MATHEMATICS OF COMPUTATION
Volume 68, Number 227, Pages 1067–1078
S 0025-5718(99)01059-5
Article electronically published on February 5, 1999

ERROR ESTIMATION OF HERMITE SPECTRAL METHOD FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Hermite approximation is investigated. Some inverse inequalities, imbedding inequalities and approximation results are obtained. A Hermite spectral scheme is constructed for Burgers equation. The stability and convergence of the proposed scheme are proved strictly. The techniques used in this paper are also applicable to other nonlinear problems in unbounded domains.

1. INTRODUCTION

A number of physical problems are set in unbounded domains. Some conditions at infinity are given by certain asymptotic behaviors for solutions. When we use the finite difference method or the finite element method to solve such problems numerically, we often restrict calculations to some bounded domains, and impose certain conditions on artificial boundaries. They cause numerical errors usually. If we use spectral methods associated with some orthogonal systems in unbounded domains, then the above troubles could be avoided. While the spectral methods provide numerical solutions with high accuracies. Maday, Pernaud-Thomas and Vandeven [1], Coulaud, Funaro and Kavian [2], and Funaro [3] used the Laguerre spectral method for several linear partial differential equations. Iranzo and Falquès [4] provided some Laguerre pseudospectral schemes and Laguerre tau schemes. Mavriplis [5] and Black [6] developed the Laguerre spectral element method. Also, Funaro and Kavian [7], and Weideman [8] considered the Hermite spectral method and the Hermite pseudospectral method. In particular, Funaro and Kavian [7] proved the convergence of a spectral scheme using the Hermite functions for some linear problems. But so far, there is no paper concerning error estimates of the Hermite spectral method using Hermite polynomials. Another spectral method for partial differential equations in unbounded domains is based on the rational basis functions, see Christov [9], Boyd [10], Iranzo and Falquès [4], and Weideman [8]. The purpose of this paper is to study spectral approximation using Hermite polynomials and their applications to nonlinear problems. Some inverse inequalities, imbedding inequalities and approximation results are given, which play important roles in analysis of the Hermite spectral method. We use the Burgers equation as an example showing how to construct Hermite spectral schemes for nonlinear problems. The generalized stability and the convergence of the proposed scheme are proved

Received by the editor October 16, 1997 and, in revised form, January 2, 1998.
1991 *Mathematics Subject Classification.* Primary 65N30, 76D99.
Key words and phrases. Hermite approximation, Burgers equation, error estimations.

strictly. The main idea and techniques used in this paper are also applicable to various nonlinear problems arising in fluid dynamics, quantum mechanics and other fields.

2. Hermite approximation

Let $\Lambda = \{x \mid -\infty < x < \infty\}$ and $\omega(x) = e^{-x^2}$. For $1 \le p \le \infty$, set

$$L^p_\omega(\Lambda) = \{v \mid v \text{ is measurable and } \| v \|_{L^p_\omega(\Lambda)} < \infty\},$$

where

$$\| v \|_{L^p_\omega(\Lambda)} = \begin{cases} \left(\int_\Lambda |v(x)|^p \omega(x)\mathrm{d}x\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{x\in\Lambda} |v(x)|, & p = \infty. \end{cases}$$

In particular, $L^2_\omega(\Lambda)$ is a Hilbert space with the inner product

$$(u, v)_{L^2_\omega(\Lambda)} = \int_\Lambda u(x)v(x)w(x)\mathrm{d}x.$$

Further, let $\partial_x v = \frac{\partial v}{\partial x}$, and for any non-negative integer m,

$$H^m_\omega(\Lambda) = \{v \mid \partial^k_x v \in L^2_\omega(\Lambda), 0 \le k \le m\}.$$

The semi-norm and the norm of $H^m_\omega(\Lambda)$ are given by

$$|v|_{H^m_\omega(\Lambda)} = \| \partial^m_x v \|_{L^2_\omega(\Lambda)}, \quad \| v \|_{H^m_\omega(\Lambda)} = \left(\sum_{k=0}^m |v|^2_{H^k_\omega(\Lambda)}\right)^{\frac{1}{2}}.$$

For any real $r \ge 0$, we define the space $H^r_\omega(\Lambda)$ with the norm $\| v \|_{H^r_\omega(\Lambda)}$ by the space interpolation as in Adams [11]. For simplicity, we denote the inner product $(u, v)_{L^2_\omega(\Lambda)}$, the semi-norm $|v|_{H^r_\omega(\Lambda)}$, the norms $\| v \|_{H^r_\omega(\Lambda)}$ and $\| v \|_{L^p_\omega(\Lambda)}$, by $(u, v)_\omega$, $|v|_{r,\omega}$, $\| v \|_{r,\omega}$ and $\| v \|_{L^p_\omega}$, respectively. In particular, $\| v \|_\omega = \| v \|_{0,\omega}$. Besides, let c denote a generic positive constant in this paper.

The Hermite polynomial of degree l is defined by

$$H_l(x) = (-1)^l e^{x^2} \partial^l_x (e^{-x^2}).$$

It is the l-th eigenfunction of a singular Liouville problem

$$\partial_x(e^{-x^2}\partial_x v(x)) + \lambda e^{-x^2} v(x) = 0, \ x \in \Lambda. \tag{2.1}$$

The corresponding eigenvalue $\lambda_l = 2l$. Clearly $H_0(x) = 1$ and $H_1(x) = 2x$. The Hermite polynomials satisfy the recurrence relations

$$H_{l+1}(x) - 2xH_l(x) + 2lH_{l-1}(x) = 0, \ l \ge 1, \tag{2.2}$$

and

$$\partial_x H_l(x) = 2lH_{l-1}(x), \ l \ge 1. \tag{2.3}$$

The set of Hermite polynomials is an orthogonal system with the weight function $\omega(x)$ on the whole line Λ, namely,

$$\int_\Lambda H_l(x)H_m(x)\omega(x)\mathrm{d}x = 2^l l! \sqrt{\pi}\delta_{l,m}. \tag{2.4}$$

By (2.3), the set of $\partial_x H_l(x)$ is also an orthogonal system with the same weight, i,e.,

$$\int_\Lambda \partial_x H_l(x)\partial_x H_m(x)\omega(x)\mathrm{d}x = 2^{l+1} l l! \sqrt{\pi}\delta_{l,m}. \tag{2.5}$$

For any function $v \in L^2_\omega(\Lambda)$,

$$v(x) = \sum_{l=0}^{\infty} \hat{v}_l H_l(x),$$

where $\hat{v}_l$ is the Hermite coefficient,

$$\hat{v}_l(x) = \frac{1}{2^l l! \sqrt{\pi}} \int_\Lambda v(x) H_l(x) \omega(x) \mathrm{d}x, \ l \geq 0.$$

We now consider the Hermite approximation. Let N be any positive integer and $\mathcal{P}_N$ be the set of polynomials of degree at most N. In numerical analysis of the Hermite spectral method, we need some inverse inequalities. The first is due to Nessel and Wilmes [12], stated in the following lemma.

Lemma 2.1. *For any $\phi \in \mathcal{P}_N$ and $1 \leq p \leq q \leq \infty$,*

$$\| \phi \|_{L^q_\omega} \leq c N^{\frac{5}{6}(\frac{1}{p} - \frac{1}{q})} \| \phi \|_{L^p_\omega} .$$

The next lemma gives another inverse inequality.

Lemma 2.2. *For any $\phi \in \mathcal{P}_N$,*

$$|\phi|_{1,\omega} \leq \sqrt{2N} \| \phi \|_\omega .$$

Proof. By (2.3),

$$\partial_x \phi(x) = 2 \sum_{l=1}^{N} l \hat{\phi}_l H_{l-1}(x).$$

Thus (2.4) leads to

$$|\phi|^2_{1,\omega} \leq 4 \sum_{l=0}^{N-1} 2^l (l+1)^2 l! \sqrt{\pi} \hat{\phi}^2_{l+1} \leq 2N \| \phi \|^2_\omega .$$

Some imbedding inequalities are useful in numerical analysis of the Hermite spectral method. We list two of them.

Lemma 2.3. *For any $v \in H^1_\omega(\Lambda)$,*

$$\| xv \|_\omega \leq \| v \|_{1,\omega} .$$

Proof. Integrating by parts, we obtain that

$$\int_\Lambda x v^2(x) \omega(x) \mathrm{d}x = \int_\Lambda v(x) \partial_x v(x) \omega(x) \mathrm{d}x \leq \| v \|_\omega |v|_{1,\omega}.$$

Thus $xv^2(x)\omega(x) \to 0$ as $|x| \to \infty$. By integrating by parts and the Cauchy inequality,

$$\begin{aligned} \| xv \|^2_\omega &= \frac{1}{2} \int_\Lambda v^2(x) \omega(x) \mathrm{d}x + \int_\Lambda x v(x) \partial_x v(x) \omega(x) \mathrm{d}x \\ &\leq \frac{1}{2} \| v \|^2_\omega + \frac{1}{2} \| xv \|^2_\omega + \frac{1}{2} |v|^2_{1,\omega} \\ &= \frac{1}{2} \| v \|^2_{1,\omega} + \frac{1}{2} \| xv \|^2_\omega . \end{aligned}$$

So the desired result follows.

Lemma 2.4. *If $v \in H^1_\omega(\Lambda)$, then for any $x \in \Lambda$,*

$$|v(x)| \le e^{\frac{x^2}{2}} \parallel v \parallel_\omega^{\frac{1}{2}} (|v|_{1,\omega} + \parallel v \parallel_{1,\omega})^{\frac{1}{2}}.$$

Moreover,

$$\parallel e^{-\frac{x^2}{2}} v \parallel^4_{L^4_\omega} \le 16\sqrt{\pi} \parallel v \parallel^2_\omega \parallel v \parallel^2_{1,\omega} .$$

Proof. We have

$$\begin{aligned} e^{-x^2} v^2(x) &= \int_{-\infty}^x \partial_y (e^{-y^2} v^2(y)) \mathrm{d}y \\ &= 2\int_{-\infty}^x v(y)\partial_y v(y)\omega(y)\mathrm{d}y - 2\int_{-\infty}^x y v^2(y)\omega(y)\mathrm{d}y. \end{aligned}$$

By Lemma 2.3 and the Cauchy inequality,

$$e^{-x^2} v^2(x) \le 2 \parallel v \parallel_\omega (|v|_{1,\omega} + \parallel v \parallel_{1,\omega}).$$

This leads to the first conclusion. Moreover,

$$\parallel e^{-\frac{x^2}{2}} v \parallel^4_{L^4_\omega} \le 16 \parallel v \parallel^2_\omega \parallel v \parallel^2_{1,\omega} \int_\Lambda \omega(x)\mathrm{d}x = 16\sqrt{\pi} \parallel v \parallel^2_\omega \parallel v \parallel^2_{1,\omega} .$$

The proof is completed.

The $L^2_\omega(\Lambda)$-orthogonal projection $P_N : L^2_\omega(\Lambda) \to \mathcal{P}_N$ is such a mapping that for any $v \in L^2_\omega(\Lambda)$,

$$(v - P_N v, \phi)_\omega = 0, \ \forall \phi \in \mathcal{P}_N,$$

or equivalently,

$$P_N v(x) = \sum_{l=0}^{N} \hat{v}_l H_l(x).$$

Lemma 2.5. *For any $v \in H^r_\omega(\Lambda)$ and $r \ge 0$,*

$$\parallel v - P_N v \parallel_\omega \le cN^{-\frac{r}{2}} \parallel v \parallel_{r,\omega} .$$

Proof. We have from (2.4) that

$$\parallel v - P_N v \parallel^2_\omega = \sqrt{\pi} \sum_{l=N+1}^{\infty} 2^l l! \hat{v}_l^2.$$

According to (2.1), we define the operator A by

$$Av(x) = -e^{x^2}\partial_x(e^{-x^2}\partial_x v(x)) = -\partial_x^2 v(x) + 2x\partial_x v(x).$$

By Lemma 2.3, A is a continuous mapping from $H^{\beta+2}_\omega(\Lambda)$ into $H^\beta_\omega(\Lambda)$, where β is any non-negative integer. When r is an even integer, we have from (2.1) and integrating by parts that

$$\begin{aligned} &\int_\Lambda v(x)H_l(x)\omega(x)\mathrm{d}x \\ &\quad = \frac{1}{2l}\int_\Lambda Av(x)H_l(x)\omega(x)\mathrm{d}x = \cdots = (2l)^{-\frac{r}{2}} \int_\Lambda A^{\frac{r}{2}} v(x)H_l(x)\omega(x)\mathrm{d}x. \end{aligned}$$

Thus

$$|\hat{v}_l| = (2l)^{-\frac{r}{2}} \parallel H_l \parallel_\omega^{-2} | \int_\Lambda A^{\frac{r}{2}} v(x)H_l(x)\omega(x)\mathrm{d}x|. \tag{2.6}$$

Hence

$$\begin{aligned}\| v - P_N v \|_\omega^2 &\le (2N)^{-r} \sum_{l=N+1}^{\infty} \| H_l \|_\omega^{-2} \, | \int_\Lambda A^{\frac{r}{2}} v(x) H_l(x)\omega(x) \mathrm{d}x|^2 \\ &= (2N)^{-r} \| A^{\frac{r}{2}} v \|_\omega^2 \le cN^{-r} \| v \|_{r,\omega}^2 .\end{aligned}$$

When r is an odd integer, we obtain that

$$\begin{aligned}\int_\Lambda v(x)H_l(x)\omega(x)\mathrm{d}x &= (2l)^{-\frac{r-1}{2}} \int_\Lambda A^{\frac{r-1}{2}} v(x)H_l(x)\omega(x)\mathrm{d}x \\ &= (2l)^{-\frac{r+1}{2}} \int_\Lambda \partial_x (A^{\frac{r-1}{2}} v(x)) \partial_x H_l(x)\omega(x)\mathrm{d}x.\end{aligned}$$

Finally, we derive from (2.4) and (2.5) that

$$\begin{aligned}\| v - P_N v \|_\omega^2 &\le \frac{1}{\sqrt{\pi}} \sum_{l=N+1}^{\infty} (2^{l+r+1} l^{r+1} l!)^{-1} (\int_\Lambda \partial_x (A^{\frac{r-1}{2}} v(x)) \partial_x H_l(x)\omega(x)\mathrm{d}x)^2 \\ &\le (2N)^{-r} \| \partial_x (A^{\frac{r-1}{2}} v) \|_\omega^2 \le cN^{-r} \| v \|_{r,\omega}^2 .\end{aligned}$$

Theorem 2.1. *For any $v \in H_\omega^r(x)$ and $0 \le \mu \le r$,*

$$\| v - P_N v \|_{\mu,\omega} \le cN^{\frac{\mu}{2}-\frac{r}{2}} \| v \|_{r,\omega} .$$

Proof. We first consider the case with integer μ. We shall use the induction. Obviously Lemma 2.5 implies the desired result for $\mu = 0$. Assume that it is true for $\mu - 1$. Then

$$\| v - P_N v \|_{\mu,\omega} \le \| v - P_N v \|_\omega + \| \partial_x v - P_N \partial_x v \|_{\mu-1,\omega} + \| P_N \partial_x v - \partial_x P_N v \|_{\mu-1,\omega} .$$

We know from Lemma 2.5 that

$$\| \partial_x v - P_N \partial_x v \|_{\mu-1,\omega} \le cN^{\frac{\mu-r}{2}} \| \partial_x v \|_{r-1,\omega} \le cN^{\frac{\mu-r}{2}} \| v \|_{r,\omega} .$$

On the other hand, (2.3) leads to

$$P_N \partial_x v - \partial_x P_N v = 2(N+1)\hat{v}_{N+1} H_N(x).$$

Using Lemma 2.2 and (2.4), we get that

$$\| H_N \|_{\mu-1,\omega}^2 \le c 2^N N^{\mu-1} N!.$$

Moreover by (2.6),

$$|\hat{v}_{N+1}|^2 \le c(2^{N+1} N^r (N+1)!)^{-1} \| v \|_{r,\omega}^2 .$$

Therefore

$$\| P_N \partial_x v - \partial_x P_N v \|_{\mu-1,\omega}^2 \le cN^{\mu-r} \| v \|_{r,\omega}^2 .$$

So the induction is completed. The previous results with space interpolation lead to the conclusion for any $r \ge 0$.

In order to obtain the optimal error estimation in the Hermite spectral method for partial differential equations, we need the $H_\omega^1(\Lambda)$-orthogonal projection $P_N^1 : H_\omega^1(\Lambda) \to \mathcal{P}_N$. It means that for any $v \in H_\omega^1(\Lambda)$,

$$(2.7) \qquad (\partial_x(v - P_N^1 v), \partial_x \phi)_\omega = 0, \ \forall \phi \in \mathcal{P}_N.$$

Let $\hat{v}_l$ be the coefficients of the Hermite expansion for $v(x)$, and

$$P_N^1 v(x) = \sum_{l=0}^{N} \hat{a}_l H_l(x).$$

By (2.3),

$$\partial_x P_N^1 v(x) = 2\sum_{l=0}^{N-1}(l+1)\hat{a}_{l+1}H_l(x).$$

Similarly

$$\partial_x v(x) = 2\sum_{l=0}^{\infty}(l+1)\hat{v}_{l+1}H_l(x).$$

By (2.3) and (2.7), we know that $\hat{a}_l = \hat{v}_l$ for $0 \le l \le N$. Thus the projection P_N^1 is exactly the same as P_N.

3. Application to Burgers equation on the whole line

In this section, we consider the Hermite spectral method for Burgers equation on the whole line. We first change it to a new representation by the similarity transformation, which is suitable for the Hermite approximation. We shall prove the stability and the convergence of the designed scheme strictly.

Let $\tilde{\Lambda} = \{y \mid -\infty < y < \infty\}$ and $\mu > 0$ be the kinetic viscosity, while $g(y,s)$ and $V_0(y,s)$ are the source term and the initial value, respectively. T is a fixed positive number. We consider the following problem

$$\tag{3.1} \begin{cases} \partial_s V + \frac{1}{2}\partial_y(V^2) - \mu\partial_y^2 V = g, & y \in \tilde{\Lambda}, 0 < s \le T, \\ V(y,0) = V_0(s), & y \in \tilde{\Lambda}. \end{cases}$$

In addition, V and $\partial_y V$ satisfy certain conditions at infinity. Let

$$a_\omega(u,v) = \int_{\tilde{\Lambda}} \partial_y u(y)\partial_y(v(y)\omega(y))\mathrm{d}y.$$

A weak formulation of (3.1) is to find $v \in L^2(0,T;H_\omega^1(\tilde{\Lambda})) \cap L^\infty(0,T;L_\omega^2(\tilde{\Lambda}))$ such that

$$\begin{cases} (\partial_s V(s), v)_{L_\omega^2(\tilde{\Lambda})} - \frac{1}{2}(V^2(s), \partial_y(v(s)\omega))_{L^2(\tilde{\Lambda})} & \\ \qquad +\mu a_\omega(V(s), v) = (g,v)_{L_\omega^2(\tilde{\Lambda})}, & \forall v \in H_\omega^1(\tilde{\Lambda}),\ 0 < s \le T, \\ V = V_0, & s = 0. \end{cases}$$

It can be checked that

$$\begin{aligned} a_\omega(v,v) &= \| \partial_y v \|_{L_\omega^2(\tilde{\Lambda})}^2 - 2\int_{\Lambda} y v(y)\partial_y v(y)\omega(y)\mathrm{d}y \\ &= \| \partial_y v \|_{L_\omega^2(\tilde{\Lambda})}^2 + \| v \|_{L_\omega^2(\tilde{\Lambda})}^2 - 2\int_{\tilde{\Lambda}} y^2 v^2(y)\omega(y)\mathrm{d}y. \end{aligned}$$

It is not clear whether the bilinear form $a_\omega(v,v)$ is non-negative or not. Thus the above weak formulation is not suitable for the Hermite spectral method. To remedy this trouble, we try to reform it. Let $W(x,t) = V(y,s), \tilde{g}(x,t) = g(y,s)$ and make the similarity transformation

$$\tag{3.2} x = \frac{y}{2\sqrt{\mu(1+s)}},\quad t = \ln(1+s).$$

Then (3.1) becomes

(3.3)

$$\begin{cases} \partial_t W - \frac{1}{2}x\partial_x W + \frac{1}{4\sqrt{\mu}}e^{\frac{t}{2}}\partial_x(W^2) - \frac{1}{4}\partial_x^2 W = e^t\tilde{g}, & x \in \Lambda, 0 < t \le \ln(1+T), \\ W = W_0, & t = 0. \end{cases}$$

Further let $U = e^{x^2}W$ and $f = e^{x^2+t}\tilde{g}$. Then we obtain the following problem

(3.4)

$$\begin{cases} \partial_t U + \frac{1}{2}U + \frac{1}{2}x\partial_x U \\ \qquad + \frac{1}{4\sqrt{\mu}}e^{x^2+\frac{t}{2}}\partial_x(e^{-2x^2}U^2) - \frac{1}{4}\partial_x^2 U = f, & x \in \Lambda, 0 < t \le \ln(1+T), \\ U = U_0, & t = 0. \end{cases}$$

In addition, U and $\partial_x U$ satisfy some conditions as $|x| \to \infty$. Let

$$B(u, z, v) = -\frac{1}{4\sqrt{\mu}}e^{\frac{t}{2}}(e^{-x^2}uz, \partial_x v)_\omega.$$

The weak formulation of (3.4) is to find $U \in L^2(0, \ln(1+T); H^1_\omega(\Lambda)) \cap L^\infty(0, \ln(1+T); L^2_\omega(\Lambda))$ such that

(3.5)

$$\begin{cases} (\partial_t U(t), v)_\omega + \frac{1}{2}(U(t), v)_\omega + B(U(t), U(t), v) \\ \qquad + \frac{1}{4}(\partial_x U(t), \partial_x v)_\omega = (f(t), v)_\omega, & \forall v \in H^1_\omega(\Lambda), 0 < t \le \ln(1+T), \\ U = U_0, & t = 0. \end{cases}$$

As in Maday, Pernaud-Thomas and Vandeven [1], we suppose that V_0 and g fulfill some conditions such that for certain $\alpha \ge 0$,

$$\lim_{|y|\to\infty} e^{\alpha y^2}(|V(y,s)| + |\partial_y V(y,s)|) = 0, \ 0 \le s \le T.$$

Then

$$\lim_{|x|\to\infty} e^{4\alpha\mu e^t x^2}(|W(x,t)| + |\partial_x W(x,t)|) = 0, \ 0 \le t \le \ln(1+T)$$

and so

$$\lim_{|x|\to\infty} e^{(4\alpha\mu e^t - 1)x^2}(|U(x,t)| + |\partial_x U(x,t)|) = 0, \ 0 \le t \le \ln(1+T).$$

If $\alpha > \frac{1}{8\mu}$, then we have that for all $t \ge 0$, $4\alpha\mu e^t - 1 > -\frac{1}{2}$. By Lemma 2.3, $U \in H^1_\omega(\Lambda)$ and so we can use the Hermite approximation for (3.5).

The Hermite spectral scheme for (3.5) is to find $u_N(t) \in \mathcal{P}_N$ for $0 \le t \le \ln(1+T)$, such that

(3.6)

$$\begin{cases} (\partial_t u_N(t), \phi)_\omega + \frac{1}{2}(u_N(t), \phi)_\omega + B(u_N(t), u_N(t), \phi) \\ \qquad + \frac{1}{4}(\partial_x u_N(t), \partial_x \phi)_\omega = (f(t), \phi)_\omega, & \forall \phi \in \mathcal{P}_N, \ 0 < t \le \ln(1+T), \\ u_N = u_{N,0} = P_N U_0, & t = 0. \end{cases}$$

We now consider the stability of (3.6). Since (3.6) is nonlinear, it is not possible to prove the stability in the sense of Courant, Friedrichs and Lewy [13]. But it will be shown that it is still stable in the sense of Guo [14, 15] and Stetter [16].

To do this, we assume that f and $u_{N,0}$ have the errors $\tilde{f}$ and $\tilde{u}_{N,0}$, respectively. They induce the error of numerical solution u_N, denoted by $\tilde{u}_N$. Then we get the following equation:

(3.7)
$$\begin{cases} (\partial_t \tilde{u}_N(t), \phi)_\omega + \frac{1}{2}(\tilde{u}_N(t), \phi)_\omega + B(\tilde{u}_N(t), \tilde{u}_N(t), \phi) + 2B(\tilde{u}_N(t), u_N(t), \phi) \\ \qquad + \frac{1}{4}(\partial_x \tilde{u}_N(t), \partial_x \phi)_\omega = (\tilde{f}(t), \phi)_\omega, \ \forall \phi \in \mathcal{P}_N, \ 0 < t \le \ln(1+T), \\ \tilde{u}_N = \tilde{u}_{N,0}, \ t = 0. \end{cases}$$

By taking $\phi = 2\tilde{u}_N$ in (3.7), it follows that

$$\begin{aligned} &\frac{d}{dt} \| \tilde{u}_N(t) \|_\omega^2 + \frac{1}{2} \| \tilde{u}_N(t) \|_{1,\omega}^2 \\ &\qquad + 2B(\tilde{u}_N(t), \tilde{u}_N(t), \tilde{u}_N(t)) + 4B(\tilde{u}_N(t), u_N(t), \tilde{u}_N(t)) \\ &\quad \le 2 \| \tilde{f}(t) \|_\omega^2 . \end{aligned} \tag{3.8}$$

Using Lemma 2.4, we deduce that for $0 \le t \le \ln(1+T)$,

(3.9)
$$\begin{aligned} |2B(\tilde{u}_N(t), \tilde{u}_N(t), \tilde{u}_N(t))| &\le \frac{1}{2\sqrt{\mu}} e^{\frac{t}{2}} |\tilde{u}_N(t)|_{1,\omega} \| e^{-\frac{x^2}{2}} \tilde{u}_N(t) \|_{L^4_\omega}^2 \\ &\le c_1(T) \| \tilde{u}_N(t) \|_\omega \| \tilde{u}_N(t) \|_{1,\omega}^2, \end{aligned}$$

where

$$c_1(T) = \frac{2\sqrt[4]{\pi(1+T)^2}}{\sqrt{\mu}}.$$

Furthermore, for any $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, we know from Hardy, Littlewood and Pólya [17] that

$$|ab| \le \frac{|a|^p}{p} + \frac{|b|^q}{q}. \tag{3.10}$$

Thus by using Lemma 2.4 and (3.10), we assert that

(3.11)
$$\begin{aligned} |4B(\tilde{u}_N(t), u_N(t), \tilde{u}_N(t))| &\le \frac{1}{\sqrt{\mu}} e^{\frac{t}{2}} |\tilde{u}_N(t)|_{1,\omega} \| e^{-x^2} u_N(t) \tilde{u}_N(t) \|_\omega \\ &\le \frac{4\sqrt[4]{\pi}}{\sqrt{\mu}} e^{\frac{t}{2}} \| u_N(t) \|_\omega^{\frac{1}{2}} \| u_N(t) \|_{1,\omega}^{\frac{1}{2}} \| \tilde{u}_N(t) \|_\omega^{\frac{1}{2}} \| \tilde{u}_N(t) \|_{1,\omega}^{\frac{3}{2}} \\ &\le \frac{1}{4} \| \tilde{u}_N(t) \|_{1,\omega}^2 + c_2(u_N, T) \| \tilde{u}_N(t) \|_\omega^2, \end{aligned}$$

where

$$c_2(u_N, T) = \frac{12^3 \pi (1+T)^2}{\mu^2} \| u_N \|_{L^\infty(0,\ln(1+T); L^2_\omega(\Lambda))}^2 \| u_N \|_{L^\infty(0,\ln(1+T); H^1_\omega(\Lambda))}^2 .$$

By substituting (3.9) and (3.11) into (3.8), and integrating the resulting inequality, we find that

$$\begin{aligned} &\| \tilde{u}_N(t) \|_\omega^2 + \int_0^t (\frac{1}{4} - c_1(T) \| \tilde{u}_N(\eta) \|_\omega) \| \tilde{u}_N(\eta) \|_{1,\omega}^2 \, d\eta \\ &\quad \le \rho(\tilde{u}_{N,0}, \tilde{f}, t) + c_2(u_N, T) \int_0^t \| \tilde{u}_N(\eta) \|_\omega^2 \, d\eta, \end{aligned} \tag{3.12}$$

where

$$\rho(\tilde{u}_{N,0}, \tilde{f}, t) = \| \tilde{u}_{N,0} \|_\omega^2 + 2\int_0^t \| \tilde{f}(\eta) \|_\omega^2 \, d\eta.$$

We need the following lemma.

Lemma 3.1. *Assume that*
(i) *the constants* $b_1 > 0, b_2 \geq 0, b_3 \geq 0$ *and* $d \geq 0$,
(ii) $Z(t)$ *and* $A(t)$ *are non-negative functions of* t,
(iii) $d \leq \frac{b_1^2}{b_2^2} e^{-b_3 t_1}$ *for certain* $t_1 > 0$,
(iv) *for all* $t \leq t_1$,

$$Z(t) + \int_0^t (b_1 - b_2 Z^{\frac{1}{2}}(\eta)) A(\eta) d\eta \leq d + b_3 \int_0^t Z(\eta) d\eta.$$

Then for all $t \leq t_1$,

$$Z(t) \leq d e^{b_3 t}.$$

Proof. Consider the function $Y(t)$ satisfying

$$Y(t) = d + b_3 \int_0^t Y(\eta) d\eta.$$

Then for all $t \leq t_1$,

$$Y(t) = d e^{b_3 t} \leq \frac{b_1^2}{b_2^2}.$$

Clearly $Z(t) \leq Y(t)$ for $t \leq t_1$, and so the conclusion is valid.

Applying Lemma 3.1 to (3.12), we obtain the following result.

Theorem 3.1. *Let* $\alpha > \frac{1}{8\mu}$ *and* $u_N(t)$ *be the solution of* (3.6). *If for certian* t_1,

$$\rho(\tilde{u}_{N,0}, \tilde{f}, t_1) \leq \frac{(1-a)^2}{16 c_1^2(T)} e^{-c_2(u_N,T)t_1}, \quad a \geq 0,$$

then for all $t \leq t_1$,

$$\| \tilde{u}_N(t) \|_\omega^2 + \frac{a}{4} \int_0^t \| \tilde{u}_N(\eta) \|_{1,\omega}^2 \, d\eta \leq \rho(\tilde{u}_{N,0}, \tilde{f}, t) e^{c_2(u_N,T)t}.$$

Theorem 3.1 indicates that the error of the numerical solution is controlled by the errors of the data $u_{N,0}$ and f, provided that the average error $\rho(\tilde{u}_{N,0}, \tilde{f}, t)$ does not exceed certain critical value. It means that (3.6) is of generalized stability in the sense of Guo [14, 15], and of restricted stability in the sense of Stetter [16].

Next we deal with the convergence of scheme (3.6). Let U be the solution of (3.5), and $U_N = P_N U$. We derive from (3.5) that

$$\begin{aligned} &(\partial_t U_N(t), \phi)_\omega + \frac{1}{2}(U_N(t), \phi)_\omega + B(U_N(t), U_N(t), \phi) \\ &\quad + \frac{1}{4}(\partial_x U_N(t), \partial_x \phi)_\omega + G(t, \phi) = (f(t), \phi)_\omega, \\ &\forall \phi \in \mathcal{P}_N, \ 0 < t \leq \ln(1+T), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} G(t,\phi) &= G_1(t,\phi)+G_2(t,\phi)+G_3(t,\phi),\\ G_1(t,\phi) &= (\partial_t U(t)-\partial_t U_N(t),\phi)_\omega,\\ G_2(t,\phi) &= \frac{1}{2}(U(t)-U_N(t),\phi)_\omega,\\ G_3(t,\phi) &= B(U(t),U(t),\phi)-B(U_N(t),U_N(t),\phi). \end{aligned}$$

Let u_N be the solution of (3.6), and $\tilde{U}_N = u_N - U_N$. By subtracting (3.13) from (3.6), we obtain that

(3.14)

$$\begin{aligned} &(\partial_t \tilde{U}_N(t),\phi)_\omega + \frac{1}{2}(\tilde{U}_N(t),\phi)_\omega + B(\tilde{U}_N(t),\tilde{U}_N(t),\phi)\\ &\qquad + 2B(\tilde{U}_N(t),U_N(t),\phi) + \frac{1}{4}(\partial_x\tilde{U}_N(t),\partial_x\phi)_\omega = G(t,\phi)_\omega,\\ &\qquad \forall\phi\in\mathcal{P}_N,\ 0<t\le\ln(1+T). \end{aligned}$$

In addition, $\tilde{U}_N(0)=0$. Comparing (3.14) to (3.7), we can derive a result similar to that of Theorem 3.1. But $u_N, \tilde{u}_N, \tilde{u}_{N,0}$ and $\tilde{f}$ are now replaced by $U_N, \tilde{U}_N, \tilde{U}_{N,0}$ and $G(t,\phi)$, respectively. Therefore we only have to estimate the term $|G(t,\tilde{U}_N(t))|$. We first have from Theorem 2.1 that for $r\ge 1$,

$$\begin{aligned} |G_1(t,\tilde{U}_N(t))| &\le cN^{-\frac{r}{2}} \| \partial_t U(t)\|_{r,\omega} \| \tilde{U}_N(t)\|_\omega,\\ |G_2(t,\tilde{U}_N(t))| &\le cN^{-\frac{r}{2}} \| U(t)\|_{r,\omega} \| \tilde{U}_N(t)\|_\omega . \end{aligned}$$

An argument, as in the derivation of (3.9), leads to that for all $t<\ln(1+T)$,

$$\begin{aligned} |G_3(t,\tilde{U}_N(t))| &\le \frac{1}{4\sqrt{\mu}} e^{\frac{t}{2}} |(e^{-x^2}(U_N(t)+U(t))(U_N(t)-U(t)),\tilde{U}_N(t))_\omega|\\ &\le \frac{1}{2}c_1(T)\| U_N(t)+U(t)\|_\omega^{\frac{1}{2}} \| U_N(t)+U(t)\|_{1,\omega}^{\frac{1}{2}} \| U_N(t)-U(t)\|_\omega^{\frac{1}{2}}\\ &\quad \times \| U_N(t)-U(t)\|_{1,\omega}^{\frac{1}{2}} \| \tilde{U}_N(t)\|_{1,\omega} . \end{aligned}$$

By Theorem 2.1,

$$|G_3(t,\tilde{U}_N(t))| \le \frac{1}{8}\| \tilde{U}_N(t)\|_{1,\omega}^2 + cc_1^2(T)N^{\frac{1}{2}-r}\| U_N(t)\|_{r,\omega}^4 .$$

Hence

(3.15)

$$\begin{aligned} |G(t,\tilde{U}_N(t))| \le\ & \frac{1}{8}\| \tilde{U}_N(t)\|_{1,\omega}^2 + \| \tilde{U}_N(t)\|_\omega^2\\ &+c(c_1^2(T)+1)N^{\frac{1}{2}-r}(\| U_N(t)\|_{r,\omega}^4 + \| \partial_t U(t)\|_{r-\frac{1}{2},\omega}^2). \end{aligned}$$

Obviously, the last term in (3.15) tends to zero as N goes to infinity. Therefore we obtain the following result.

Theorem 3.2. *If* $\alpha > \frac{1}{8\mu}$ *and*

$$U\in L^2(0,\ln(1+T);H_\omega^r(\Lambda))\cap H^1(0,\ln(1+T);H_\omega^{r-\frac{1}{2}}(\Lambda))$$

with $r\ge 1$, *then for all* $t\le\ln(1+T)$,

$$\| u_N(t)-U(t)\|_\omega^2 + \int_0^t \| u_N(\eta)-U(\eta)\|_{1,\omega}^2\,\mathrm{d}\eta \le c^* N^{\frac{1}{2}-r},$$

where c^* *is a positive constant depending only on* μ, T *and the norms of* U *in the space mentioned above.*

Remark 3.1. In the proof of Theorem 3.1 and Theorem 3.2, we require that $U \in H^1_\omega(\Lambda)$ and so $e^{-\frac{x^2}{2}}(|U(x,t)| + |\partial_x U(x,t)|) \to 0$ as $|x| \to \infty$. A sufficient condition is that for certain $\alpha > \frac{1}{8\mu}$, $e^{\alpha y^2}(|V(y,s)| + |\partial_y V(y,s)|) \to 0$, as $|y| \to \infty$. It means that $V(y,s)$ should decay fast enough. It agrees with the experience in actual computations as described in Funaro and Kavian [7] and other papers.

Remark 3.2. In this paper, we use the variable transformation (3.2) and so obtain the error estimations. In fact, a similiar transformation was used in actual computations by Funaro and Kavian [7]. This trick can be generalized to other problems, such as the two-dimensional heat equation and the Navier-Stokes equations.

In actual computations, we need to discretize the term $\partial_t u_N$ in (3.6). We can use Lemmas 2.1–2.4, Theorem 2.1 and an argument as in the proof of Theorems 3.1 and 3.2, to prove the generalized stability and the convergence of a fully discrete scheme, provided that the value of τN satisfies certain reasonable conditions, where τ is the step size in time t, and N is the number of terms used in Hermite approximations. For instance, by Lemma 2.2, τN should be bounded in the case of explicite schemes.

We can also approximate nonlinear partial differential equations by the base functions

$$\tilde{H}_l(x) = (2^l l! \sqrt{\pi})^{-\frac{1}{2}} e^{-ax^2} H_l(x), \; a \geq 0, \; l \geq 0.$$

The set of $\tilde{H}_l(x)$ is an orthogonal system associated with the weight $e^{(2a-1)x^2}$. For the application to linear problems with $a = 1$, we refer to Funaro and Kavian [7].

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