

THE DISCRETE PLATEAU PROBLEM: CONVERGENCE RESULTS

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ABSTRACT. We solve the problem of finding and justifying an optimal fully discrete finite element procedure for approximating minimal, including unstable, surfaces. In a previous paper we introduced the general framework and some preliminary estimates, developed the algorithm and give the numerical results. In this paper we prove the convergence estimate.

1. INTRODUCTION

We recall from [DH4] that a *minimal surface* or *solution of the Plateau Problem* can be characterised in a number of different ways. For our purposes it is convenient to begin with the following formulation, which we restate more precisely later, cf. (3).

Let D be the unit disc in \mathbb{R}^2 and Γ a smooth Jordan curve in \mathbb{R}^n . Let \mathcal{F} be the class of *harmonic* maps $u: \overline{D} \rightarrow \mathbb{R}^n$ such that $u|_{\partial D}: \partial D \rightarrow \Gamma$ is monotone and satisfies a certain integral “three-point condition”, cf. (1). The function $u \in \mathcal{F}$ is said to be a *minimal surface* if u is stationary in \mathcal{F} for the Dirichlet energy $\mathcal{D}(u) = \frac{1}{2} \int_D |\nabla u|^2$. Such maps u provide an harmonic conformal parametrisation of the corresponding minimal surface.

Let D_h be a quasi-uniform triangulation of D with grid size controlled by h . Let \mathcal{F}_h be the class of *discrete harmonic* maps $u_h: \overline{D}_h \rightarrow \mathbb{R}^n$ for which $u_h(\phi_j) \in \Gamma$ whenever ϕ_j is a boundary node of D_h , and which satisfy an analogue of the previous integral “three-point condition”. Note that we do not require “monotonicity” of $u_h|_{\partial D_h}$. The function $u_h \in \mathcal{F}_h$ is said to be a *discrete minimal surface* if u_h is stationary within \mathcal{F}_h for the Dirichlet energy $\mathcal{D}(u_h) = \frac{1}{2} \int_{D_h} |\nabla u_h|^2$; cf. (27) for the precise formulation.

The main result (Theorem 5.5) is that if u is a nondegenerate minimal surface spanning Γ then there exist discrete minimal surfaces u_h , unique in a ball of “almost” constant radius $\epsilon_0 |\log h|^{-1}$, such that $\|u - u_h\|_{H^1(D_h)} \leq ch$, where c depends on Γ and the nondegeneracy constant λ for u but is independent of h . Recall that nondegeneracy corresponds to the absence of zero eigenvalues for the second variation of the Dirichlet energy at u , and generically corresponds to the absence of branch points. The constant c blows up as $\lambda \rightarrow 0$, and this is consistent with the numerical results in [DH4]. We give an outline of the proof of the main result in Section 2.3.

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2. FORMULATION OF THE PROBLEM

2.1. The energy functional. We first recall some notation from [DH4] Section 3, to which we refer for further discussion. See Section 3 of the present paper for the properties of $H^{1/2}$.

Let D be the open unit disc in \mathbb{R}^2 , with boundary ∂D . It will be convenient to let S^1 denote another, distinct, copy of the unit circle. Let Γ be a Jordan curve in \mathbb{R}^n with regular C^r -parametrisation $\gamma: S^1 \rightarrow \Gamma$ where $r \geq 3$.

For $f: \partial D \rightarrow \mathbb{R}^n$ we denote by $\Phi(f): \bar{D} \rightarrow \mathbb{R}^n$ its unique harmonic extension to D specified by

$$\Delta \Phi(f) = 0 \quad \text{in } D, \quad \Phi(f) = f \quad \text{on } \partial D.$$

Then $\Phi: H^{1/2}(\partial D, \mathbb{R}^n) \rightarrow H^1(D, \mathbb{R}^n)$ is a bounded linear map with bounded inverse.

Harmonic maps are uniquely determined by the associated boundary maps. We will use the Hilbert space H of functions defined by

$$H = \{ \xi: \partial D \rightarrow \mathbb{R} \mid |\xi|_{H^{1/2}} < \infty \text{ and (1) is satisfied} \},$$

where

$$(1) \quad \int_0^{2\pi} \xi(\phi) d\phi = 0, \quad \int_0^{2\pi} \xi(\phi) \cos \phi d\phi = 0, \quad \int_0^{2\pi} \xi(\phi) \sin \phi d\phi = 0.$$

The norm on H is the usual norm $\|\cdot\|_{H^{1/2}}$. The corresponding affine space of maps $s: \partial D \rightarrow S^1$ such that $s(\phi) = \phi + \sigma(\phi)$ for some $\sigma \in H$ is denoted by \mathcal{H} . We also need the Banach space T defined by $T = H \cap C^0(\partial D, \mathbb{R})$ with norm $\|\xi\|_T = \|\xi\|_{H^{1/2}} + \|\xi\|_{C^0}$. The corresponding affine space \mathcal{T} is defined by $\mathcal{T} = \mathcal{H} \cap C^0(\partial D, S^1)$. With some abuse of standard notation, we write $\|s\| = 1 + \|\sigma\|$ for various norms on σ .

The energy functional E is defined on \mathcal{H} by

$$(2) \quad E(s) = \frac{1}{2} \int_D |\nabla \Phi(\gamma \circ s)|^2 = \mathcal{D}(\Phi(\gamma \circ s)).$$

Finiteness of E follows from (8). We say the harmonic function $u = \Phi(\gamma \circ s)$ is a *minimal surface spanning* Γ , or a *solution of the Plateau Problem for* Γ , if and only if s is monotone and stationary for E , i.e.

$$(3) \quad \langle E'(s), \xi \rangle = 0 \quad \forall \xi \in T.$$

As discussed in [DH4], this is equivalent to other formulations of the notion of a minimal surface.

For the proof of asymptotic convergence of the numerical method we need the following regularity result, which follows from standard regularity results (cf. [DH4] Theorem 3.2) and the Implicit Function Theorem used to write Γ locally as the graph of a $C^{k,\alpha}$ function.

Proposition 2.1. *If $\gamma \in C^{k,\alpha}$ where $k \geq 1$ and $0 < \alpha < 1$, and $s \in \mathcal{T}$ is monotone and stationary for E , then*

$$\|s\|_{C^{k,\alpha}} \leq c = c(\|\gamma\|_{C^{k,\alpha}}, \|\gamma'\|^{-1}\|_{L^\infty}).$$

We next recall some properties of the energy functional from [DH4] Section 3.3. Using the notation

$$(4) \quad u = \Phi(\gamma \circ s), \quad v = \Phi(\gamma' \circ s \xi), \quad w = \Phi(\gamma'' \circ s \xi^2),$$

we get by formal computation

$$(5) \quad E(s) = \frac{1}{2} \int_D |\nabla u|^2,$$

$$(6) \quad \langle E'(s), \xi \rangle = \left. \frac{d}{dt} \right|_{t=0} E(s + t\xi) = \int_D \nabla u \nabla v,$$

$$(7) \quad E''(s)(\xi, \xi) = \left. \frac{d^2}{dt^2} \right|_{t=0} E(s + t\xi) = \int_D \nabla u \nabla w + \int_D |\nabla v|^2,$$

with an analogous expression for $E''(s)(\xi, \eta)$ obtained by bilinearity in the case of distinct variations.

If $\gamma \in C^3$ then $E \in C^2(\mathcal{T}, \mathbb{R})$, and the Fréchet derivatives are given by (6) and (7). The functional E is *not* differentiable on \mathcal{H} , but if γ and s are as smooth as is necessary for the following estimates, then one has

$$(8) \quad E(s) \leq c \|\gamma\|_{C^1}^2 \|s\|_{H^{1/2}}^2,$$

$$(9) \quad |\langle E'(s), \xi \rangle| \leq c \|\gamma\|_{C^2}^2 \|s\|_{C^1}^2 \|\xi\|_{H^{1/2}},$$

$$(10) \quad |E''(s)(\xi, \eta)| \leq c \|\gamma\|_{C^2}^2 \|s\|_{C^1}^2 \|\xi\|_{H^{1/2}} \|\eta\|_{H^{1/2}}.$$

In particular, these will be used in case s is stationary for E .

It will be important to consider the behaviour of the second derivatives of E near a stationary point $s \in \mathcal{T}$. The second derivative $E''(s)$ can then be interpreted as a self-adjoint bounded map $\nabla^2 E(s): H \rightarrow H$. Let

$$(11) \quad H = H^- \oplus H^0 \oplus H^+, \quad \xi = \xi^- + \xi^0 + \xi^+ \quad \text{if } \xi \in H,$$

be the orthogonal decomposition generated by the eigenfunctions of $\nabla^2 E(s)$ having negative, zero and positive eigenvalues respectively.

Proposition 2.2. *Suppose $\gamma \in C^{3,\alpha}$. Suppose s is monotone and stationary for E . Then H^- and H^0 are finite dimensional. If $\xi \in H^- \oplus H^0$ then $\xi \in H^{3/2}(\partial D)$ and*

$$(12) \quad \|\xi\|_{H^{3/2}} \leq \nu \|\xi\|_{H^{1/2}}$$

with $\nu = \nu(\|\gamma\|_{C^{3,\alpha}}, \|\gamma'\|^{-1}\|_{L^\infty})$.

Proof. The finite dimensionality of H^- and H^0 follow from elliptic theory; see the proof of [St, Proposition II.5.6]. It is also shown there, with a different notation, that

$$\begin{aligned} \|\Phi(\gamma' \circ s \xi)\|_{H^2(D)} &\leq c \left(\|\gamma\|_{C^3}, \|\gamma'\|^{-1}\|_{L^\infty}, \|s\|_{C^3}, \|\xi\|_{H^{1/2}} \right) \\ &\leq c \left(\|\gamma\|_{C^{3,\alpha}}, \|\gamma'\|^{-1}\|_{L^\infty}, \|\xi\|_{H^{1/2}} \right) \end{aligned}$$

by Proposition 2.1. But

$$\begin{aligned} \|\xi\|_{H^{3/2}} &\leq \left\| \frac{\gamma' \circ s}{|\gamma' \circ s|^2} \right\|_{C^2} \|\gamma' \circ s \xi\|_{H^{3/2}} \quad \text{by (52)} \\ &\leq c \left(\|\gamma\|_{C^3}, \|\gamma'\|^{-1}\|_{L^\infty} \right) \|\Phi(\gamma' \circ s \xi)\|_{H^2(D)} \quad \text{from (52) and (40)}. \end{aligned}$$

Hence

$$\|\xi\|_{H^{3/2}} \leq c \left(\|\gamma\|_{C^{3,\alpha}}, \|\gamma'\|^{-1}\|_{L^\infty}, \|\xi\|_{H^{1/2}} \right),$$

and so the required result follows by scaling. □

Higher regularity on γ implies higher regularity on ξ . In particular, $\gamma \in C^{4,\alpha}$ implies $\xi \in H^{5/2}(\partial D)$; see the proof of [St, Proposition II.5.6].

If s is monotone and stationary for E , we say s is *nondegenerate* if $H^0 = \{0\}$. The corresponding minimal surface $u = \Phi(\gamma \circ s)$ is also said to be *nondegenerate*. If s is nondegenerate it follows that there exists a $\lambda > 0$ such that for $\xi \in H$,

$$(13) \quad E''(s)(\xi, \xi^+ - \xi^-) = E''(s)(\xi^+, \xi^+) - E''(s)(\xi^-, \xi^-) \geq \lambda \|\xi\|_{H^{1/2}}^2.$$

We call λ the *nondegeneracy constant* for s .

2.2. The discrete energy functional. We recall the necessary notation from [DH4] Section 4 and prove some preliminary estimates.

Let \mathcal{G}_h be a quasi-uniform triangulation of D with grid size comparable to h . Let

$$D_h = \bigcup \{G \mid G \in \mathcal{G}_h\},$$

$$\partial D_h = \bigcup \{E_j \mid 1 \leq j \leq M\} \quad \text{where the } E_j \text{ are the boundary edges,}$$

and let

$$\mathcal{B}_h = \{\phi_1, \dots, \phi_M\} \quad \text{be the set of boundary nodes.}$$

The projection $\pi: \partial D \rightarrow \partial D_h$ is defined by

$$(14) \quad \pi \left(e^{i((1-t)\phi_j + t\phi_{j+1})} \right) = (1-t)e^{i\phi_j} + te^{i\phi_{j+1}}$$

for $0 \leq t \leq 1, 1 \leq j \leq M$.

In order to have a discrete analogue E_h of the functional E we define the following discrete analogues of $H^1(D; \mathbb{R}^n), H^{1/2}(\partial D; \mathbb{R}^n), H, T, \mathcal{H}$ and \mathcal{T} :

- (15) $X_h^n = \{u_h \in C^0(D_h; \mathbb{R}^n) \mid u_h \in P_1(G) \text{ for } G \in \mathcal{G}_h\},$
- (16) $x_h^n = \{f_h \in C^0(\partial D_h; \mathbb{R}^n) \mid f_h \in P_1(E_j) \text{ for } 1 \leq j \leq M\},$
- (17) $H_h = \{\xi_h \in C^0(\partial D; \mathbb{R}) \mid \xi_h \in P_1(\pi^{-1}(E_j)) \text{ if } 1 \leq j \leq M, \xi_h \text{ satisfies (1)}\},$
- (18) $\mathcal{H}_h = \{s_h \in C^0(\partial D; S^1) \mid s_h(\phi) = \phi + \sigma_h(\phi) \text{ for some } \sigma_h \in H_h\}.$

Thus $H_h \subset T \subset H, \mathcal{H}_h \subset \mathcal{T} \subset \mathcal{H}$, and the space of variations at any $s_h \in \mathcal{H}_h$ is naturally identified with H_h . Setting $n = 1$, we similarly define X_h and x_h .

We will make frequent use of the following inverse-type estimates:

Proposition 2.3. *If $\xi_h \in H_h$ then*

$$(19) \quad \|\xi_h\|_{H^1} \leq ch^{-1/2} \|\xi_h\|_{H^{1/2}},$$

$$(20) \quad \|\xi_h\|_{C^0} \leq c|\log h|^{1/2} \|\xi_h\|_{H^{1/2}}.$$

Proof. The first estimate is standard. The second is shown in [DH1] Proposition 5.3. □

Suppose $f \in C^0(\partial D; \mathbb{R}^n)$. One defines the “linear interpolants”

$$I_h f \in x_h^n, \quad I_h f \left((1-t)e^{i\phi_j} + te^{i\phi_{j+1}} \right) = (1-t)f(e^{i\phi_j}) + tf(e^{i\phi_{j+1}}),$$

$$I_h^{\partial D} f \in C^0(\partial D; \mathbb{R}^n), \quad I_h^{\partial D} f \left((1-t)e^{i\phi_j} + te^{i\phi_{j+1}} \right) = (1-t)f(e^{i\phi_j}) + tf(e^{i\phi_{j+1}}),$$

where $0 \leq t \leq 1$, $1 \leq j \leq M$. Here and elsewhere, $\phi_{M+1} = \phi_1$. Note the different domains of $I_h f$ and $I_h^{\partial D} f$. Note also that the image of $I_h(\gamma \circ s)$ is a polygonal approximation to Γ , and $I_h(\gamma \circ s)(\phi_j) = \gamma \circ s(\phi_j) \in \Gamma$ for $\phi_j \in \mathcal{B}_h$. Finally,

$$(21) \quad I_h^{\partial D} f = I_h f \circ \pi.$$

Another type of approximation operator we require is a map $p_h : T(\mathcal{T}) \rightarrow H_h(\mathcal{H}_h)$. The usual interpolation operator does not preserve the normalisation conditions (1). However, if we first interpolate and then project onto H_h , the resulting operator still satisfies all the usual estimates. The proof of the following is essentially given in [DH1] Proposition 5.2.

Proposition 2.4. *There is a bounded linear operator $p_h : T \rightarrow H_h$ such that, in particular,*

$$(22) \quad \|\xi - p_h \xi\|_{H^s} \leq ch^{k-s} \|\xi\|_{H^k}$$

for $s = 0, \frac{1}{2}, 1$ and $k = 1, \frac{3}{2}, 2$. Moreover,

$$(23) \quad \|\xi - p_h \xi\|_{C^{0,1}} \leq ch \|\xi\|_{C^2}, \quad \|p_h \xi\|_{C^{0,1}} \leq c \|\xi\|_{C^{0,1}},$$

$$(24) \quad \|\xi - p_h \xi\|_{C^0} \leq ch^2 \|\xi\|_{C^2}, \quad \|\xi - p_h \xi\|_{C^0} \leq ch \|\xi\|_{C^1}.$$

If $s \in \mathcal{T}$ and $s(\phi) = \phi + \sigma(\phi)$, then $p_h s$ is defined by $p_h s(\phi) = \phi + p_h \sigma(\phi)$, $s - p_h s := \sigma - p_h \sigma$, and hence $p_h s$ satisfies estimates similar to those for $p_h \xi$.

For $f_h \in x_h$ the discrete harmonic extension $\Phi_h f_h \in X_h$ is defined by

$$(25) \quad \Delta_h \Phi_h f_h = 0 \quad \text{in } D_h, \quad \Phi_h f_h = f_h \quad \text{on } \partial D_h,$$

where Δ_h is the discrete Laplacian and the first equation in (25) is interpreted as $\int_{D_h} \nabla(\Phi_h f_h) \nabla \psi_h = 0$ for all $\psi_h \in X_h$ such that $\psi_h = 0$ on ∂D_h . If $f_h \in x_h^n$ the discrete harmonic extension $\Phi_h f_h$ is defined componentwise.

For $s_h \in \mathcal{H}_h$, the discrete energy functional E_h is defined by

$$(26) \quad E_h(s_h) = \frac{1}{2} \int_{D_h} |\nabla \Phi_h I_h(\gamma \circ s_h)|^2 = \mathcal{D}_h(\Phi_h I_h(\gamma \circ s_h)).$$

Note that E_h is of course *not* the restriction of E to \mathcal{H}_h . The discrete harmonic function $u_h = \Phi_h I_h(\gamma \circ s_h)$ is said to be a *discrete minimal surface spanning Γ* , or a *solution of the discrete Plateau Problem for Γ* , if and only if

$$(27) \quad \langle E'_h(s_h), \xi_h \rangle = 0 \quad \forall \xi_h \in H_h.$$

Note that we do not require monotonicity of s_h , as is the case for s in (3).

The derivatives of E_h , cf. (5)–(7), are given by

$$(28) \quad E_h(s_h) = \frac{1}{2} \int_{D_h} |\nabla u_h|^2,$$

$$(29) \quad \langle E'_h(s_h), \xi_h \rangle = \int_{D_h} \nabla u_h \nabla v_h,$$

$$(30) \quad E''_h(s_h)(\xi_h, \xi_h) = \int_{D_h} \nabla u_h \nabla w_h + \int_{D_h} |\nabla v_h|^2,$$

where

$$(31) \quad u_h = \Phi_h I_h(\gamma \circ s_h), \quad v_h = \Phi_h I_h(\gamma' \circ s_h \xi_h), \quad w_h = \Phi_h I_h(\gamma'' \circ s_h \xi_h^2).$$

For $\xi_h \in H_h$ and s stationary for E , we define the projection of the decomposition (11):

$$(32) \quad \xi_h^{(-)} = p_h \xi_h^-, \quad \xi_h^{(0)} = p_h \xi_h^0, \quad \xi_h^{(+)} = p_h \xi_h^+.$$

Note that ξ_h^- , ξ_h^0 and ξ_h^+ do not normally belong to H_h ; in particular, the first two are smooth functions. However, if $H^- = H^0 = \{0\}$ then $\xi_h = \xi_h^+ = \xi_h^{(+)}$.

Proposition 2.5. *If $\xi_h \in H_h$ then*

$$(33) \quad \xi_h = \xi_h^{(-)} + \xi_h^{(0)} + \xi_h^{(+)}$$

and $\xi_h^{(-)}, \xi_h^{(0)}, \xi_h^{(+)}$ $\in H_h$. Moreover, with ν as in (12),

$$(34) \quad \|\xi_h^- - \xi_h^{(-)}\|_{H^{1/2}} \leq ch\nu\|\xi_h\|_{H^{1/2}}, \quad \|\xi_h^- - \xi_h^{(-)}\|_{H^1} \leq ch^{1/2}\nu\|\xi_h\|_{H^{1/2}},$$

$$(35) \quad \|\xi_h^0 - \xi_h^{(0)}\|_{H^{1/2}} \leq ch\nu\|\xi_h\|_{H^{1/2}}, \quad \|\xi_h^0 - \xi_h^{(0)}\|_{H^1} \leq ch^{1/2}\nu\|\xi_h\|_{H^{1/2}},$$

$$(36) \quad \|\xi_h^+ - \xi_h^{(+)}\|_{H^{1/2}} \leq ch\nu\|\xi_h\|_{H^{1/2}}, \quad \|\xi_h^+ - \xi_h^{(+)}\|_{H^1} \leq ch^{1/2}\nu\|\xi_h\|_{H^{1/2}}.$$

Proof. Since $p_h \xi_h = \xi_h$, (33) follows from (11). Next, from (22) and (12),

$$\|\xi_h^- - \xi_h^{(-)}\|_{H^{1/2}} = \|\xi_h^- - p_h \xi_h^-\|_{H^{1/2}} \leq ch\|\xi_h^-\|_{H^{3/2}} \leq ch\nu\|\xi_h^-\|_{H^{1/2}}.$$

The proof of the other inequality in (34), and of (35), is similar.

For (36) just note that, from (33) and (11),

$$\xi_h^+ - \xi_h^{(+)} = -(\xi_h^- - \xi_h^{(-)}) - (\xi_h^0 - \xi_h^{(0)}),$$

and use the previous estimates. □

We remark that the decomposition (33), unlike (11), is *not* an orthogonal decomposition. We also note that the powers of h in (34)–(36) can be increased by 1/2 if we assume $\|\xi\|_{H^2} \leq \nu\|\xi\|_{H^{1/2}}$; this latter holds if $\gamma \in C^{4,\alpha}$, cf. the remarks after Proposition 2.2.

2.3. Structure of the proof. The main error estimates, Theorems 5.4 and 5.5, are proved in Section 5. To motivate our approach and the need for the various preliminary estimates in Sections 4 and 5, we outline in an informal manner the structure of the proof of Theorem 5.4. See Figure 1.

Assume s is a nondegenerate stationary point for E with nondegeneracy constant λ . One applies the Inverse Function Theorem to the derivative $E'_h : \mathcal{H}_h \rightarrow H'_h$ (the dual space of H_h) in a neighbourhood of the point $p_h s$; remember that H_h is the tangent space for \mathcal{H}_h , i.e. the space of variations at any $t_h \in \mathcal{H}_h$. Identify E''_h with the derivative $(E'_h)'$ of E'_h ; thus $(E'_h)'(t_h) : H_h \rightarrow H'_h$ for $t_h \in \mathcal{H}_h$.

There are three estimates to be proved.

A: $\|E'_h(p_h s)\| \leq c_1 h$ for h sufficiently small. This follows immediately from the consistency estimate (Proposition 4.2) for the first derivative. See (109).

B₁: $\|((E'_h)'(p_h s), \xi_h)\| \geq \frac{\lambda}{2}\|\xi_h\|$ for h sufficiently small; i.e. “the slope of E'_h at $p_h s$ is bounded away from 0 by $\lambda/2$ ”. This is a nondegeneracy estimate on $E''_h(p_h s)$ and is established in Proposition 5.3, using the one-sided consistency estimate (Proposition 4.3) for the second derivative, and the regularity of members of the negative eigenspaces to compensate for the one-sided nature of that estimate. See (114).

B₂: $\|(E'_h)'(p_h s) - (E'_h)'(p_h s + \eta_h)\| \leq \lambda/4$ for h sufficiently small and η_h sufficiently close to $p_h s$, more precisely for $\eta_h \leq \epsilon_0/|\log h|$; i.e. “the slope of E'_h near $p_h s$

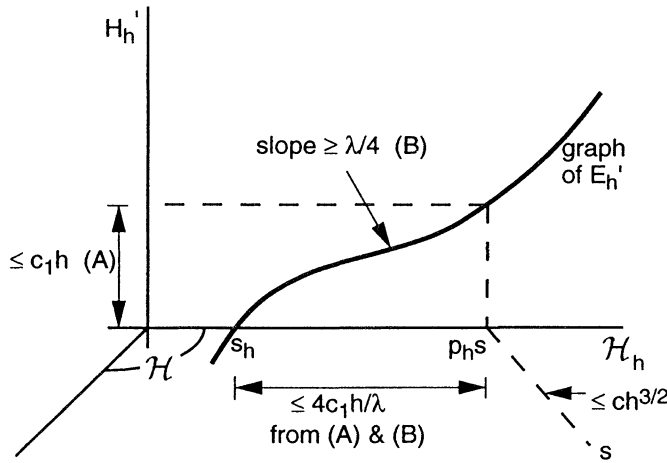


FIGURE 1

differs from that at $p_h s$ by at most $\lambda/4$. This is a consequence of Proposition 5.2. See (115). As indicated in the diagram, it follows that there exists a unique stationary point s_h for E_h such that $\|p_h s - s_h\|_{H^{1/2}} \leq 4c_1 h/\lambda$, and hence such that $\|s - s_h\|_{H^{1/2}} \leq ch/\lambda$.

3. PRELIMINARY ESTIMATES

One can define the $H^s(\partial D)$ and $H^s(D)$ norms for any real s ; but apart from non-negative integers s we will only need the following cases.

For $f: \partial D \rightarrow \mathfrak{R}$ the $H^{1/2}(\partial D)$ seminorm is defined by

$$(37) \quad |f|_{H^{1/2}(\partial D)}^2 = \int_{\partial D} \int_{\partial D} \frac{|f(\phi) - f(\bar{\phi})|^2}{|\phi - \bar{\phi}|^2} d\phi d\bar{\phi},$$

and for $u: D \rightarrow \mathfrak{R}$ the $H^{1/2}(D)$ seminorm is defined by

$$(38) \quad |u|_{H^{1/2}(D)}^2 = \int_D \int_D \frac{|u(x) - u(\bar{x})|^2}{|x - \bar{x}|^3} dx d\bar{x}.$$

In both cases the corresponding norm is given by

$$(39) \quad \|\cdot\|_{H^{1/2}}^2 = \|\cdot\|_{L^2}^2 + |\cdot|_{H^{1/2}}^2.$$

Also,

$$|f|_{H^{3/2}(\partial D)} = |f'|_{H^{1/2}(\partial D)}, \quad |u|_{H^{3/2}(D)} = |\nabla u|_{H^{1/2}(D)},$$

$$\|\cdot\|_{H^{3/2}}^2 = \|\cdot\|_{L^2}^2 + |\cdot|_{H^{3/2}}^2.$$

If $u \in H^{s+1/2}(D)$ for $s = 1/2, 1, 3/2$ (in fact for $s > 0$), then u has a well-defined trace f on ∂D and

$$(40) \quad \|f\|_{H^s(\partial D)} \leq c \|u\|_{H^{s+1/2}(D)}.$$

More precisely, the previous estimate is true for $u \in C^\infty(\bar{D})$, and the definition of trace is extended by continuity and density to the general case. Such an estimate is not true for smooth u if $s = 0$, but in this case if $\Delta u \in L^2(D)$ the following

estimate holds, and then the trace operator is again defined by continuity and density to satisfy

$$(41) \quad \|f\|_{L^2(\partial D)} \leq c (\|u\|_{H^{1/2}(D)} + \|\Delta u\|_{L^2(D)}).$$

See [LM], Chapter 1, Section 9.2, page 193 (remarks before Theorem 8.1) and page 187, Theorem 7.3. *Note* that by considering kernels it is clear that in the case $s = 1/2$ one can replace the norms in (40) by the corresponding seminorms. We will frequently use this fact without further remark.

Conversely, if $f \in H^s(\partial D)$ for $s = 0, 1/2, 1, 3/2$ (with similar results for any real s), then there is a unique harmonic function $\Phi(f)$ defined on D with trace f as before, and in particular

$$(42) \quad \|\Phi(f)\|_{H^{s+1/2}(D)} \leq c \|f\|_{H^s(\partial D)}.$$

See [LM], Theorem 7.4, page 188. *Note* that for $s = 1/2, 1, 3/2$ the norms can be replaced by seminorms, as follows again by considering kernels; we will use this fact without further comment.

In the case of L^2 boundary data one also has, with $u = \Phi(f)$,

$$(43) \quad \lim_{r \rightarrow 1^-} u(r\theta) = f(\theta) \quad \text{for a.e. } \theta \in \partial D,$$

$$(44) \quad \left\| \sup_{0 \leq r < 1} |u(r \cdot)| \right\|_{L^2(\partial D)} \leq c \|f\|_{L^2(\partial D)}.$$

See [JK], Theorem (1.16), page 11. In particular, $u(r \cdot) \rightarrow f$ in $L^2(\partial D)$ as $r \rightarrow 1^-$. (The formulae (43) and (44) are established using the Poisson integral representation of u from f . The function u agrees with $\Phi(f)$ in (42), since it agrees for f in the dense subset $C^\infty(\partial D) \subset L^2(\partial D)$ by the maximum principle, and since the Poisson integral map and Φ are both bounded as maps from $L^2(\partial D)$ into $L^2(D)$, for example.)

If $f: \partial D \rightarrow \mathfrak{R}$ has the Fourier series expansion

$$f(\phi) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi),$$

then one can define

$$(45) \quad \|f\|_{L^2(\partial D)}^2 = a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2),$$

$$(46) \quad |f|_{H^s(\partial D)}^2 = \sum_{n=1}^{\infty} n^{2s} (a_n^2 + b_n^2) \quad s > 0,$$

$$(47) \quad \|f\|_{H^s(\partial D)}^2 = \|f\|_{L^2(\partial D)}^2 + |f|_{H^s(\partial D)}^2.$$

These norms and seminorms are equivalent to the usual definitions in case $s \in \mathbb{N}$, and the previous definitions in case $s = 1/2$ or $3/2$. The harmonic extension of f is given by

$$(\Phi f)(r, \phi) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\phi + b_n \sin n\phi).$$

Proposition 3.1. *Suppose $f, g: \partial D \rightarrow \mathfrak{R}$. Then*

$$(48) \quad |fg|_{H^{1/2}} \leq \|f\|_{C^0} |g|_{H^{1/2}} + |f|_{H^{1/2}} \|g\|_{C^0},$$

$$(49) \quad \|fg\|_{H^{1/2}} \leq c \|f\|_{C^{0,1}} \|g\|_{H^{1/2}},$$

$$(50) \quad |fg|_{H^1} \leq \|f\|_{C^0} |g|_{H^1} + |f|_{H^1} \|g\|_{C^0},$$

$$(51) \quad \|fg\|_{H^1} \leq c \|f\|_{C^{0,1}} \|g\|_{H^1},$$

$$(52) \quad \|fg\|_{H^{3/2}} \leq c \|f\|_{C^2} \|g\|_{H^{3/2}}.$$

Proof. The first two inequalities follow from (37). The next two inequalities are standard. The last inequality follows from (49). \square

The next proposition will be applied in case g is (a component of) γ , γ' or γ'' , and s is either smooth or piecewise linear and continuous. In particular, g will be at least C^1 but s may be only $C^{0,1}$ in some cases.

Proposition 3.2. *Suppose $s = id + \sigma: \partial D \rightarrow S^1$, $g: S^1 \rightarrow \mathfrak{R}$. Then*

$$(53) \quad \|g \circ s\|_{C^0} \leq \|g\|_{C^0} \leq \|g\|_{C^0} \|s\|_{C^0},$$

$$(54) \quad \|g \circ s\|_{C^{0,1}} \leq c \|g\|_{C^1} \|s\|_{C^{0,1}},$$

$$(55) \quad \|g \circ s\|_{C^1} \leq c \|g\|_{C^1} \|s\|_{C^1},$$

$$(56) \quad \|g \circ s\|_{C^2} \leq c \|g\|_{C^2} \|s\|_{C^2}^2,$$

$$(57) \quad \|g \circ s\|_{L^2} \leq c \|g\|_{C^0} \leq c \|g\|_{C^0} \|s\|_{L^2},$$

$$(58) \quad \|g \circ s\|_{H^{1/2}} \leq c \|g\|_{C^1} \|s\|_{H^{1/2}},$$

$$(59) \quad \|g \circ s\|_{H^1} \leq c \|g\|_{C^1} \|s\|_{H^1}.$$

Proof. Recall that $\|s\|_{C^0} = 1 + \|\sigma\|_{C^0} \geq 1$. Then (53) is immediate, and similarly for (57). Also (54) is immediate. For (55), (56) and (59), note that

$$(g \circ s)' = g' \circ s \, s', \quad (g \circ s)'' = g'' \circ s \, (s')^2 + g' \circ s \, s''.$$

Inequality (58) follows from (37). \square

The following proposition will typically be applied in case g is γ , γ' or γ'' and in particular is C^1 ; and either $s_1 = s_0$ and $s_2 = p_h s_0$, or $s_1 = p_h s_0$ and $s_2 = p_h s_0 + \eta_h$ for an arbitrary $\eta_h \in H_h$. Note that in (61) only the C^0 norm of $s_1 - s_2$ is required, and in case $s_1 - s_2 \in H_h$ this will be estimated by the inverse estimate (20).

Proposition 3.3. *Suppose $s_i = id + \sigma_i: \partial D \rightarrow S^1$ for $i = 1, 2$ and $g: S^1 \rightarrow \mathfrak{R}$. Then*

$$(60) \quad \|g \circ s_1 - g \circ s_2\|_{L^2} \leq c \|g\|_{C^1} \|s_1 - s_2\|_{L^2},$$

$$(61) \quad |g \circ s_1 - g \circ s_2|_{H^{1/2}} \leq c \|g\|_{C^2} (\|s_1\|_{C^{0,1}} + \|s_1 - s_2\|_{C^0}) \|s_1 - s_2\|_{H^{1/2}},$$

$$(62) \quad |g \circ s_1 - g \circ s_2|_{H^1} \leq c \|g\|_{C^2} \|s_1\|_{C^{0,1}} \|s_1 - s_2\|_{H^1},$$

$$(63) \quad |g \circ s_1 - g \circ s_2|_{C^{0,1}} \leq c \|g\|_{C^2} \|s_1\|_{C^{0,1}} \|s_1 - s_2\|_{C^{0,1}}.$$

Proof. The proof of (60) is immediate.

For (61) write

$$\eta(\phi) = s_2(\phi) - s_1(\phi).$$

Then

$$\begin{aligned} & \left| \left(g(s_1(\phi)) - g(s_1(\phi) + \eta(\phi)) \right) - \left(g(s_1(\bar{\phi})) - g(s_1(\bar{\phi}) + \eta(\bar{\phi})) \right) \right| \\ &= \left| - \int_0^1 g'(s_1(\phi) + t\eta(\phi)) dt \eta(\phi) + \int_0^1 g'(s_1(\bar{\phi}) + t\eta(\bar{\phi})) dt \eta(\bar{\phi}) \right| \\ &\leq \int_0^1 |g'(s_1(\phi) + t\eta(\phi))| dt |\eta(\phi) - \eta(\bar{\phi})| \\ &\quad + \int_0^1 |g'(s_1(\phi) + t\eta(\phi)) - g'(s_1(\bar{\phi}) + t\eta(\bar{\phi}))| dt |\eta(\bar{\phi})| \\ &\leq \|g\|_{C^1} |\eta(\phi) - \eta(\bar{\phi})| + \|g\|_{C^2} \left(|s_1(\phi) - s_1(\bar{\phi})| + |\eta(\phi) - \eta(\bar{\phi})| \right) |\eta(\bar{\phi})|. \end{aligned}$$

It now follows from (37) that

$$\begin{aligned} & |g \circ s_1 - g \circ (s_1 + \eta)|_{H^{1/2}} \\ &\leq \|g\|_{C^1} \|\eta\|_{H^{1/2}} + \|g\|_{C^2} (c \|s_1\|_{C^{0,1}} \|\eta\|_{L^2} + \|\eta\|_{H^{1/2}} \|\eta\|_{C^0}) \\ &\leq c \|g\|_{C^2} (\|s_1\|_{C^{0,1}} + \|\eta\|_{C^0}) \|\eta\|_{H^{1/2}}, \end{aligned}$$

recalling that $\|s_1\|_{C^{0,1}} \geq 1$. This establishes (61).

For (62) we compute

$$\begin{aligned} & \|(g \circ s_1 - g \circ s_2)'\|_{L^2} \\ &= \|g' \circ s_1 s'_1 - g' \circ s_2 s'_2\|_{L^2} \\ &\leq \|g' \circ s_1 - g' \circ s_2\|_{L^2} \|s'_1\|_{L^\infty} + \|g' \circ s_2\|_{C^0} \|s'_1 - s'_2\|_{L^2} \\ &\leq c \|g\|_{C^2} \|s_1 - s_2\|_{L^2} \|s_1\|_{C^{0,1}} + \|g\|_{C^1} \|s_1 - s_2\|_{H^1} \quad \text{from (60)} \\ &\leq c \|g\|_{C^2} \|s_1\|_{C^{0,1}} \|s_1 - s_2\|_{H^1}. \end{aligned}$$

For (63) we similarly estimate $\|(g \circ s_1 - g \circ s_2)'\|_{L^\infty}$. □

The next proposition will be used repeatedly in the consistency and non-degeneracy estimates of Sections 4 and 5, particularly in case $s = 1$. In Corollary 3.5 we see that the $H^1(D_h)$ seminorm of the discrete harmonic extension of a *discrete* function f can be estimated by the $H^1(D)$ seminorm of the smooth harmonic extension of the function. This is not true for general smooth f . The corollary is due to Bramble, Pasciak and Schatz [BPS].

Proposition 3.4. *If $f \in H^s(\partial D, \mathbb{R}^n)$ where $s = 1, 3/2$, then*

$$(64) \quad |\Phi(f) - \Phi_h I_h(f)|_{H^1(D_h)} \leq ch^{s-1/2} |f|_{H^s(\partial D)},$$

$$(65) \quad |\Phi_h I_h(f)|_{H^1(D_h)} \leq |f|_{H^{1/2}(\partial D)} + ch^{s-1/2} |f|_{H^s(\partial D)}.$$

Proof. It is sufficient to take $n = 1$. Let

$$u = \Phi(f), \quad u_h = \Phi_h I_h(f).$$

Note that $I_h f$ is well-defined, as $f \in C^0(\partial D_h)$. Since $u_h = I_h u$ on ∂D_h , it follows in the usual way from the weak form of Laplace's equation and (25) that

$$(66) \quad 0 = \int_{D_h} (\nabla u_h - \nabla u)(\nabla u_h - \nabla I_h u),$$

and so

$$(67) \quad \int_{D_h} |\nabla u - \nabla u_h|^2 = \int_{D_h} (\nabla u - \nabla u_h)(\nabla u - \nabla I_h u).$$

Hence

$$\begin{aligned} |u - u_h|_{H^1(D_h)} &\leq |u - I_h u|_{H^1(D_h)} \\ &\leq ch^{s-1/2} |u|_{H^{s+1/2}(D_h)} \leq ch^{s-1/2} |u|_{H^{s+1/2}(D)} \\ &\leq ch^{s-1/2} |u|_{H^s(\partial D)}, \end{aligned}$$

from a standard interpolation result and (42). This gives (64), and (65) then follows from (42). \square

Corollary 3.5. *Suppose $f_h \in H_h$. Then*

$$(68) \quad |\Phi_h I_h(f_h)|_{H^1(D_h)} \leq c |\Phi(f_h)|_{H^1(D)}$$

Proof. This follows from (65) with $s = 1$, the inverse estimate (19), and (40). \square

Proposition 3.7 will allow us to estimate various quantities involving a harmonic function and the discrete disc D_h , in terms of the trace of the harmonic function on D . But first we need an elementary lemma.

Lemma 3.6. *Suppose $u \in H^1(D)$, and $\pi: \partial D \rightarrow \partial D_h$ is as in (14). Then*

$$(69) \quad \|u - u \circ \pi\|_{L^2(\partial D)} \leq ch |u|_{H^1(D \setminus D_h)}$$

Proof. Let L_θ be the straight line segment joining $\theta \in \partial D$ to $\pi(\theta) \in \partial D_h$. Then

$$|u(\theta) - u \circ \pi(\theta)|^2 \leq \left(\int_{L_\theta} |\nabla u| \right)^2 \leq ch^2 \int_{L_\theta} |\nabla u|^2.$$

Each $z \in D \setminus D_h$ can be written uniquely as $(\theta, y) \in \partial D \times \mathbb{R}^+$, where $z \in L_\theta$ and y is the distance of z from $\theta \in \partial D$. The corresponding map $(\theta, y) \mapsto z$ has Jacobian J satisfying $1 - ch^2 \leq J \leq 1$. Hence

$$\int_{\partial D} |u - u \circ \pi|^2 \leq ch^2 \int_{\partial D} d\theta \int_{L_\theta} |\nabla u|^2 \leq ch^2 \int_{D \setminus D_h} |\nabla u|^2.$$

\square

In the following proposition and elsewhere, $\frac{\partial u}{\partial \nu}$ and $\frac{\partial u}{\partial \tau}$ denote normal and tangential derivatives on the relevant curve.

Proposition 3.7. *Suppose u is harmonic in D with trace $u|_{\partial D} \in L^2(\partial D) \cap H^1(\partial D)$ as appropriate. Then*

$$(70) \quad \|u\|_{L^2(D \setminus D_h)} \leq ch \|u\|_{L^2(\partial D)},$$

$$(71) \quad \|\nabla u\|_{L^2(D \setminus D_h)} \leq ch |u|_{H^1(\partial D)},$$

$$(72) \quad \|u - u \circ \pi\|_{L^2(\partial D)} \leq ch^2 |u|_{H^1(\partial D)},$$

$$(73) \quad \left\| \frac{\partial u}{\partial \nu} \right\|_{L^2(\partial D_h)} \leq c |u|_{H^1(\partial D)}.$$

Proof. Let $D(\rho)$ be the disc of radius ρ . From (44)

$$\|u\|_{L^2(\partial D(\rho))} \leq c \|u\|_{L^2(\partial D)}.$$

Integrating the square of this inequality with respect to ρ from $1 - ch^2$ to 1 now establishes (70).

Since ∇u is also harmonic, it follows that

$$\|\nabla u\|_{L^2(D \setminus D_h)} \leq ch \|\nabla u\|_{L^2(\partial D)}.$$

But

$$(74) \quad \int_{\partial D} \left| \frac{\partial u}{\partial \nu} \right|^2 = \int_{\partial D} \left| \frac{\partial u}{\partial \tau} \right|^2,$$

from a Fourier series expansion, and so (71) follows.

Inequality (72) follows from (71) and the previous lemma.

For (73) we have that if $\Delta u = 0$ and $u|_{\partial D} \in H^1(\partial D)$, then $\Delta \nabla u = 0$ and $\nabla u \in H^{1/2}(D)$ by (42). Hence $\nabla u|_{\partial D}$ is well-defined, and $\nabla u|_{\partial D} \in L^2(\partial D)$ from (41). From (44) applied to ∇u ,

$$\|\nabla u\|_{L^2(\partial D_h)} \leq c \|\nabla u\|_{L^2(\partial D)}$$

and in particular, using (74),

$$\left\| \frac{\partial u}{\partial \nu} \right\|_{L^2(\partial D_h)} \leq c |u|_{H^1(\partial D)}.$$

This establishes (73). □

4. CONSISTENCY ESTIMATES FOR THE ENERGY

In this section we compare E and its derivatives at s with E_h and its derivatives at $p_h s$. Apart from their intrinsic significance, these estimates will be needed in the next section to establish the main convergence results.

Remarks on the Proofs. In the proof of each of the three propositions in this section, Proposition 3.4 is used to estimate the difference between a harmonic function and the corresponding discrete harmonic function, and also to estimate various discrete harmonic “error” terms.

Interpolation results in terms of the H^2 piecewise seminorm, together with inverse estimates, are used to estimate the L^2 norm of the quantities

$$\begin{aligned} &\gamma \circ p_h s - I_h^{\partial D}(\gamma \circ p_h s), \\ &\gamma' \circ p_h s \xi_h - I_h^{\partial D}(\gamma' \circ p_h s \xi_h), \end{aligned}$$

and

$$\gamma'' \circ p_h s \xi_h^2 - I_h^{\partial D}(\gamma'' \circ p_h s \xi_h)^2.$$

Since piecewise second derivatives of $p_h s$ and ξ_h vanish, this enables us to gain an extra power of h over what one might at first expect.

In each of the three proofs the term I_2 is estimated with an integration by parts. In Proposition 4.1 this improves the order of convergence from $O(h)$ to $O(h^2)$. In the other two propositions, one could not otherwise expect any order of convergence; see also the remark preceding Proposition 4.3 concerning $\|\nabla(v - v_h)\|_{L^2(D_h)}$.

Proposition 4.1. *Let $s \in \mathcal{H} \cap C^2$, $\gamma \in C^2$. Then*

$$|E_h(p_h s) - E(s)| \leq ch^2 \|\gamma\|_{C^2}^2 \|s\|_{C^2}^4.$$

Proof. Let

$$(75) \quad u = \Phi(\gamma \circ s), \quad u_h = \Phi_h I_h(\gamma \circ p_h s).$$

Then

$$\begin{aligned}
 E_h(p_h s) - E(s) &= \frac{1}{2} \int_{D_h} |\nabla u_h|^2 - \frac{1}{2} \int_D |\nabla u|^2 \\
 &= \frac{1}{2} \int_{D_h} |\nabla(u - u_h)|^2 - \int_{D_h} \nabla u \nabla(u - u_h) - \frac{1}{2} \int_{D \setminus D_h} |\nabla u|^2 \\
 (76) \quad &= I_1 + I_2 + I_3.
 \end{aligned}$$

We will estimate these terms separately.

For I_1 ,

$$\begin{aligned}
 (2I_1)^{1/2} &= |\Phi(\gamma \circ s) - \Phi_h I_h(\gamma \circ p_h s)|_{H^1(D_h)} \\
 &\leq |\Phi(\gamma \circ s) - \Phi_h I_h(\gamma \circ s)|_{H^1(D_h)} + |\Phi_h I_h(\gamma \circ s - \gamma \circ p_h s)|_{H^1(D_h)} \\
 &\leq ch|\gamma \circ s|_{H^{3/2}} + |\gamma \circ s - \gamma \circ p_h s|_{H^{1/2}} + ch^{1/2}|\gamma \circ s - \gamma \circ p_h s|_{H^1} \\
 &\quad \text{from (64) with } s = 3/2 \text{ and (65) with } s = 1.
 \end{aligned}$$

But

$$h|\gamma \circ s|_{H^{3/2}} \leq ch|\gamma \circ s|_{C^2} \leq ch\|\gamma\|_{C^2}\|s\|_{C^2}^2 \quad \text{from (56);}$$

also

$$\begin{aligned}
 |\gamma \circ s - \gamma \circ p_h s|_{H^{1/2}} &\leq c\|\gamma\|_{C^2}(\|s\|_{C^{0,1}} + \|p_h s\|_{C^{0,1}})\|s - p_h s\|_{H^{1/2}} \text{ by (61)} \\
 &\leq ch^{3/2}\|\gamma\|_{C^2}\|s\|_{C^1}\|s\|_{H^2} \text{ by (23) and (22)} \\
 &\leq ch^{3/2}\|\gamma\|_{C^2}\|s\|_{C^2}^2,
 \end{aligned}$$

and

$$\begin{aligned}
 h^{1/2}|\gamma \circ s - \gamma \circ p_h s|_{H^1} &\leq ch^{1/2}\|\gamma\|_{C^2}\|s\|_{C^1}\|s - p_h s\|_{H^1} \text{ from (62)} \\
 &\leq ch^{3/2}\|\gamma\|_{C^2}\|s\|_{C^2}^2 \text{ from (22)}.
 \end{aligned}$$

Hence

$$(77) \quad I_1 \leq ch^2\|\gamma\|_{C^2}^2\|s\|_{C^2}^4.$$

For I_2 one has

$$\begin{aligned}
 |I_2| &= \left| \int_{\partial D_h} \frac{\partial u}{\partial \nu}(u - u_h) \right| \\
 &\leq \left\| \frac{\partial u}{\partial \nu} \right\|_{L^2(\partial D_h)} \|u - u_h\|_{L^2(\partial D_h)} \\
 &\leq c\|u\|_{H^1(\partial D)} \|u - u_h\|_{L^2(\partial D_h)} \text{ by (73)} \\
 &\leq c\|\gamma\|_{C^1}\|s\|_{C^1}\|u \circ \pi - u_h \circ \pi\|_{L^2(\partial D)} \text{ by (59) and (14)} \\
 &\leq c\|\gamma\|_{C^1}\|s\|_{C^1} (\|u \circ \pi - u\|_{L^2(\partial D)} + \|u - u_h \circ \pi\|_{L^2(\partial D)}).
 \end{aligned}$$

But

$$\|u \circ \pi - u\|_{L^2(\partial D)} \leq ch^2\|u\|_{H^1(\partial D)} = ch^2|\gamma \circ s|_{H^1} \leq ch^2\|\gamma\|_{C^1}\|s\|_{C^1}$$

by (72) and (59). Also

$$\begin{aligned}
 \|u - u_h \circ \pi\|_{L^2(\partial D)} &= \|\gamma \circ s - I_h^{\partial D}(\gamma \circ p_h s)\|_{L^2} \text{ by (75) and (21)} \\
 &\leq \|\gamma \circ s - \gamma \circ p_h s\|_{L^2} + \|\gamma \circ p_h s - I_h^{\partial D}(\gamma \circ p_h s)\|_{L^2} \\
 &\leq \|\gamma\|_{C^1}\|s - p_h s\|_{L^2} + ch^2|\gamma \circ p_h s|_{H^2},
 \end{aligned}$$

by (60) and a standard interpolation estimate, where the H^2 semi-norm is to be understood in a piecewise sense. On each arc segment, $(\gamma \circ p_h s)'' = \gamma'' \circ p_h s ((p_h s)')^2$ since $(p_h s)'' = 0$, so we can continue the estimate by

$$\leq c \|\gamma\|_{C^1} h^2 \|s\|_{C^2} + ch^2 \|\gamma\|_{C^2} \|s\|_{C^1}^2 \leq ch^2 \|\gamma\|_{C^2} \|s\|_{C^2}^2,$$

using (24) and (23). Hence

$$(78) \quad |I_2| \leq ch^2 \|\gamma\|_{C^2}^2 \|s\|_{C^2}^3.$$

Finally,

$$(79) \quad \begin{aligned} |I_3| &= \frac{1}{2} \|\nabla u\|_{L^2(D \setminus D_h)}^2 \leq ch^2 |u|_{H^1(\partial D)}^2 \quad \text{by (71)} \\ &= ch^2 |\gamma \circ s|_{H^1}^2 \leq ch^2 \|\gamma\|_{C^1}^2 \|s\|_{C^1}^2. \end{aligned}$$

The proposition now follows from (76), (77), (78) and (79). □

The next proposition shows that $E'_h(p_h s)$ is an $O(h)$ approximation to $E'(s)|_{H_h}$ in the H'_h norm for γ and s sufficiently smooth. The actual power of h is important in the proof of Theorem 5.4.

Proposition 4.2. *Let $s \in \mathcal{H} \cap C^2$, $\gamma \in C^3$. Then for any $\xi_h \in H_h$*

$$|\langle E'(s), \xi_h \rangle - \langle E'_h(p_h s), \xi_h \rangle| \leq ch \|\gamma\|_{C^3}^2 \|s\|_{C^3}^3 \|\xi_h\|_{H^{1/2}}.$$

Proof. Define u and u_h as in (75), and for $\xi_h \in H_h$ define

$$(80) \quad v = \Phi(\gamma' \circ s \xi_h), \quad v_h = \Phi_h I_h(\gamma' \circ p_h s \xi_h).$$

Then from (29),

$$(81) \quad \begin{aligned} &\langle E'(s), \xi_h \rangle - \langle E'_h(p_h s), \xi_h \rangle \\ &= \int_D \nabla u \nabla v - \int_{D_h} \nabla u_h \nabla v_h \\ &= \int_{D_h} (\nabla u \nabla v - \nabla u_h \nabla v_h) + \int_{D \setminus D_h} \nabla u \nabla v \\ &= \int_{D_h} \nabla(u - u_h) \nabla v_h + \int_{D_h} \nabla u \nabla(v - v_h) + \int_{D \setminus D_h} \nabla u \nabla v \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Now

$$\begin{aligned} |I_1| &\leq \|\nabla(u - u_h)\|_{L^2(D_h)} \|\nabla v_h\|_{L^2(D_h)} \\ &\leq ch \|\gamma\|_{C^2} \|s\|_{C^2}^2 \|\nabla v_h\|_{L^2(D_h)} \quad \text{from (77)}. \end{aligned}$$

Also

$$\begin{aligned} \|\nabla v_h\|_{L^2(D_h)} &= |\Phi_h I_h(\gamma' \circ p_h s \xi_h)|_{H^1(D_h)} \\ &\leq |\gamma' \circ p_h s \xi_h|_{H^{1/2}} + ch^{1/2} |\gamma' \circ p_h s \xi_h|_{H^1} \quad \text{by (65)}. \end{aligned}$$

The first term is estimated by

$$\begin{aligned} |\gamma' \circ p_h s \xi_h|_{H^{1/2}} &\leq c \|\gamma' \circ p_h s\|_{C^{0,1}} \|\xi_h\|_{H^{1/2}} \quad \text{from (49)} \\ &\leq c \|\gamma\|_{C^2} \|s\|_{C^1} \|\xi_h\|_{H^{1/2}} \quad \text{using (23);} \end{aligned}$$

the second by

$$\begin{aligned}
 |\gamma' \circ p_h s \xi_h|_{H^1} &\leq c \|\gamma' \circ p_h s\|_{C^{0,1}} \|\xi_h\|_{H^1} \quad \text{by (51)} \\
 &\leq c \|\gamma\|_{C^2} \|p_h s\|_{C^{0,1}} \|\xi_h\|_{H^1} \quad \text{by (54)} \\
 (82) \qquad &\leq ch^{-1/2} \|\gamma\|_{C^2} \|s\|_{C^1} \|\xi_h\|_{H^{1/2}} \quad \text{by (23) and (19)}.
 \end{aligned}$$

So altogether,

$$(83) \qquad |I_1| \leq ch \|\gamma\|_{C^2}^2 \|s\|_{C^1}^3 \|\xi_h\|_{H^{1/2}}.$$

For I_2 , as for I_2 in the proof of the previous proposition,

$$\begin{aligned}
 |I_2| &= \left| \int_{\partial D_h} \frac{\partial u}{\partial \nu} (v - v_h) \right| \\
 &\leq c \|\gamma\|_{C^1} \|s\|_{C^1} (\|v \circ \pi - v\|_{L^2(\partial D)} + \|v - v_h \circ \pi\|_{L^2(\partial D)}).
 \end{aligned}$$

From (72) and the argument for (82),

$$\begin{aligned}
 \|v \circ \pi - v\|_{L^2(\partial D)} &\leq ch^2 |v|_{H^1(\partial D)} = ch^2 |\gamma' \circ s \xi_h|_{H^1} \\
 &\leq ch^{3/2} \|\gamma\|_{C^2} \|s\|_{C^1} \|\xi_h\|_{H^{1/2}}.
 \end{aligned}$$

Also,

$$\begin{aligned}
 &\|v - v_h \circ \pi\|_{L^2(\partial D)} \\
 &= \|\gamma' \circ s \xi_h - I_h^{\partial D}(\gamma' \circ p_h s \xi_h)\|_{L^2} \\
 &\leq \|(\gamma' \circ s - \gamma' \circ p_h s) \xi_h\|_{L^2} + \|\gamma' \circ p_h s \xi_h - I_h^{\partial D}(\gamma' \circ p_h s \xi_h)\|_{L^2} \\
 &\leq \|\gamma\|_{C^2} \|s - p_h s\|_{C^0} \|\xi_h\|_{L^2} + ch^2 |\gamma' \circ p_h s \xi_h|_{H^2} \\
 &\quad \text{where the } H^2 \text{ semi-norm is understood in the piecewise sense} \\
 &\leq ch^2 \|\gamma\|_{C^2} \|s\|_{C^2} \|\xi_h\|_{L^2} \quad \text{from (24)} \\
 &\quad + ch^2 (\|\gamma\|_{C^3} \|s\|_{C^1}^2 \|\xi_h\|_{L^2} + \|\gamma\|_{C^2} \|s\|_{C^1} \|\xi_h\|_{H^1}) \quad \text{since on each arc} \\
 &\quad \text{segment, } (\gamma' \circ p_h s \xi_h)'' = \gamma''' \circ p_h s ((p_h s)')^2 \xi_h + 2\gamma'' \circ p_h s (p_h s)' \xi_h' \\
 &\leq ch^2 \|\gamma\|_{C^3} \|s\|_{C^2}^2 \|\xi_h\|_{H^1} \\
 &\leq ch^{3/2} \|\gamma\|_{C^3} \|s\|_{C^2}^2 \|\xi_h\|_{H^{1/2}} \quad \text{by (19)}.
 \end{aligned}$$

Altogether,

$$(84) \qquad |I_2| \leq ch^{3/2} \|\gamma\|_{C^3}^2 \|s\|_{C^2}^3 \|\xi_h\|_{H^{1/2}}.$$

Finally,

$$\begin{aligned}
 |I_3| &\leq \|\nabla u\|_{L^2(D \setminus D_h)} \|\nabla v\|_{L^2(D \setminus D_h)} \\
 &\leq ch \|\gamma\|_{C^1} \|s\|_{C^1} \|\nabla v\|_{L^2(D \setminus D_h)} \quad \text{by (79)} \\
 &\leq ch^2 \|\gamma\|_{C^1} \|s\|_{C^1} |v|_{H^1(\partial D)} \quad \text{by (71)} \\
 &\leq ch^2 \|\gamma\|_{C^1} \|s\|_{C^1} |\gamma' \circ s \xi_h|_{H^1} \\
 &\leq ch^2 \|\gamma\|_{C^1} \|s\|_{C^1} \|\gamma\|_{C^2} \|s\|_{C^1} \|\xi_h\|_{H^1} \quad \text{by (51) and (55)} \\
 (85) \qquad &\leq ch^{3/2} \|\gamma\|_{C^2}^2 \|s\|_{C^1}^2 \|\xi_h\|_{H^{1/2}} \quad \text{by (19)}.
 \end{aligned}$$

The proposition follows from (81), (83), (84) and (85). □

Remark. The following proposition, and the related Proposition 5.3, are essential for the proof of the main theorem. Assuming γ and s are sufficiently smooth, we see from (87) that if $E''(s)$ is positive definite (for example, if s is a strictly stable

local minimum for E) then so is $E''_h(p_h s)$ for h sufficiently small, with the same “ellipticity” constant up to $O(h^{1/2}|\log h|^{1/2})$.

One might hope to estimate the positive term

$$(86) \quad \|\nabla(v - v_h)\|_{L^2(D_h)}^2 = \|\Phi(\gamma' \circ s \xi_h) - \Phi_h I_h(\gamma' \circ p_h s \xi_h)\|_{H^1(D_h)}^2$$

in (87) by $ch^\alpha \|\xi_h\|_{H^{1/2}}^2$ for some $\alpha > 0$. But we cannot expect this, as we see by considering the “model” estimates:

$$\|\Phi(\xi_h) - \Phi_h I_h(\xi_h)\|_{H^1(D_h)} \leq ch^{1/2} \|\xi_h\|_{H^1} \leq c \|\xi_h\|_{H^{1/2}}$$

from (64) and (19) — these estimates cannot be improved.

Nonetheless, in Proposition 5.3 we *do* estimate (86) in case $\xi_h = p_h \xi$, where ξ is in the negative eigenspace of $E''(s)$. The point is that ξ is then smooth by elliptic regularity theory (although ξ_h is of course not smooth), and in fact we show in (106) that in this case

$$\|\nabla(v - v_h)\|_{L^2(D_h)}^2 \leq ch \|\xi_h\|_{H^{1/2}}^2.$$

As a consequence we deduce in Proposition 5.3 that if $E''(s)$ is nondegenerate then so is $E''_h(p_h s)$ for h sufficiently small, with the same nondegeneracy constant up to $O(h^{1/2}|\log h|^{1/2})$.

Proposition 4.3. *Let $s \in C^2 \cap \mathcal{H}$ and $\gamma \in C^3$. Then for any $\xi_h \in H_h$*

$$(87) \quad E''_h(p_h s)(\xi_h, \xi_h) - E''(s)(\xi_h, \xi_h) = \|\nabla(v - v_h)\|_{L^2(D_h)}^2 + R$$

where

$$v = \Phi(\gamma' \circ s \xi_h), \quad v_h = \Phi_h I_h(\gamma' \circ p_h s \xi_h),$$

and

$$|R| \leq ch^{1/2} |\log h|^{1/2} \|\gamma\|_{C^3}^2 \|s\|_{C^2}^3 \|\xi_h\|_{H^{1/2}}^2.$$

Proof. Let u and u_h be as in (75). Let

$$(88) \quad w = \Phi(\gamma'' \circ s \xi_h^2), \quad w_h = \Phi_h I_h(\gamma'' \circ p_h s \xi_h^2).$$

From (30),

$$\begin{aligned} & (E''(s) - E''_h(p_h s))(\xi_h, \xi_h) \\ &= \int_D |\nabla v|^2 + \int_D \nabla u \nabla w - \int_{D_h} |\nabla v_h|^2 - \int_{D_h} \nabla u_h \nabla w_h \\ &= - \int_{D_h} |\nabla(v - v_h)|^2 + 2 \int_{D_h} \nabla v \nabla(v - v_h) + \int_{D_h} \nabla(u - u_h) \nabla w_h \\ &\quad + \int_{D_h} \nabla(w - w_h) \nabla u + \int_{D \setminus D_h} |\nabla v|^2 + \int_{D \setminus D_h} \nabla u \nabla w \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned}$$

We estimate the terms other than I_1 .

For I_2 we have

$$\begin{aligned} |I_2| &= 2 \left| \int_{D_h} \nabla v \nabla(v - v_h) \right| = 2 \left| \int_{\partial D_h} \frac{\partial v}{\partial \nu} (v - v_h) \right| \\ &\leq 2 \left\| \frac{\partial v}{\partial \nu} \right\|_{L^2(\partial D_h)} \|v - v_h\|_{L^2(\partial D_h)}. \end{aligned}$$

The first factor is estimated with the use of (73) and arguing as for (82):

$$\left\| \frac{\partial v}{\partial \nu} \right\|_{L^2(\partial D_h)} \leq c|v|_{H^1(\partial D)} = c|\gamma' \circ s \xi_h|_{H^1} \leq ch^{-1/2} \|\gamma\|_{C^2} \|s\|_{C^1} \|\xi_h\|_{H^{1/2}}.$$

The second factor was estimated in the estimate for I_2 in the proof of Proposition 4.2:

$$\|v - v_h\|_{L^2(\partial D_h)} \leq ch^{3/2} \|\gamma\|_{C^3} \|s\|_{C^2}^2 \|\xi_h\|_{H^{1/2}}.$$

Altogether for I_2 ,

$$(89) \quad |I_2| \leq ch \|\gamma\|_{C^3}^2 \|s\|_{C^2}^3 \|\xi_h\|_{H^{1/2}}^2.$$

For I_3 ,

$$\begin{aligned} |I_3| &= \left| \int_{D_h} \nabla(u - u_h) \nabla w_h \right| \leq \|\nabla(u - u_h)\|_{L^2(D_h)} \|\nabla w_h\|_{L^2(D_h)} \\ &\leq ch \|\gamma\|_{C^2} \|s\|_{C^2}^2 \|\nabla w_h\|_{L^2(D_h)} \quad \text{from (77)}. \end{aligned}$$

From (65) we get

$$\begin{aligned} \|\nabla w_h\|_{L^2(D_h)} &= |\Phi_h I_h(\gamma'' \circ p_h s \xi_h^2)|_{H^1(D_h)} \\ &\leq |\gamma'' \circ p_h s \xi_h^2|_{H^{1/2}} + ch^{1/2} |\gamma'' \circ p_h s \xi_h^2|_{H^1}. \end{aligned}$$

But

$$\begin{aligned} |\gamma'' \circ p_h s \xi_h^2|_{H^{1/2}} &\leq c \|\gamma\|_{C^3} \|s\|_{C^1} \|\xi_h^2\|_{H^{1/2}} \quad \text{from (49) and (54)} \\ &\leq c \|\gamma\|_{C^3} \|s\|_{C^1} \|\xi_h\|_{C^0} \|\xi_h\|_{H^{1/2}} \quad \text{from (48)} \\ &\leq c |\log h|^{1/2} \|\gamma\|_{C^3} \|s\|_{C^1} \|\xi_h\|_{H^{1/2}}^2 \quad \text{from (20)}. \end{aligned}$$

Similarly,

$$\begin{aligned} |\gamma'' \circ p_h s \xi_h^2|_{H^1} &\leq c \|\gamma\|_{C^3} \|s\|_{C^1} \|\xi_h^2\|_{H^1} \quad \text{using (51)} \\ &\leq c \|\gamma\|_{C^3} \|s\|_{C^1} \|\xi_h\|_{C^0} \|\xi_h\|_{H^1} \quad \text{from (50)} \end{aligned} \tag{90}$$

from (20) and (19). Hence

$$\|\nabla w_h\|_{L^2(D_h)} \leq c |\log h|^{1/2} \|\gamma\|_{C^3} \|s\|_{C^1} \|\xi_h\|_{H^{1/2}}^2,$$

and so

$$(91) \quad |I_3| \leq ch |\log h|^{1/2} \|\gamma\|_{C^3}^2 \|s\|_{C^2}^3 \|\xi_h\|_{H^{1/2}}^2.$$

For I_4 we have

$$I_4 = \int_{D_h} \nabla(w - w_h) \nabla u = \int_{D_h} (w - w_h) \frac{\partial u}{\partial \nu},$$

and so, as for I_2 in the proof of Proposition 4.1,

$$(92) \quad |I_4| \leq c \|\gamma\|_{C^1} \|s\|_{C^1} (\|w \circ \pi - w\|_{L^2(\partial D)} + \|w - w_h \circ \pi\|_{L^2(\partial D)}).$$

But

$$\begin{aligned} \|w \circ \pi - w\|_{L^2(\partial D)} &\leq ch^2 |w|_{H^1(\partial D)} \quad \text{by (72)} \\ &\leq ch^2 |\gamma'' \circ s \xi_h^2|_{H^1} \\ &\leq ch^{3/2} |\log h|^{1/2} \|\gamma\|_{C^3} \|s\|_{C^1} \|\xi_h\|_{H^{1/2}}^2 \quad \text{as for (90)}. \end{aligned}$$

Also

$$\begin{aligned}
 & \|w - w_h \circ \pi\|_{L^2(\partial D)} \\
 &= \|\gamma'' \circ s \xi_h^2 - I_h^{\partial D}(\gamma'' \circ p_h s \xi_h^2)\|_{L^2} \quad \text{from (88) and (21)} \\
 &\leq \|\gamma'' \circ s \xi_h^2 - \gamma'' \circ p_h s \xi_h^2\|_{L^2} + \|\gamma'' \circ p_h s \xi_h^2 - I_h^{\partial D}(\gamma'' \circ p_h s \xi_h^2)\|_{L^2} \\
 &\leq \|\gamma'' \circ s - \gamma'' \circ p_h s\|_{C^0} \|\xi_h\|_{L^4}^2 + ch|\gamma'' \circ p_h s \xi_h^2|_{H^1} \\
 &\leq c\|\gamma\|_{C^3} \|s - p_h s\|_{C^0} \|\xi_h\|_{H^{1/2}}^2 + ch^{1/2} |\log h|^{1/2} \|\gamma\|_{C^3} \|s\|_{C^1} \|\xi_h\|_{H^{1/2}}^2 \\
 &\quad \text{using a Sobolev embedding theorem and (90)} \\
 &\leq ch^{1/2} |\log h|^{1/2} \|\gamma\|_{C^3} \|s\|_{C^1} \|\xi_h\|_{H^{1/2}}^2 \quad \text{from (24)}.
 \end{aligned}$$

Hence

$$(93) \quad |I_4| \leq ch^{1/2} |\log h|^{1/2} \|\gamma\|_{C^3}^2 \|s\|_{C^1}^2 \|\xi_h\|_{H^{1/2}}^2.$$

From the final part of the argument for I_3 in the proof of the previous proposition,

$$(94) \quad I_5 = \|\nabla v\|_{L^2(D \setminus D_h)}^2 \leq ch \|\gamma\|_{C^2}^2 \|s\|_{C^1}^2 \|\xi_h\|_{H^{1/2}}^2.$$

Finally,

$$\begin{aligned}
 |I_6| &\leq \|\nabla u\|_{L^2(D \setminus D_h)} \|\nabla w\|_{L^2(D \setminus D_h)} \\
 &\leq ch^2 |u|_{H^1(\partial D)} |w|_{H^1(\partial D)} \quad \text{by (71)} \\
 &= ch^2 |\gamma \circ s|_{H^1} |\gamma'' \circ s \xi_h^2|_{H^1} \\
 &\leq ch^2 \|\gamma\|_{C^1} \|s\|_{C^1} |\gamma'' \circ s \xi_h^2|_{H^1} \quad \text{by (59)} \\
 (95) \quad &\leq ch^{3/2} |\log h|^{1/2} \|\gamma\|_{C^3}^2 \|s\|_{C^1}^2 \|\xi_h\|_{H^{1/2}}^2 \quad \text{as for (90)}.
 \end{aligned}$$

Now we collect the estimates (89), (91), (93), (94) and (95), and prove the proposition. \square

5. THE MAIN ERROR ESTIMATES

We will apply the following quantitative version of the Inverse Function Theorem with $\mathcal{X} = \mathcal{H}_h$ and $Y = H'_h$ in the proof of Theorem 5.4.

Lemma 5.1. *Let \mathcal{X} be an affine Banach space with Banach space X as tangent space, and let Y be a Banach space. Suppose $x_0 \in \mathcal{X}$ and $f \in C^1(\mathcal{X}, Y)$. Assume there are positive constants α, β, δ and ϵ such that*

$$(96) \quad \|f(x_0)\|_Y \leq \delta,$$

$$(97) \quad \|f'(x_0)^{-1}\|_{L(Y, X)} \leq \alpha^{-1},$$

$$(98) \quad \|f'(x) - f'(x_0)\|_{L(X, Y)} \leq \beta \quad \text{for all } x \in \overline{B}_\epsilon(x_0),$$

where

$$(99) \quad \beta < \alpha, \quad \delta \leq (\alpha - \beta)\epsilon.$$

Then there exists a unique $x_* \in \overline{B}_\epsilon(x_0)$ such that $f(x_*) = 0$.

Proof. This follows from the proof of the Inverse Function Theorem in [Be, pp. 113–114]. The modifications necessary since \mathcal{X} is an affine space are trivial. \square

The next proposition establishes that for any $\beta > 0$,

$$\|E''_h(p_h s) - E''_h(p_h s + \eta_h)\|_{(H_h \times H_h)'} \leq \beta$$

provided $|\log h| \|\eta_h\|_{H^{1/2}}$ is sufficiently small (depending on β but independent of h). This will be used to establish the appropriate version of (98) in the proof of Proposition 5.4.

Remark on the Proof. The precise form of estimate (61), and the inverse estimate (20), are used to control $|g \circ s_h - g \circ (s_h + \eta_h)|_{H^{1/2}}$ for $g = \gamma, \gamma'$ and γ'' .

Proposition 5.2. *Let $s \in \mathcal{H} \cap C^1$ and $\gamma \in C^4$. Then for any $\eta_h, \xi_h \in H_h$*

$$\begin{aligned} & \left| \left(E_h''(p_h s) - E_h''(p_h s + \eta_h) \right) (\xi_h, \xi_h) \right| \\ & \leq c \left(1 + |\log h|^{1/2} \|\eta_h\|_{H^{1/2}} \right)^2 |\log h| \|\eta_h\|_{H^{1/2}} \|\gamma\|_{C^4}^2 \|s\|_{C^1}^2 \|\xi_h\|_{H^{1/2}}^2. \end{aligned}$$

Proof. For simplicity we write s_h for $p_h s$.

From (30),

$$\begin{aligned} & \left(E_h''(s_h) - E_h''(s_h + \eta_h) \right) (\xi_h, \xi_h) \\ & = \int_{D_h} |\nabla v_h|^2 + \int_{D_h} \nabla u_h \nabla w_h - \int_{D_h} |\nabla \bar{v}_h|^2 - \int_{D_h} \nabla \bar{u}_h \nabla \bar{w}_h, \end{aligned}$$

with

$$\begin{aligned} u_h &= \Phi_h I_h(\gamma \circ s_h), & \bar{u}_h &= \Phi_h I_h(\gamma \circ (s_h + \eta_h)), \\ v_h &= \Phi_h I_h(\gamma' \circ s_h \xi_h), & \bar{v}_h &= \Phi_h I_h(\gamma' \circ (s_h + \eta_h) \xi_h), \\ w_h &= \Phi_h I_h(\gamma'' \circ s_h \xi_h^2), & \bar{w}_h &= \Phi_h I_h(\gamma'' \circ (s_h + \eta_h) \xi_h^2). \end{aligned}$$

So

$$\begin{aligned} & \left| \left(E_h''(s_h) - E_h''(s_h + \eta_h) \right) (\xi_h, \xi_h) \right| \\ & \leq \int_{D_h} |\nabla(v_h - \bar{v}_h)| (|\nabla v_h| + |\nabla \bar{v}_h|) + \int_{D_h} |\nabla(u_h - \bar{u}_h)| |\nabla w_h| \\ & \quad + \int_{D_h} |\nabla \bar{u}_h| |\nabla(w_h - \bar{w}_h)| \\ & \leq \|\nabla(u_h - \bar{u}_h)\|_{L^2(D_h)} \|\nabla w_h\|_{L^2(D_h)} \\ & \quad + \|\nabla(v_h - \bar{v}_h)\|_{L^2(D_h)} (\|\nabla v_h\|_{L^2(D_h)} + \|\nabla \bar{v}_h\|_{L^2(D_h)}) \\ & \quad + \|\nabla(w_h - \bar{w}_h)\|_{L^2(D_h)} \|\nabla \bar{u}_h\|_{L^2(D_h)} \\ (100) \quad & = A_1 B_1 + A_2 (B_{21} + B_{22}) + A_3 B_3. \end{aligned}$$

Now we estimate these terms separately.

By (65)

$$\begin{aligned} A_1 &= \left| \Phi_h I_h(\gamma \circ s_h - \gamma \circ (s_h + \eta_h)) \right|_{H^1(D_h)} \\ &\leq |\gamma \circ s_h - \gamma \circ (s_h + \eta_h)|_{H^{1/2}} + ch^{1/2} |\gamma \circ s_h - \gamma \circ (s_h + \eta_h)|_{H^1} \\ &\leq c \|\gamma\|_{C^2} (\|s_h\|_{C^{0,1}} + \|\eta_h\|_{C^0}) \|\eta_h\|_{H^{1/2}} + ch^{1/2} \|\gamma\|_{C^2} \|s_h\|_{C^{0,1}} \|\eta_h\|_{H^1} \\ &\quad \text{by (61) and (62)} \\ &\leq c \|\gamma\|_{C^2} \|s\|_{C^1} \|\eta_h\|_{H^{1/2}} \left(1 + |\log h|^{1/2} \|\eta_h\|_{H^{1/2}} \right) \\ &\quad + c \|\gamma\|_{C^2} \|s\|_{C^1} \|\eta_h\|_{H^{1/2}} \quad \text{by (20), (19) and (23)} \\ &\leq c \left(1 + |\log h|^{1/2} \|\eta_h\|_{H^{1/2}} \right) \|\eta_h\|_{H^{1/2}} \|\gamma\|_{C^2} \|s\|_{C^1}. \end{aligned}$$

Again from (65)

$$\begin{aligned}
A_2 &= \left| \Phi_h \mathcal{I}_h \left(\gamma' \circ s_h \xi_h - \gamma' \circ (s_h + \eta_h) \xi_h \right) \right|_{H^1(D_h)} \\
&\leq |(\gamma' \circ s_h - \gamma' \circ (s_h + \eta_h)) \xi_h|_{H^{1/2}} \\
&\quad + ch^{1/2} |(\gamma' \circ s_h - \gamma' \circ (s_h + \eta_h)) \xi_h|_{H^1} \\
&\leq |\gamma' \circ s_h - \gamma' \circ (s_h + \eta_h)|_{H^{1/2}} \|\xi_h\|_{C^0} \\
&\quad + \|\gamma' \circ s_h - \gamma' \circ (s_h + \eta_h)\|_{C^0} |\xi_h|_{H^{1/2}} \quad \text{by (48)} \\
&\quad + ch^{1/2} \|\gamma' \circ s_h - \gamma' \circ (s_h + \eta_h)\|_{C^0} |\xi_h|_{H^1} \\
&\quad + ch^{1/2} |\gamma' \circ s_h - \gamma' \circ (s_h + \eta_h)|_{H^1} \|\xi_h\|_{C^0} \quad \text{by (50)} \\
&\leq c \left(1 + |\log h|^{1/2} \|\eta_h\|_{H^{1/2}} \right) \|\eta_h\|_{H^{1/2}} \|\gamma\|_{C^3} \|s\|_{C^1} \|\xi_h\|_{C^0} \\
&\quad + c \|\gamma\|_{C^2} \|\eta_h\|_{C^0} |\xi_h|_{H^{1/2}} + ch^{1/2} \|\gamma\|_{C^2} \|\eta_h\|_{C^0} |\xi_h|_{H^1} \quad \text{using the argu-} \\
&\quad \quad \quad \text{ment for } A_1 \text{ on the first and last terms in the preceding inequality} \\
&\leq c \left(1 + |\log h|^{1/2} \|\eta_h\|_{H^{1/2}} \right) |\log h|^{1/2} \|\eta_h\|_{H^{1/2}} \|\gamma\|_{C^3} \|s\|_{C^1} \|\xi_h\|_{H^{1/2}},
\end{aligned}$$

again using the inverse estimates (20) and (19) on η_h and ξ_h .

In the same way we proceed for A_3 to first obtain the analogue of the next-to-last inequality above:

$$\begin{aligned}
A_3 &= \left| \Phi_h \mathcal{I}_h (\gamma'' \circ s_h \xi_h^2 - \gamma'' \circ (s_h + \eta_h) \xi_h^2) \right|_{H^1(D_h)} \\
&\leq c \left(1 + |\log h|^{1/2} \|\eta_h\|_{H^{1/2}} \right) \|\eta_h\|^{1/2} \|\gamma\|_{C^4} \|s\|_{C^1} \|\xi_h^2\|_{C^0} \\
&\quad + c \|\gamma\|_{C^3} \|\eta_h\|_{C^0} |\xi_h^2|_{H^{1/2}} + ch^{1/2} \|\gamma\|_{C^3} \|\eta_h\|_{C^0} |\xi_h^2|_{H^1} \\
&\leq c \left(1 + |\log h|^{1/2} \|\eta_h\|_{H^{1/2}} \right) |\log h| \|\eta_h\|_{H^{1/2}} \|\gamma\|_{C^4} \|s\|_{C^1} \|\xi_h\|_{H^{1/2}}^2,
\end{aligned}$$

from the inverse estimates (19) and (20) applied to η_h and ξ_h , and since

$$\|\xi_h^2\|_{H^{1/2}} \leq 2\|\xi_h\|_{C^0} \|\xi_h\|_{H^{1/2}}, \quad \|\xi_h^2\|_{H^1} \leq 2\|\xi_h\|_{C^0} \|\xi_h\|_{H^1}.$$

For B_3 ,

$$\begin{aligned}
B_3 &= |\Phi_h \mathcal{I}_h (\gamma \circ (s_h + \eta_h))|_{H^1(D_h)} \\
&\leq |\gamma \circ (s_h + \eta_h)|_{H^{1/2}} + ch^{1/2} |\gamma \circ (s_h + \eta_h)|_{H^1} \quad \text{by (65)} \\
&\leq \|\gamma\|_{C^1} (\|s_h\|_{H^{1/2}} + \|\eta_h\|_{H^{1/2}}) + ch^{1/2} \|\gamma\|_{C^1} (\|s_h\|_{H^1} + \|\eta_h\|_{H^1}) \\
&\leq c \|\gamma\|_{C^1} (\|s_h\|_{H^{1/2}} + \|\eta_h\|_{H^{1/2}}) \quad \text{by (19)} \\
&\leq c(1 + \|\eta_h\|_{H^{1/2}}) \|\gamma\|_{C^1} \|s\|_{C^1}.
\end{aligned}$$

For B_{22} ,

$$\begin{aligned}
 B_{22} &= |\Phi_h I_h(\gamma' \circ (s_h + \eta_h) \xi_h)|_{H^1(D_h)} \\
 &\leq |\gamma' \circ (s_h + \eta_h) \xi_h|_{H^{1/2}} + ch^{1/2} |\gamma' \circ (s_h + \eta_h) \xi_h|_{H^1} \quad \text{by (65)} \\
 &\leq \|\gamma' \circ (s_h + \eta_h)\|_{C^0} \|\xi_h\|_{H^{1/2}} + |\gamma' \circ (s_h + \eta_h)|_{H^{1/2}} \|\xi_h\|_{C^0} \\
 &\quad + ch^{1/2} \left(\|\gamma' \circ (s_h + \eta_h)\|_{C^0} \|\xi_h\|_{H^1} + |\gamma' \circ (s_h + \eta_h)|_{H^1} \|\xi_h\|_{C^0} \right) \\
 &\quad \text{by (48)} \\
 &\leq c \|\gamma\|_{C^1} \|\xi_h\|_{H^{1/2}} + c \|\gamma\|_{C^2} \|s_h + \eta_h\|_{H^{1/2}} \|\xi_h\|_{C^0} \\
 &\quad + ch^{1/2} \left(\|\gamma\|_{C^1} \|\xi_h\|_{H^1} + \|\gamma\|_{C^2} \|s_h + \eta_h\|_{H^1} \|\xi_h\|_{C^0} \right) \\
 &\leq c \|\gamma\|_{C^2} (\|s_h + \eta_h\|_{H^{1/2}}) |\log h|^{1/2} \|\xi_h\|_{H^{1/2}} \quad \text{by (19) and (20)} \\
 &\leq c(1 + \|\eta_h\|_{H^{1/2}}) |\log h|^{1/2} \|\gamma\|_{C^2} \|s\|_{C^1} \|\xi_h\|_{H^{1/2}}.
 \end{aligned}$$

For B_{21} we have the same estimate with η_h set equal to 0:

$$B_{21} \leq c |\log h|^{1/2} \|\gamma\|_{C^2} \|s\|_{C^1} \|\xi_h\|_{H^{1/2}}.$$

Finally,

$$\begin{aligned}
 B_1 &= |\Phi_h I_h(\gamma'' \circ s_h \xi_h^2)|_{H^1(D_h)} \\
 &\leq |\gamma'' \circ s_h \xi_h^2|_{H^{1/2}} + ch^{1/2} |\gamma'' \circ s_h \xi_h^2|_{H^1} \quad \text{by (65)} \\
 &\leq \|\gamma'' \circ s_h\|_{C^0} \|\xi_h^2\|_{H^{1/2}} + |\gamma'' \circ s_h|_{H^{1/2}} \|\xi_h^2\|_{C^0} \\
 &\quad + ch^{1/2} \left(\|\gamma'' \circ s_h\|_{C^0} \|\xi_h^2\|_{H^1} + |\gamma'' \circ s_h|_{H^1} \|\xi_h^2\|_{C^0} \right) \\
 &\leq c \|\gamma\|_{C^2} \|\xi_h\|_{C^0} \|\xi_h\|_{H^{1/2}} + c \|\gamma\|_{C^3} \|s_h\|_{H^{1/2}} \|\xi_h\|_{C^0}^2 \\
 &\quad + ch^{1/2} \left(\|\gamma\|_{C^2} \|\xi_h\|_{C^0} \|\xi_h\|_{H^1} + \|\gamma\|_{C^3} \|s_h\|_{H^1} \|\xi_h\|_{C^0}^2 \right) \\
 &\leq c |\log h| \|\gamma\|_{C^3} \|s\|_{C^1} \|\xi_h\|_{H^{1/2}}^2.
 \end{aligned}$$

Substituting the previous estimates in (100), we get

$$\begin{aligned}
 &\left| \left(E_h''(s_h) - E_h''(s_h + \eta_h) \right) (\xi_h, \xi_h) \right| \\
 &\leq c \left(1 + |\log h|^{1/2} \|\eta_h\|_{H^{1/2}} \right) \|\eta_h\|_{H^{1/2}} \|\gamma\|_{C^2} \|s\|_{C^1} \\
 &\quad \times |\log h| \|\gamma\|_{C^3} \|s\|_{C^1} \|\xi_h\|_{H^{1/2}}^2 \\
 &+ c \left(1 + |\log h|^{1/2} \|\eta_h\|_{H^{1/2}} \right) |\log h|^{1/2} \|\eta_h\|_{H^{1/2}} \|\gamma\|_{C^3} \|s\|_{C^1} \|\xi_h\|_{H^{1/2}} \\
 &\quad \times \left(1 + \|\eta_h\|_{H^{1/2}} \right) |\log h|^{1/2} \|\gamma\|_{C^2} \|s\|_{C^1} \|\xi_h\|_{H^{1/2}} \\
 &+ c \left(1 + \|\eta_h\|_{H^{1/2}} \right) \|\gamma\|_{C^1} \|s\|_{C^1} \\
 &\quad \times \left(1 + |\log h|^{1/2} \|\eta_h\|_{H^{1/2}} \right) |\log h| \|\eta_h\|_{H^{1/2}} \|\gamma\|_{C^4} \|s\|_{C^1} \|\xi_h\|_{H^{1/2}}^2 \\
 &\leq c \left(1 + |\log h|^{1/2} \|\eta_h\|_{H^{1/2}} \right)^2 |\log h| \|\eta_h\|_{H^{1/2}} \|\gamma\|_{C^4}^2 \|s\|_{C^1}^2 \|\xi_h\|_{H^{1/2}}^2.
 \end{aligned}$$

□

It follows from the next result that $E_h''(p_h s)$ is nondegenerate with nondegeneracy constant arbitrarily close to λ , provided h is sufficiently small. This will be used to establish the appropriate version of (97) in the proof of Theorem 5.4.

Remark on the Proof. (See also the Remark preceding Proposition 4.3). Nondegeneracy for “interpolants” of functions from the positive and negative spaces for $E''(s)$ must be treated separately—see the separate treatment of I_1 and I_2 below. In the first case, nondegeneracy follows more or less directly from Proposition 4.3. In the second case, the term $\|\nabla(v - v_h)\|_{L^2(D_h)}^2$, where $v = \Phi(\gamma' \circ s \xi_h^{(-)})$ and $v_h = \Phi_h I_h(\gamma' \circ s_h \xi_h^{(-)})$, must be estimated. The main points are to apply the regularity results (34) and Proposition 2.2 to members of the negative space.

Proposition 5.3. *Let $s \in \mathcal{H} \cap C^2$ and $\gamma \in C^3$. Suppose $E''(s)$ is nondegenerate with nondegeneracy constant λ , as in (13). Let ν be as in (12). Then*

$$E''_h(p_h s)(\xi_h, \xi_h^{(+)} - \xi_h^{(-)}) \geq (\lambda - ch^{1/2} |\log h|^{1/2} \|\gamma\|_{C^3}^2 \nu^2 \|s\|_{C^2}^4) \|\xi_h\|_{H^{1/2}}^2$$

for every $\xi_h \in H_h$.

Proof. We again denote $p_h s$ by s_h and split

$$\begin{aligned} E''_h(s_h)(\xi_h, \xi_h^{(+)} - \xi_h^{(-)}) &= E''_h(s_h)(\xi_h^{(+)}, \xi_h^{(+)}) - E''_h(s_h)(\xi_h^{(-)}, \xi_h^{(-)}) \\ (101) \qquad \qquad \qquad &= I_1 + I_2. \end{aligned}$$

It follows from (87), after discarding a positive term, that

$$\begin{aligned} I_1 &\geq E''(s)(\xi_h^{(+)}, \xi_h^{(+)}) - ch^{1/2} |\log h|^{1/2} \|\gamma\|_{C^3}^2 \|s\|_{C^2}^3 \|\xi_h^{(+)}\|_{H^{1/2}}^2 \\ &= E''(s)(\xi_h^+, \xi_h^+) + E''(s)((\xi_h^{(+)}, \xi_h^{(+)}) - (\xi_h^+, \xi_h^+)) \\ &\quad - ch^{1/2} |\log h|^{1/2} \|\gamma\|_{C^3}^2 \|s\|_{C^2}^3 \|\xi_h^{(+)}\|_{H^{1/2}}^2. \end{aligned}$$

Using the notation of (13), there exists $\lambda^+ > 0$ such that

$$E''(s)(\xi_h^+, \xi_h^+) \geq \lambda^+ \|\xi_h^+\|_{H^{1/2}}^2.$$

Also

$$\begin{aligned} &|E''(s)((\xi_h^{(+)}, \xi_h^{(+)}) - (\xi_h^+, \xi_h^+))| \\ &= E''(s)(\xi_h^{(+)} - \xi_h^+, \xi_h^{(+)} + \xi_h^+) \\ &\leq c \|\gamma\|_{C^2}^2 \|s\|_{C^1}^2 \|\xi_h^{(+)} - \xi_h^+\|_{H^{1/2}} \|\xi_h^{(+)} + \xi_h^+\|_{H^{1/2}} \quad \text{by (10)} \\ &\leq ch \|\gamma\|_{C^2}^2 \nu^2 \|s\|_{C^1}^2 \|\xi_h\|_{H^{1/2}}^2 \quad \text{by (36)}. \end{aligned}$$

Using (36) again, I_1 can then be estimated from below by

$$(102) \quad I_1 \geq \lambda^+ \|\xi_h^+\|_{H^{1/2}}^2 - ch^{1/2} |\log h|^{1/2} \|\gamma\|_{C^3}^2 \nu^2 \|s\|_{C^2}^3 \|\xi_h\|_{H^{1/2}}^2.$$

With similar arguments, but now keeping all terms in (87) and using the notation of (13), we get

$$(103) \quad \begin{aligned} I_2 &\geq \lambda^- \|\xi_h^-\|_{H^{1/2}}^2 - ch^{1/2} |\log h|^{1/2} \|\gamma\|_{C^3}^2 \nu^2 \|s\|_{C^2}^3 \|\xi_h\|_{H^{1/2}}^2 \\ &\quad - \|\nabla(v - v_h)\|_{L^2(D_h)}^2, \end{aligned}$$

where, here,

$$(104) \quad v = \Phi(\gamma' \circ s \xi_h^{(-)}), \quad v_h = \Phi_h I_h(\gamma' \circ s_h \xi_h^{(-)}).$$

From (101), (102), (103), and using (13),

$$(105) \quad \begin{aligned} E_h''(s_h)(\xi_h, \xi_h^{(+)} - \xi_h^{(-)}) &\geq \left(\lambda - ch^{1/2} |\log h|^{1/2} \|\gamma\|_{C^3}^2 \nu^2 \|s\|_{C^2}^3 \right) \|\xi_h\|_{H^{1/2}}^2 \\ &\quad - \|\nabla(v - v_h)\|_{L^2(D_h)}^2. \end{aligned}$$

It remains to estimate the last term. For this it is important to notice that ξ_h^- is a smooth function. We have

$$\begin{aligned} &\|\nabla(v - v_h)\|_{L^2(D_h)} \\ &= |\Phi(\gamma' \circ s \xi_h^{(-)}) - \Phi_h I_h(\gamma' \circ s_h \xi_h^{(-)})|_{H^1(D_h)} \\ &\leq |\Phi(\gamma' \circ s (\xi_h^{(-)} - \xi_h^-))|_{H^1(D_h)} + |\Phi_h I_h(\gamma' \circ s_h (\xi_h^{(-)} - \xi_h^-))|_{H^1(D_h)} \\ &\quad + |\Phi(\gamma' \circ s \xi_h^-) - \Phi_h I_h(\gamma' \circ s \xi_h^-)|_{H^1(D_h)} \\ &\quad + |\Phi_h I_h((\gamma' \circ s - \gamma' \circ s_h) \xi_h^-)|_{H^1(D_h)} \\ &= I_3 + I_4 + I_5 + I_6. \end{aligned}$$

But

$$\begin{aligned} I_3 &\leq |\gamma' \circ s (\xi_h^{(-)} - \xi_h^-)|_{H^{1/2}} \\ &\leq \|\gamma' \circ s\|_{C^1} \|\xi_h^{(-)} - \xi_h^-\|_{H^{1/2}} \\ &\leq ch \|\gamma\|_{C^2} \|s\|_{C^1} \nu \|\xi_h\|_{H^{1/2}} \quad \text{by (34)}. \end{aligned}$$

And

$$\begin{aligned} I_4 &\leq |\gamma' \circ s_h (\xi_h^- - \xi_h^{(-)})|_{H^{1/2}} + ch^{1/2} |\gamma' \circ s_h (\xi_h^- - \xi_h^{(-)})|_{H^1} \quad \text{by (65)} \\ &\leq c \|\gamma' \circ s_h\|_{C^1} \left(\|\xi_h^- - \xi_h^{(-)}\|_{H^{1/2}} + h^{1/2} \|\xi_h^- - \xi_h^{(-)}\|_{H^1} \right) \\ &\leq ch \|\gamma\|_{C^2} \|s\|_{C^1} \nu \|\xi_h\|_{H^{1/2}} \quad \text{from (34)}. \end{aligned}$$

Also

$$\begin{aligned} I_5 &\leq ch^{1/2} |\gamma' \circ s \xi_h^-|_{H^1} \quad \text{by (64)} \\ &\leq ch^{1/2} \|\gamma' \circ s\|_{C^1} \|\xi_h^-\|_{H^1} \\ &\leq ch^{1/2} \|\gamma\|_{C^2} \|s\|_{C^1} \nu \|\xi_h\|_{H^{1/2}} \quad \text{by (12)}. \end{aligned}$$

Finally,

$$\begin{aligned} I_6 &\leq c |(\gamma' \circ s - \gamma' \circ s_h) \xi_h^-|_{H^{1/2}} + ch^{1/2} |(\gamma' \circ s - \gamma' \circ s_h) \xi_h^-|_{H^1} \quad \text{by (65)} \\ &\leq c \|\gamma' \circ s - \gamma' \circ s_h\|_{C^{0,1}} \left(\|\xi_h^-\|_{H^{1/2}} + h^{1/2} \|\xi_h^-\|_{H^1} \right) \quad \text{by (49) and (51)} \\ &\leq c \|\gamma\|_{C^3} \|s\|_{C^1} \|s - s_h\|_{C^{0,1}} \nu \|\xi_h\|_{H^{1/2}} \quad \text{by (63) and (12)} \\ &\leq ch \|\gamma\|_{C^3} \|s\|_{C^2}^2 \nu \|\xi_h\|_{H^{1/2}} \quad \text{from (23)}. \end{aligned}$$

If we now put together the estimates for I_3, I_4, I_5 and I_6 , we get

$$(106) \quad \|\nabla(v - v_h)\|_{L^2(D_h)} \leq ch^{1/2} \|\gamma\|_{C^3} \nu \|s\|_{C^2}^2 \|\xi_h\|_{H^{1/2}}.$$

This together with (105) proves the proposition. □

The next result gives the main error estimates for discrete maps s_h which are stationary for E_h . As remarked at the beginning of this section, the proof uses Lemma 5.1. The necessary estimates were established in Propositions 4.2, 5.2 and 5.3.

Theorem 5.4. *Assume $\gamma \in C^4$. Let s be a monotone nondegenerate stationary point for E , with nondegeneracy constant λ as in (13).*

Then there exist positive constants h_0 and c_0 depending on $\|\gamma\|_{C^4}$ and $\|\gamma'\|^{-1}\|_{L^\infty}$, and λ in the case of h_0 , such that if $0 < h \leq h_0$ then there exists $s_h \in \mathcal{H}_h$ which is stationary for E_h and satisfies

$$(107) \quad \|s - s_h\|_{H^{1/2}} \leq c_0 \lambda^{-1} h.$$

Moreover, there exists $\epsilon_0 = \epsilon_0(\|\gamma\|_{C^4}, \|\gamma'\|^{-1}\|_{L^\infty}, \lambda) > 0$ such that s_h is the unique stationary point for E_h satisfying

$$(108) \quad \|s - s_h\|_{H^{1/2}} \leq \epsilon_0 |\log h|^{-1}.$$

Proof. We will apply Lemma 5.1 with $\mathcal{X} = \mathcal{H}_h$, $X = H_h$, $Y = H'_h$ (the dual space of H_h), $f = E'_h$ and $x_0 = p_h s$.

Note that

$$E'_h : \mathcal{H}_h \rightarrow H'_h.$$

From Propositions 4.2 and 2.1, since $E'(s) = 0$,

$$(109) \quad \|E'_h(p_h s)\|_{H'_h} \leq c_1 h$$

where $c_1 = c_1(\|\gamma\|_{C^3}, \|\gamma'\|^{-1}\|_{L^\infty})$.

The derivative $(E'_h)'$ of E'_h is a map

$$(E'_h)' : \mathcal{H}_h \rightarrow L(H_h, H'_h)$$

and is naturally identified in the usual way with E''_h via

$$(110) \quad \langle \langle (E'_h)'(t_h), \xi_h \rangle, \eta_h \rangle = E''_h(t_h)(\xi_h, \eta_h)$$

for all $t_h \in \mathcal{H}_h$ and $\xi_h, \eta_h \in \mathcal{H}_h$. From Propositions 5.3, 2.1 and 2.2, since $h^{1/2} |\log h|^{1/2} \rightarrow 0$ as $h \rightarrow 0$,

$$(111) \quad E''_h(p_h s)(\xi_h, \xi_h^{(+)} - \xi_h^{(-)}) \geq \frac{3\lambda}{4} \|\xi_h\|_{H^{1/2}}^2$$

for all $\xi_h \in H_h$, provided $0 < h \leq h_0 = h_0(\|\gamma\|_{C^4}, \|\gamma'\|^{-1}\|_{L^\infty}, \lambda)$. But

$$(112) \quad \begin{aligned} \|\xi_h^{(+)} - \xi_h^{(-)}\|_{H^{1/2}} &\leq \|\xi_h^+ - \xi_h^-\|_{H^{1/2}} + \|\xi_h^+ - \xi_h^{(+)}\|_{H^{1/2}} + \|\xi_h^- - \xi_h^{(-)}\|_{H^{1/2}} \\ &\leq (1 + ch\nu) \|\xi_h\|_{H^{1/2}} \end{aligned}$$

from (36) and (34), and since $\|\xi_h^+ - \xi_h^-\|_{H^{1/2}} = \|\xi_h\|_{H^{1/2}}$. Hence from (110), (111) and (112),

$$(113) \quad \langle \langle (E'_h)'(p_h s), \xi_h \rangle, \xi_h^{(+)} - \xi_h^{(-)} \rangle \geq \frac{\lambda}{2} \|\xi_h\|_{H^{1/2}} \|\xi_h^{(+)} - \xi_h^{(-)}\|_{H^{1/2}}$$

for all $\xi \in H_h$, provided $0 < h \leq h_0$ for a new h_0 with the same dependencies as before. Thus for each $\xi_h \in H_h$ with $\|\xi_h\|_{H^{1/2}} = 1$, the map $\langle \langle (E'_h)'(p_h s), \xi_h \rangle, \xi_h \rangle$ ($\in H'_h$) has norm $\geq \lambda/2$. It follows, since H_h and H'_h have equal finite dimension, that $(E'_h)'(p_h s)$ is invertible and

$$(114) \quad \left\| \langle \langle (E'_h)'(p_h s) \rangle^{-1} \right\|_{L(H'_h, H_h)} \leq \left(\frac{\lambda}{2} \right)^{-1}.$$

Next note that from (110) and Proposition 5.2, by the symmetry of the second derivative,

$$\begin{aligned} & \| (E'_h)'(p_h s) - (E'_h)'(p_h s + \eta_h) \|_{L(H'_h, H_h)} \\ & \leq c_2 \left(1 + |\log h|^{1/2} \|\eta_h\|_{H^{1/2}} \right)^2 |\log h| \|\eta_h\|_{H^{1/2}} \end{aligned}$$

for all $\eta_h \in H_h$, where $c_2 = c_2(\|\gamma\|_{C^4}, \|\gamma'\|^{-1}_{L^\infty})$ from Proposition 2.1.

We can now choose $\epsilon_0 = \epsilon_0(\|\gamma\|_{C^4}, \|\gamma'\|^{-1}_{L^\infty}, \lambda)$ so that $|\log h| \|\eta_h\|_{H^{1/2}} < \epsilon_0$ implies

$$(115) \quad \| (E'_h)'(p_h s) - (E'_h)'(p_h s + \eta_h) \|_{L(H'_h, H_h)} \leq \frac{\lambda}{4}.$$

By further restricting h_0 , again with the same dependencies, we can ensure that, see (109),

$$(116) \quad 0 < h \leq h_0 \quad \text{implies} \quad c_1 h \leq \frac{\lambda}{4} \epsilon_0 |\log h|^{-1}.$$

From (109), (114), (115), (116) and Lemma 5.1 with $\alpha = \lambda/2$, $\beta = \lambda/4$, $\delta = c_1 h$ and $\epsilon = \epsilon_0 |\log h|^{-1}$, it follows that for $0 < h \leq h_0$ there is a unique $s_h \in \mathcal{H}_h$ which is stationary for E_h and such that

$$(117) \quad \|s_h - p_h s\|_{H^{1/2}} \leq \epsilon_0 |\log h|^{-1}.$$

Next apply Lemma 5.1 with $\alpha = \lambda/2$, $\beta = \lambda/4$, $\delta = c_1 h$ and $\epsilon = 4c_1 h/\lambda$. Note that $\delta = (\alpha - \beta)\epsilon (= c_1 h)$. Moreover, if $0 < h \leq h_0$ then $\epsilon \leq \epsilon_0 |\log h|^{-1}$ from (116). It follows that the unique stationary s_h as in (117) satisfies

$$(118) \quad \|s_h - p_h s\|_{H^{1/2}} \leq \frac{c_0}{\lambda} h,$$

where $c_0 = 4c_1$.

Since

$$\|s - p_h s\|_{H^{1/2}} \leq ch^{3/2} \|s\|_{H^2} \leq ch^{3/2} \|s\|_{C^2} \leq ch^{3/2} \|\gamma\|_{C^3},$$

we may replace $p_h s$ by s in (117) and (118), after further restricting h_0 , ϵ_0 and c_0 if necessary. □

Remark. Suppose $u = \Phi(\gamma \circ s): D \rightarrow \mathfrak{R}^n$ is harmonic. Then

$$\begin{aligned} \|u\|_{H^1(D \setminus D_h)} & \leq ch \|u\|_{H^1(\partial D)} \quad \text{from (70) and (71)} \\ & \leq ch \|\gamma\|_{C^1} \|s\|_{C^1} \quad \text{from (57) and (59)} \\ & \leq ch, \end{aligned}$$

where $c = c(\|\gamma\|_{C^2})$. Thus the contribution to $\|u\|_{H^1(D)}$ from the boundary strip $D \setminus D_h$ is $O(h)$. This is consistent with the order of approximation in the following theorem.

Theorem 5.5. *Assume $\gamma \in C^4$. Let u be a nondegenerate minimal surface spanning Γ with nondegeneracy constant λ as in (13).*

Then there exist positive constants h_0 and c_0 depending on $\|\gamma\|_{C^4}$ and $\|\gamma'\|^{-1}_{L^\infty}$, and λ in the case of h_0 , such that if $0 < h \leq h_0$, then there is a discrete minimal surface u_h satisfying

$$(119) \quad \|u - u_h\|_{H^1(D_h)} \leq c_0 \lambda^{-1} h.$$

Moreover, there exists $\epsilon_0 = \epsilon_0(\|\gamma\|_{C^4}, \|\gamma'\|^{-1}\|_{L^\infty}, \lambda) > 0$ such that if $u = \Phi(\gamma \circ s)$ and $u_h = \Phi_h I_h(\gamma \circ s_h)$, then u_h is the unique discrete minimal surface satisfying

$$(120) \quad \|s - s_h\|_{H^{1/2}} \leq \epsilon_0 |\log h|^{-1}.$$

Proof. Let

$$u = \Phi(\gamma \circ s), \quad u_h = \Phi_h I_h(\gamma \circ s_h),$$

where s and s_h are as in Proposition 5.4. We begin with h_0, ϵ_0 and c_0 as in Proposition 5.4.

Now

$$\begin{aligned} & |u - u_h|_{H^1(D_h)} \\ &= |\Phi(\gamma \circ s) - \Phi_h I_h(\gamma \circ s_h)|_{H^1(D_h)} \\ &\leq |\Phi(\gamma \circ s) - \Phi_h I_h(\gamma \circ p_h s)|_{H^1(D_h)} + |\Phi_h I_h(\gamma \circ p_h s - \gamma \circ s_h)|_{H^1(D_h)} \\ (121) \quad &= A + B. \end{aligned}$$

But

$$(122) \quad A \leq ch \|\gamma\|_{C^2} \|s\|_{C^2}^2,$$

by the estimate (77) for I_1 in the proof of Proposition 4.1.

Also,

$$B \leq c \left(1 + |\log h|^{1/2} \|s_h - p_h s\|_{H^{1/2}} \right) \|s_h - p_h s\|_{H^{1/2}} \|\gamma\|_{C^2} \|s\|_{C^1}$$

by the estimate for A_1 in the proof of Proposition 5.2, with s_h and η_h there replaced by $p_h s$ and $s_h - p_h s$, respectively. Hence

$$(123) \quad B \leq \frac{c_0}{\lambda} h \quad \text{from (118)}$$

provided $0 < h \leq h_0$, for a new h_0 and c_0 with the same dependencies.

It follows from (121), (122) and (123), possibly again with new c_0 and h_0 with the same dependencies as before, that

$$(124) \quad |u - u_h|_{H^1(D_h)} \leq \frac{c_0}{\lambda} h$$

if $0 < h \leq h_0$.

Now

$$(125) \quad \|u - u_h\|_{H^1(D_h)} \leq c \left(|u - u_h|_{H^1(D_h)} + \|u - u_h\|_{L^2(\partial D_h)} \right),$$

as follows easily by integrating along rays (c depends only on n). It is routine to estimate $\|u - u_h\|_{L^2(\partial D_h)}$. Let

$$\tilde{u}_h = \Phi_h I_h(\gamma \circ p_h s).$$

Then

$$\begin{aligned} & \|u - u_h\|_{L^2(\partial D_h)} \leq \|u - \tilde{u}_h\|_{L^2(\partial D_h)} + \|\tilde{u}_h - u_h\|_{L^2(\partial D_h)} \\ & \leq ch^2 \|\gamma\|_{C^2} \|s\|_{C^2}^2 + \|\tilde{u}_h - u_h\|_{L^2(\partial D_h)} \\ (126) \quad & = ch^2 \|\gamma\|_{C^2} \|s\|_{C^2}^2 + D \end{aligned}$$

from the argument used to estimate I_2 in the proof of Proposition 4.1, with u_h there replaced by \tilde{u}_h . Moreover,

$$\begin{aligned}
 D &= \|I_h(\gamma \circ p_h s - \gamma \circ s_h)\|_{L^2(\partial D_h)} \\
 &\leq \|I_h^{\partial D}(\gamma \circ p_h s - \gamma \circ s_h)\|_{L^2(\partial D)} \quad \text{by (21) as } |\nabla\pi| \leq 1 \\
 &\leq \|\gamma \circ p_h s - \gamma \circ s_h\|_{L^2} + ch\|\gamma \circ p_h s - \gamma \circ s_h\|_{H^1} \\
 &\leq c\|\gamma\|_{C^1}\|p_h s - s_h\|_{L^2} + ch\|\gamma\|_{C^2}\|s\|_{C^1}\|p_h s - s_h\|_{H^1} \\
 &\quad \text{by (60), (62) and (23)} \\
 &\leq c\|\gamma\|_{C^2}\|s\|_{C^1}\|p_h s - s_h\|_{L^2} \quad \text{by a standard inverse estimate} \\
 (127) \quad &\leq \frac{c_0}{\lambda}h \quad \text{from (118)}
 \end{aligned}$$

with a new c_0 but with the same dependencies, for $0 < h \leq h_0$. From (124), (125), (126) and (127) it follows that if $0 < h \leq h_0$ then

$$\|u - u_h\|_{H^1(D_h)} \leq \frac{c_0}{\lambda}h,$$

where $c_0 = c_0(\|\gamma\|_{C^3}, \|\gamma'\|^{-1}\|_{L^\infty})$ and $h_0 = h_0(\|\gamma\|_{C^3}, \|\gamma'\|^{-1}\|_{L^\infty}, \lambda)$. \square

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