4x^7+5x^6-4x^5+4x^4-x^3-x^2-1$	1	T_{23}	
$-6647387=-83.283^2$	-283	$x^8-2x^6+x^5+5x^4-7x^3+3x^2+x-1$	1	T_{23}	
$-6668032=-2^8\,7.61^2$	-976	$x^8-3x^6+2x^4+2x^3-2x-1$	1	T_{23}	
		$(r,s)=(4,2)$			
$15243125=5^4\,29^3$	725	$x^8+x^7-3x^6-3x^5+3x^4+6x^3-2x^2-3x+1$	7	T_{17}	
$15297613=37.643^2$	-643	$x^8-2x^7+x^4+5x^3-x^2-4x-1$	1	T_{23}	*
$16324589=149.331^2$	-331	$x^8-2x^7-3x^6+5x^5+5x^4-6x^3-x^2+3x-1$	1	T_{23}	
$16643125=5^4\,26629$	5	$x^8-x^7-2x^6+2x^5-x^3+x+1$	1	T_{47}	
$17238125=5^4\,27581$	5	$x^8-x^7-3x^6+2x^5+2x^4+2x^3-3x-1$	1	T_{47}	
$17318125=5^4\,11^2\,229$	-275	$x^8-x^7-3x^6+5x^5+3x^4-11x^3+8x-1$	23	T_{35}	
$18340381=229.283^2$	-283	$x^8-4x^6+x^5+5x^4-x^3-4x^2+1$	1	T_{23}	
$18660737=233.283^2$	-283	$x^8-3x^6-4x^5+5x^4+4x^3-3x^2+1$	16	T_{23}	
$19268125=5^4\,30829$	5	$x^8-4x^7+3x^6+3x^5-4x^3-x+1$	1	T_{47}	
$19360000=2^8\,5^4\,11^2$	4400	$x^8-3x^6+3x^4-3x^2+1$	1	$+T_{18}$	
$20262517=11.23.283^2$	-283	$x^8-4x^7+5x^6-4x^5+5x^4+x^3-7x^2+x+1$	9	T_{23}	
$20268125=5^4\,32429$	5	$x^8-2x^7-5x^6+7x^5+2x^4-7x^3+2x^2+2x-1$	1	T_{47}	
$20493125=5^4\,32789$	5	$x^8-x^7+3x^5-3x^4+2x^2-2x-1$	1	T_{47}	
$20502784=2^8\,283^2$	-283	$x^8-x^6+4x^4-4x^2+1$	1	$+T_{39}$	
$20993125=5^4\,33589$	5	$x^8-3x^6+x^5+2x^4-4x^3+3x-1$	1	T_{47}	
$21543941=269.283^2$	-283	$x^8+x^7-6x^6-5x^5+10x^4+10x^3-4x^2-5x-1$	1	T_{23}	
$21550625=5^4\,29^2\,41$	725	$x^8-x^7-x^6-2x^5+3x^4+4x^3-4x^2-2x+1$	1	T_{35}	
$22974464=2^{12}\,71.79$	8	$x^8-4x^7+6x^6-6x^5+4x^4-x^2+2x-1$	1	T_{47}	
$23040000=2^{12}\,5^4\,3^2$	1600	$x^8-2x^7-x^6+4x^5+5x^4-8x^3-5x^2+6x-1$	89	$+T_{11}$	
$23643125=5^4\,11.19.181$	5	$x^8-3x^7+x^6+3x^5-5x^4+2x^3+x^2-2x+1$	1	T_{47}	
$24212981=13.17.331^2$	-331	$x^8-x^7+2x^6+x^5-7x^4+5x^3+3x^2-4x+1$	1	T_{23}	
		$(r,s)=(6,1)$			
$-68856875=5^4\,29^2\,131$	725	$x^8-2x^7-4x^6+6x^5+7x^4-3x^3-6x^2-x+1$	1	T_{35}	
$-73061875=5^4\,29^2\,139$	725	$x^8+4x^7+x^6-11x^5-8x^4+7x^3+6x^2-1$	1	T_{35}	
$-74671875=3^4\,5^6\,59$	1125	$x^8-x^7-6x^6+3x^5+9x^4-4x^3-4x^2+2x+1$	1	T_{27}	
$-74906875=5^4\,119851$	5	$x^8-x^7-3x^6+4x^5-2x^4-x^3+5x^2-x-1$	1	T_{47}	
$-84356875=5^4\,71.1901$	5	$x^8-3x^6+x^5-4x^3+3x^2+2x-1$	1	T_{47}	
$-86606875=5^4\,138571$	5	$x^8-x^7-3x^6+2x^5-x^4+6x^3+3x^2-7x-1$	1	T_{47}	

We finally notice that the norm of the relative index is equal to 1 for all the fields given in Tables 1 and 2. The final results are given in Table 3, where we find (from left to right): the field discriminant and its decomposition; the discriminant of the fixed subfield; the polynomial defining the extension $K/\mathbf{Q}$, corresponding to the relative polynomial given in the previous tables; its index; and the Galois group of the Galois closure of the extension $K/\mathbf{Q}$.

An asterisk $(*)$ denotes the Euclidean fields discovered by A. Leutbecher [11].

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