CALCULATION OF VALUES OF L-FUNCTIONS ASSOCIATED TO ELLIPTIC CURVES

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ABSTRACT. We calculated numerically the values of L-functions of four typical elliptic curves in the critical strip in the range $\mathrm{Im}(s) \leq 400$. We found that all the non-trivial zeros in this range lie on the critical line $\mathrm{Re}(s) = 1$ and are simple except the one at $s = 1$. The method we employed in this paper is the approximate functional equation with incomplete gamma functions in the coefficients. For incomplete gamma functions, we continued them holomorphically to the right half plane $\mathrm{Re}(s) > 0$, which enables us to calculate for large $\mathrm{Im}(s)$. Furthermore we remark that a relation exists between Sato-Tate conjecture and the generalized Riemann Hypothesis.

1. INTRODUCTION AND THE STATEMENT OF RESULTS

The numerical calculations of the Riemann zeta function $\zeta(s)$ have a long history. In the critical strip, the Euler-Maclaurin summation formula is applicable, but on the critical line, the famous Riemann-Siegel formula is useful because it is very fast and accurate (see [3] or [8]). Using these formulas, it is known at present that the Riemann Hypothesis holds for $\mathrm{Im}(s)$ less than about 1.5×10^9 (see J. van de Lune, H. J. J. te Riele and D. T. Winter [13]; see also Odlyzko [16]). By the Euler-Maclaurin summation formula, we can also calculate the values of the Hurwitz zeta function and hence the values of the Dirichlet L-function because it is a finite sum of the Hurwitz zeta functions.

For other L-functions, we have the examples of Manin [14], [15], Yoshida [24], [25] and Fermigier [4]. In his papers, Manin developed the theory of modular symbols and applied his theory to the calculation of Fourier coefficients of cusp forms of weight 2 and the value at 1 of the corresponding Dirichlet series. In fact, four examples of modular curves were treated in [15]. See Cremona [1] for other examples.

Yoshida and Fermigier calculated the values of the L-function on the critical line. Yoshida makes use of the iteration of partial summations in order to access the speed of convergence of the Dirichlet series. He said that his method "may seem to be only speculative" (p. 89 in [24]) and "on heuristic grounds" (p. 91 in [24]), though he made some discussions on the confidence of his data. He found zeros of many L-functions, namely, the L-function associated with the Ramanujan function $\Delta(z)$, the symmetric j-th power L-functions of $\Delta(z)$ for $j = 2, 3$ and 4, L-functions associated to cusp forms, L-functions associated to Hecke characters of

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real quadratic fields and Artin's L-functions. He also observed that the generalized Riemann Hypothesis is true for $L(s, \Delta)$ in $0 \leq \text{Im}(s) \leq 100$.

On the other hand, Fermigier used an expression of an L-function by the Dirichlet series with incomplete gamma functions in the coefficients. He employed the Romberg integration method for the calculation of incomplete gamma functions. He calculated the zeros of $L(1 + it, E)$ in $0 \leq t \leq 15$ for all the elliptic curves of prime conductor $N \leq 13100$. He also observed that the generalized Riemann Hypothesis is true for these $L(s, E)$ in $0 \leq \text{Im}(s) \leq 15$.

The aim of this paper is to show a method and some examples for larger $\text{Im}(s)$ than theirs. As in Fermigier [4], we use an expression of an L-function with the incomplete gamma function $\Gamma(s, z)$. However, the second variable z is real in his case, so it seems somewhat difficult to calculate when $\text{Im}(s)$ is large because the necessary increase in the memory bank of a machine is of exponential order with respect to $\text{Im}(s)$. We continue the incomplete gamma function in the right half plane $\text{Re}(s) > 0$. After this, the increase is of polynomial order, hence we can calculate the value numerically for rather large $\text{Im}(s)$. See the paragraphs after Theorem 1 below. After the analytic continuation, we expand it to the continued fraction. Using this method, we calculated the values of $L(1 + it, E)$ for four typical examples of elliptic curves in the range $0 \leq t \leq 400$, and checked that the generalized Riemann Hypothesis holds in this range.

We also present some graphs drawn by plotting the values of an L-function on the Gaussian plane. Since $L(s, E)$ has a functional equation, we can easily determine the argument of $L(s, E)$ on the critical line. So one may think that the essential thing is the absolute value, and such graph is meaningless. But the authors believe that such a visualization gives us the beauty and sense of the Riemann Hypothesis.

To state Theorem 1, we shall recall the definition of the L-function associated to an elliptic curve. Let E be an elliptic curve given by the global minimal Weierstrass equation

$$y^2 + A_1 xy + A_3 y = x^3 + A_2 x^2 + A_4 x + A_6,$$

and let Δ be its discriminant. For each prime p, we put

$$a_p = p + 1 - \sharp E_p \left(\mathbf{Z}/p\mathbf{Z} \right),$$

where E_p is the reduction modulo p. If $p | \Delta$, then E_p has a singularity and

$$a_p = \begin{cases} 0 & \text{for the case of a cusp,} \\ 1 & \text{for the case of a split node,} \\ -1 & \text{for the case of a non-split node.} \end{cases}$$

If $p \nmid \Delta$, then we have

(1) $$|a_p| \leq 2\sqrt{p}$$

(Hasse's theorem). The L-function associated to E is defined by

$$L(s, E) = \prod_{p | \Delta} \frac{1}{1 - a_p p^{-s}} \prod_{p \nmid \Delta} \frac{1}{1 - a_p p^{-s} + p^{1-2s}}.$$

This infinite product is absolutely convergent for $\text{Re}\, s > 3/2$ by (1), and there we can expand it into the Dirichlet series $L(s, E) = \sum_{n=1}^{\infty} a_n n^{-s}$.

Assume that E has a modular parametrization of level N. In the recent works of Wiles [23], Taylor and Wiles [19] and Diamond [2], it was proved that any elliptic

curve which has semi-stable good reduction at 3 and 5 is modular. All the curves we treat in Section 5 satisfy this assumption. Let

$$f_E(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z},$$

then by Eichler-Shimura's theory, $f_E(z)$ is a primitive form of weight 2 with respect to $\Gamma_0(N)$. Furthermore we have

$$f_E(-\frac{1}{Nz}) = \mu N z^2 f_E(z) \qquad (\mu = \pm 1).$$

Now, $L(s, E)$ can be continued holomorphically to the whole complex plane and satisfies the functional equation:

$$N^{s/2}(2\pi)^{-s}\Gamma(s)L(s, E) = -\mu N^{(2-s)/2}(2\pi)^{-(2-s)}\Gamma(2-s)L(2-s, E).$$

The readers may refer to Knapp [10] for these subjects.

Then our first theorem is

Theorem 1. *Let $s = \sigma + it$ be a complex number such that $1/2 \leq \sigma \leq 3/2$ and $t > 0$ and let $\Gamma(s, z)$ be the incomplete gamma function of the second kind. We take a positive integer M satisfying $M > t\sqrt{N}/4$, and put $r = e^{i(\pi/2-\delta(t))}$, where $\delta(t)$ is a function of t with $0 < \delta(t) \leq \pi/2$. Then we have*

$$
L(s, E) = \frac{1}{\Gamma(s)} \sum_{n \leq M} \frac{a_n}{n^s} \Gamma\left(s, \frac{2\pi n r}{\sqrt{N}}\right)
$$

(2)
$$
- \frac{\mu N^{1-s}(2\pi)^{2(s-1)}}{\Gamma(s)} \sum_{n \leq M} \frac{a_n}{n^{2-s}} \Gamma\left(2-s, \frac{2\pi n}{\sqrt{N} r}\right)
$$

$$
+ (2\pi)^s \Gamma(s)^{-1} R,
$$

where the last term R satisfies the inequality

(3)
$$
|R| < e^{-\pi t/2} e^{\delta(t)(t-4M/\sqrt{N})} N^{(1-\sigma)/2} \sqrt{M} \delta(t)^{-1}
$$

$$
\times \left\{ 1 + \frac{\log M + \sigma + 1}{2t\delta(t)} + \frac{(\sigma-1)(\log M + 2)}{4(t\delta(t))^2} \right\}.
$$

The proof will be given in Section 2.

The incomplete gamma function of the second kind is defined by

(4)
$$
\Gamma(s, z) = \int_z^{\infty} e^{-t} t^{s-1} dt \qquad (z \in \mathbf{R}, \mathrm{Re}(s) > 0).
$$

(Note that this integral is denoted by $\Gamma(z, s)$ in Fermigier [4].) Following Lavrik [12], Karatsuba [9] and Turganaliev [21], we continue the function $\Gamma(s, z)$ holomorphically to the right half plane $\mathrm{Re}(z) > 0$ by making a change of variable $t \to zt$ and rotating the line of integration by the angle $\arg z$. Then we have

(5)
$$
\Gamma(s, z) = z^s \int_1^{\infty} e^{-zt} t^{s-1} dt
$$

and, when $\mathrm{Re}(s)$ is bounded and $\mathrm{Re}(z) > c$ for some positive constant c,

(6)
$$
|\Gamma(s, z)| \ll |z^s|
$$

uniformly in s and z.

The inequality (3) shows that when $\delta(t)$ is small, we must take large M to get the accuracy of data. Hence, $\delta(t) = \pi/2$ (this means $r = 1$) is the most efficient choice for calculations theoretically. But, for a technical reason, we have to choose the function $\delta(t) < \pi/2$. The function of the form $\Gamma(s, z)/\Gamma(s)$ appears in the right hand side of (2). Since $\Gamma(s)$ is of exponential decay when $|t| \to \infty$, the absolute value of each term in the sum may be very large. In fact, if $r = 1$ and t is large, each term in (2) is a huge number and it becomes impossible to compute it. In order to avoid this difficulty, we put $r = e^{i(\pi/2-\delta(t))}$ and $\delta(t) < \pi/2$. Then from (6),

$$|\Gamma(s, Ar)| \ll A^\sigma e^{-(\frac{\pi}{2}-\delta(t))t},$$

(where A is independent of t), hence the factor $e^{-\pi t/2}$ cancels the one arising from the denominator. For example, if $\delta(t) \sim 1/t$, $\Gamma(s, Ar)/\Gamma(s)$ has a polynomial order as $|t| \to \infty$, hence we can put the calculations in practice. Turganaliev [21] put $\delta(t) = 1/(1+t)$. We choose $\delta(t) = \pi/2$ for small t, $\delta(t) = 1/(1+\log^2 t)$ for a little larger t and $\delta(t) = 1/(1+t)$ otherwise.

The estimation (3) does not make sense when t is very close to 0. But, in this case, we set $\delta(t) = \pi/2$ and can get the explicit estimation from (16).

Our method is applicable to the Dirichlet series with an appropriate functional equation. Explicit estimation of the coefficients is needed to get the concrete upper bound of the error term. For example, we can treat the Dedekind zeta function or the Hecke L-function of a cusp form.

Now we shall state the next theorem. Let E be an elliptic curve defined over $\mathbf{Q}$ without complex multiplication. Define $\theta_p \in (0, \pi)$ by $a_p = 2\sqrt{p} \cos(\theta_p)$. Let p_n be the n-th prime number, and consider the real sequence $x_n = \theta_{p_n}/\pi$, $(n = 1, 2, \dots)$. Let g be a real valued strictly increasing function on the interval $[0, 1]$ with $g(0) = 0$ and $g(1) = 1$. Define the discrepancy with respect to the distribution function g by

$$(7) \qquad D_K^{(g)}(x_n) = \sup_{0 \leq \alpha \leq 1} \left| \frac{A([0, \alpha), (x_n), K)}{K} - g(\alpha) \right|,$$

where

$$A([0, \alpha), (x_n), K) = \sharp\{x_n \in [0, \alpha) \, ; 1 \leq n \leq K\}.$$

Put

$$ST(x) = x - \frac{\sin(2\pi x)}{2\pi}.$$

We call $D_K^{(ST)}$ *the discrepancy with respect to the Sato-Tate measure*. The Sato-Tate conjecture asserts that $\lim_{K \to \infty} D_K^{(ST)}(x_n) = 0$, which is easily shown to be equivalent to

$$\frac{\sharp\{\theta_p \in [\alpha, \beta) \, ; p \leq X\}}{\sharp\{p \, ; p \leq X\}} \to \frac{2}{\pi} \int_\alpha^\beta \sin^2(t)dt,$$

for any $0 \leq \alpha \leq \beta \leq \pi$. See Ogg [17] or Shahidi [18] for the Sato-Tate conjecture. Here we propose a quantitative version of their conjecture.

Conjecture 1. *For any positive ε, $D_K^{(ST)}(x_n) = O(K^{-1/2+\varepsilon})$.*

Then our second theorem is

Theorem 2. *Let E be an elliptic curve defined over $\mathbf{Q}$ which has no complex multiplication. Then the above Conjecture implies the truth of the generalized Riemann Hypothesis for $L(s, E)$.*

The proof of Theorem 2 will be given in Section 3.

In Section 5, we show numerical experiments on Conjecture 1 for some elliptic curves. These seem to support the validity of Conjecture 1.

2. APPROXIMATE FUNCTIONAL EQUATION

Let $f(z)$ be a cusp form of weight k with respect to $\Gamma_0(N)$, and let $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ be the Fourier expansion at the cusp ∞. Furthermore we assume that $f(z)$ is an eigenfunction of the involution

$$f \to f\Big|_k \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$$

with eigenvalue $\mu = \pm 1$; namely,

$$(8) \qquad f(-\frac{1}{Nz}) = \mu N^{k/2} z^k f(z).$$

Let $L(s, f)$ be the Dirichlet series defined by $L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}$. Since $a_n = O(n^{\frac{k-1}{2}+\varepsilon})$ by Deligne's theorem, $L(s, f)$ converges absolutely for $\mathrm{Re}\, s > (k+1)/2$. It is well known that the function $\Lambda(s, f) := N^{s/2}(2\pi)^{-s}\Gamma(s)L(s, f)$ is an entire function of s and satisfies the functional equation

$$(9) \qquad \Lambda(s, f) = \mu i^k \Lambda(k - s, f).$$

By modifying the standard proof of (9), we get the approximate functional equation with incomplete gamma functions. Let $I(s)$ be the function defined by

$$I(s) = \int_0^\infty y^{s-1} f(iy) dy,$$

which is equal to $(2\pi)^{-s}\Gamma(s)L(s, f)$ for $\mathrm{Re}\, s > (k+1)/2$. Let r be a complex number with $\mathrm{Re}(r) > 0$, and divide the integral of $I(s)$ as

$$I(s) = \int_0^\infty y^{s-1} f(iy) dy = \int_0^{\frac{r}{\sqrt{N}}} + \int_{\frac{r}{\sqrt{N}}}^\infty .$$

Then

$$(10) \qquad \int_{\frac{r}{\sqrt{N}}}^\infty y^{s-1} f(iy) dy = N^{-s/2} r^s \sum_{n=1}^\infty a_n \int_1^\infty y^{s-1} e^{-\frac{2\pi n r y}{\sqrt{N}}} dy.$$

On the other hand, the equation (8) gives us

$$(11) \qquad \int_0^{\frac{r}{\sqrt{N}}} y^{s-1} f(iy) dy = N^{-s} \int_{\frac{1}{\sqrt{N}r}}^\infty y^{-s-1} f(\frac{i}{Ny}) dy$$

$$= \mu i^k N^{-s/2} r^{s-k} \sum_{n=1}^\infty a_n \int_1^\infty y^{k-s-1} e^{-\frac{2\pi n y}{\sqrt{N}r}} dy.$$

This shows the analytic continuation of $I(s)$ to the whole complex plane. The last integrals in (10) and (11) can be written as

$$\left(\frac{2\pi nr}{\sqrt{N}}\right)^{-s}\Gamma\left(s,\frac{2\pi nr}{\sqrt{N}}\right) \quad \text{and} \quad \left(\frac{\sqrt{N}r}{2\pi n}\right)^{k-s}\Gamma\left(k-s,\frac{2\pi n}{\sqrt{N}r}\right),$$

respectively. Hence we obtain the following

Lemma 1. *Let M be an arbitrary positive integer, then we have*

$$L(s,f) = \frac{1}{\Gamma(s)}\sum_{n\leq M}\frac{a_n}{n^s}\Gamma\left(s,\frac{2\pi nr}{\sqrt{N}}\right)$$

$$+ \frac{\mu i^k N^{k/2-s}(2\pi)^{2s-k}}{\Gamma(s)}\sum_{n\leq M}\frac{a_n}{n^{k-s}}\Gamma\left(k-s,\frac{2\pi n}{\sqrt{N}r}\right)$$

$$+ (2\pi)^s\Gamma(s)^{-1}R,$$

where

(12)
$$R = N^{-s/2}r^s\sum_{n>M}a_n\int_1^\infty y^{s-1}e^{-\frac{2\pi nry}{\sqrt{N}}}dy$$

$$+ \mu i^k N^{-s/2}r^{s-k}\sum_{n>M}a_n\int_1^\infty y^{k-1-s}e^{-\frac{2\pi ny}{\sqrt{N}r}}dy.$$

Now let us consider the case $k=2$. We assume that $f(z)$ is a primitive form. The error term R in (12) can be written as

$$R = N^{-s/2}r^s\sum_{n>M}a_n\int_1^\infty y^{s-1}e^{-\frac{2\pi nry}{\sqrt{N}}}dy$$

$$- \mu N^{-s/2}r^{s-2}\sum_{n>M}a_n\int_1^\infty y^{1-s}e^{-\frac{2\pi ny}{\sqrt{N}r}}dy.$$

As is stated in Section 1, we put

$$r = e^{i(\pi/2-\delta(t))},$$

where $\delta(t)$ is a function of t satisfying $0 < \delta(t) \leq \pi/2$. Then

$$|R| \leq N^{-\sigma/2}e^{-(\pi/2-\delta(t))t}\left\{\sum_{n>M}|a_n|\int_1^\infty y^{\sigma-1}e^{-\frac{2\pi n\sin\delta(t)}{\sqrt{N}}y}dy\right.$$

$$\left.+ \sum_{n>M}|a_n|\int_1^\infty y^{1-\sigma}e^{-\frac{2\pi n\sin\delta(t)}{\sqrt{N}}y}dy\right\}.$$

We may assume that $1 \leq \sigma < 3/2$, and we write $\lambda = 2\pi\sin\delta(t)$ for simplicity. The integrals in the right hand side of the above inequality are evaluated as

$$\int_1^\infty y^{\sigma-1}e^{-\frac{\lambda ny}{\sqrt{N}}}dy \leq \frac{\sqrt{N}}{\lambda n}e^{-\frac{\lambda n}{\sqrt{N}}} + \frac{N(\sigma-1)}{(\lambda n)^2}e^{-\frac{\lambda n}{\sqrt{N}}},$$

and

$$\int_1^\infty y^{1-\sigma}e^{-\frac{\lambda ny}{\sqrt{N}}}dy \leq \frac{\sqrt{N}}{\lambda n}e^{-\frac{\lambda n}{\sqrt{N}}}.$$

We also have $|a_n| \leq \sqrt{n}\, d(n)$. Hence we get

$$(13) \qquad |R| \leq 2N^{(1-\sigma)/2} e^{-(\pi/2-\delta(t))t} \lambda^{-1} \sum_{n>M} \frac{d(n)}{n^{1/2}} e^{-\frac{\lambda n}{\sqrt{N}}}$$

$$+ (\sigma-1) N^{1-\sigma/2} e^{-(\pi/2-\delta(t))t} \lambda^{-2} \sum_{n>M} \frac{d(n)}{n^{3/2}} e^{-\frac{\lambda n}{\sqrt{N}}}.$$

To evaluate the summations in the right hand side, we put $D(x) = \sum_{n \leq x} d(n)$. By the elementary fact

$$x(\log x - 1) \leq D(x) \leq x(\log x + 1)$$

and partial summation, we get

$$(14) \quad \sum_{n>M} \frac{d(n)}{n^{1/2}} e^{-\frac{\lambda n}{\sqrt{N}}} = -D(M) M^{-1/2} e^{-\frac{\lambda M}{\sqrt{N}}} - \int_M^\infty D(u) \frac{d}{du} \left(u^{-1/2} e^{-\frac{\lambda u}{\sqrt{N}}} \right) du$$

$$\leq -\sqrt{M}(\log M - 1) e^{-\frac{\lambda M}{\sqrt{N}}} + \int_M^\infty (\log u + 1) \left(\frac{\lambda}{\sqrt{N}} \sqrt{u} + \frac{1}{2\sqrt{u}} \right) e^{-\frac{\lambda u}{\sqrt{N}}} du$$

$$\leq \left(\frac{\log M + 2}{\lambda} \sqrt{\frac{N}{M}} + 2\sqrt{M} \right) e^{-\frac{\lambda M}{\sqrt{N}}}.$$

Similarly, we have that

$$(15) \qquad \sum_{n>M} \frac{d(n)}{n^{3/2}} e^{-\frac{\lambda n}{\sqrt{N}}} \leq \left(\frac{2}{\sqrt{M}} + \frac{\log M + 2}{\lambda M} \sqrt{\frac{N}{M}} \right) e^{-\frac{\lambda M}{\sqrt{N}}}.$$

From (13), (14) and (15), we get

$$(16) \quad |R| \leq e^{-\frac{\pi}{2}t} e^{(t\delta(t) - \frac{\lambda M}{\sqrt{N}})} N^{(1-\sigma)/2} \left\{ \frac{4\sqrt{M}}{\lambda} + \frac{2(\log M + \sigma + 1)}{\lambda^2} \sqrt{\frac{N}{M}} \right.$$

$$\left. + \frac{(\sigma-1)(\log M + 2)}{\lambda^3 \sqrt{M}} \frac{N}{M} \right\}.$$

Now take $M \geq t\sqrt{N}/4$ and use the inequality $\lambda > 4\delta(t)$, then we get Theorem 1.

3. Proof of Theorem 2

To prove Theorem 2, we need the following lemma of Koksma.

Lemma 2 (Koksma's inequality). *Let f be a real valued function on $[0,1]$. Suppose that f has bounded variation. Let g be a real valued continuous strictly increasing function on $[0,1]$ with $g(0) = 0$ and $g(1) = 1$. Then we have*

$$\left| \frac{1}{K} \sum_{n=1}^K f(x_n) - \int_0^1 f(t) dg(t) \right| \leq D_K^{(g)}(x_n) V(f),$$

for any sequence of real numbers $(x_n)_{n=1,2,\ldots}$ in $[0,1]$. Here $V(f)$ is the total variation of f in $[0,1]$ and $D_K^{(g)}(x_n)$ is the discrepancy defined by (7).

Proof. See Kuipers and Niederreiter [11, p. 142] for the case $g(t) = t$. It is an easy exercise to generalize it to our case.

Now we prove Theorem 2. Put

$$A(s) = \prod_{p \nmid \Delta} \frac{1}{1 - a_p p^{-s} + p^{1-2s}}$$

for $\operatorname{Re} s > 3/2$. We note that

$$\log A(s) = \sum_p a_p p^{-s} + O(\sum_p p^{1-2\sigma}).$$

Since the error term is holomorphic in $\operatorname{Re} s > 1/2$, we have only to consider the sum $\sum_p a_p p^{-s} = 2 \sum_p \cos(\theta_p) p^{1/2-s}$. (Here we can neglect the bad primes.) If this sum is holomorphic in $\operatorname{Re} s > 1$, then $L(s, E)$ has no zeros in $\operatorname{Re} s > 1$, which is the generalized Riemann Hypothesis for $L(s, E)$. By partial summation, we see that if

$$(17) \qquad \sum_{p < x} \cos(\theta_p) = O(x^{1/2+\varepsilon}) \qquad \text{for any} \quad \varepsilon > 0,$$

then $\log A(s)$ is holomorphic in $\operatorname{Re} s > 1$. Now, let $f(t) = \cos(\pi t)$ and $g(t) = ST(t)$ in Lemma 2. Then we have

$$(18) \qquad \left| \frac{1}{K} \sum_{n=1}^{K} \cos(\theta_{p_n}) \right| \leq 2 D_K^{(ST)}(x_n).$$

Hence, Conjecture 1 implies that $\sum_{n=1}^{K} \cos(\theta_{p_n}) = O(K^{1/2+\varepsilon})$. Combining this and (18), we see that (17) holds, and we get the desired result.

4. Continued fraction of the incomplete gamma function

The problem is that the calculation of incomplete gamma functions is very difficult and needs much time. Here it is appropriate to explain how to evaluate it with satisfactory accuracy. Let $\Gamma(s, z)$ be the incomplete gamma function of the second kind defined by (4), and let

$$(19) \qquad \gamma(s, z) = \int_0^z e^{-t} t^{s-1} dt = \Gamma(s) - \Gamma(s, z)$$

be the incomplete gamma function of the first kind.

When z has a small absolute value, we can use the Taylor expansion:

$$\gamma(s, z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{s+n}}{n!(s+n)},$$

or the formula due to Legendre:

$$(20) \qquad \gamma(s, z) = e^{-z} \sum_{n=0}^{\infty} \frac{\Gamma(s) z^{s+n}}{\Gamma(s+n+1)}.$$

When $|z|$ is large enough, we use the formula

$$(21) \qquad \Gamma(s, z) = z^{s-1} e^{-z} \left(1 + \sum_{n=1}^{\infty} \frac{1}{z^n} (s-1)(s-2) \cdots (s-n) \right).$$

Note that the series of the right hand side is divergent and this equality should be considered as the asymptotic expansion at $z = \infty$. To calculate approximate values of $\gamma(s, z)$ and $\Gamma(s, z)$, we employ their classical continued fraction expansions:

$$(22) \qquad \gamma(s, z) = z^s e^{-z} \cfrac{1}{s - \cfrac{s \cdot z}{s + 1 + \cfrac{1 \cdot z}{s + 2 - \cfrac{(s+1)z}{s + 3 + \cfrac{2 \cdot z}{s + 4 - \cfrac{(s+2)z}{s + 5 + \cfrac{3 \cdot z}{s + 6 - \cfrac{}{\ddots}}}}}}}}$$

and

$$(23) \qquad \Gamma(s, z) = z^s e^{-z} \cfrac{1}{z + \cfrac{1 - s}{1 + \cfrac{1}{z + \cfrac{2 - s}{1 + \cfrac{2}{z + \cfrac{3 - s}{1 + \cfrac{3}{z + \ddots}}}}}}}$$

These formulas are special cases of Gauss's continued fraction method using confluent hypergeometric functions (see Jones and Thron [5], pp. 205-209 and pp. 344-348 or Wall [22]). As a formal power series of z and s, (22) (resp. (23)) is equivalent to (20) (resp. (21)). However, the continued fraction in (22) is convergent for any z and s with $\operatorname{Re} s > 0$, and the one in (23) for z with $|\arg z| < \pi$ and $s \neq 1, 2, 3, \ldots$.

To estimate the truncation error of these continued fraction expansions, we quote the result of [5]: Let θ, ξ_{-1} be real numbers with $0 < |\theta| < \pi$ and let $\{a_n\}_{n=0,1,2,\ldots}$ be a sequence of arbitrary non-zero complex numbers. We define $\xi_n = \arg a_n - \xi_{n-1} - \theta$ recursively. Let $\{b_n\}_{n=0,1,2,\ldots}$ be another sequence of complex numbers satisfying the conditions

$$0 \leq \arg b_n - \xi_n \leq \theta \quad \text{if} \quad 0 < \theta < \pi,$$
$$\theta \leq \arg b_n - \xi_n \leq 0 \quad \text{if} \quad -\pi < \theta < 0.$$

Consider the continued fraction

$$(24) \qquad \cfrac{a_0}{b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}}}$$

Then Jones and Thron (see Th. 8.8 in [5] and [6]) proved the following

Theorem 3 (Jones and Thron). *If the continued fraction* (24) *converges to a value x, then*

$$\left| x - \frac{P_n}{Q_n} \right| \leq \left| \frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} \right|, \qquad if \quad 0 < |\theta| \leq \frac{\pi}{2},$$

$$\left| x - \frac{P_n}{Q_n} \right| \leq \frac{1}{|\sin\theta|} \left| \frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} \right|, \quad if \quad \frac{\pi}{2} < |\theta| < \pi,$$

where P_n/Q_n $(n = 0, 1, 2, \ldots)$ is the n-th convergent of the continued fraction (24).

First we apply Theorem 3 to (22). Let P_n/Q_n $(n = 0, 1, 2, \ldots)$ be the n-th convergent of the continued fraction expansion of $\gamma(s, z)z^{-s}e^z$. Put

$$b_n = \begin{cases} nz, & n \quad \text{even}, \\ -(s + (n-1)/2)z, & n \quad \text{odd}. \end{cases}$$

Then it is easily seen that

$$Q_n = (s + n + 1)Q_{n-1} + b_n Q_{n-2},$$

and

$$P_n Q_{n-1} - P_{n-1} Q_n = -b_n(P_{n-1}Q_{n-2} - P_{n-2}Q_{n-1}).$$

Thus we have

$$Q_{2n-1} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(2n - k + s)}{\Gamma(s)} z^k$$

$$Q_{2n} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(2n + 1 - k + s)}{\Gamma(s)} z^k,$$

and

$$(25) \qquad \frac{P_{2n}}{Q_{2n}} - \frac{P_{2n-1}}{Q_{2n-1}} = \frac{(-1)^n n! \Gamma(s+n)}{\Gamma(s) Q_{2n} Q_{2n-1}} = \frac{(-1)^n n! \Gamma(s)}{\Gamma(s+n) Q_{2n}^* Q_{2n-1}^*},$$

where, in the right hand side of (25), we put

$$Q_{2n-1}^* = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(s+n+k)}{\Gamma(s+n)} z^{-k},$$

$$Q_{2n}^* = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(s+n+1+k)}{\Gamma(s+n)} z^{-k},$$

or in another way

$$(26) \qquad Q_{2n-1}^* = \sum_{k=0}^{n} \binom{n}{k} \left(1 + \frac{n+k-1}{s}\right) \left(1 + \frac{n+k-2}{s}\right) \cdots \left(1 + \frac{n}{s}\right) \left(-\frac{z}{s}\right)^{-k},$$

$$Q_{2n}^* = \sum_{k=0}^{n} \binom{n}{k} \left(1 + \frac{n+k}{s}\right) \left(1 + \frac{n+k-1}{s}\right) \cdots \left(1 + \frac{n}{s}\right) \left(-\frac{x}{s}\right)^{-k}.$$

The error estimate of (23) proceeds similarly. Let U_n/V_n $(n = 0, 1, 2, \dots)$ be the n-th convergent of $\Gamma(s, z)z^{-s}e^z$, then we can show by induction

$$V_{2n-1} = \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(n+1-s)}{\Gamma(k+1-s)} x^k,$$

$$V_{2n} = \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(n+1-s)}{\Gamma(k+2-s)} x^{k+1}.$$

Thus we have

$$(27) \qquad \frac{U_{2n}}{V_{2n}} - \frac{U_{2n-1}}{V_{2n-1}} = \frac{n!\Gamma(n+1-s)}{\Gamma(1-s)V_{2n}V_{2n-1}} = \frac{n!\Gamma(1-s)}{\Gamma(n+2-s)V_{2n}^* V_{2n-1}^*},$$

where

$$V_{2n-1}^* = \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(1-s)}{\Gamma(k+1-s)} z^k,$$

$$V_{2n}^* = \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(1-s)}{\Gamma(k+2-s)} z^{k+1},$$

or similarly,

$$(28) \qquad \begin{aligned} V_{2n-1}^* &= \sum_{k=0}^{n} \binom{n}{k} \frac{(-z/s)^k}{\left(1 - \frac{k}{s}\right)\left(1 - \frac{k-1}{s}\right) \cdots \left(1 - \frac{1}{s}\right)}, \\ V_{2n}^* &= \sum_{k=0}^{n} \binom{n}{k} \frac{(-z/s)^{k+1}}{\left(1 - \frac{k+1}{s}\right)\left(1 - \frac{k}{s}\right) \cdots \left(1 - \frac{1}{s}\right)}. \end{aligned}$$

Applying the Theorem of Jones and Thron, we can estimate the truncation error by (25) or (27). The expressions (26) and (28) suggest that there exists a kind of duality. Thus, to calculate the precise value of $\Gamma(s, z)$, it is better to use (23) when $|z| \geq |s|$ and (22) when $|z| < |s|$. Numerical experiments suggest that this choice is very suitable. Note that, if necessary, we must apply Theorem 3 to the sub-expression of (22) and (23) because the first finite terms may not satisfy the condition of Theorem 3.

5. EXAMPLES

In this section, we give examples of our calculations. In the graphs of zeta functions or L-functions below, the horizontal axis (resp. vertical axis) represents the real part (resp. imaginary part) of their values.

As a model, we first show two graphs of the Riemann zeta function on the critical line (Figures 1 and 2). The range of t is written under each graph. Figures 3, 4, 5, and 6 are the graphs of the Hurwitz zeta function $\zeta(1/2 + it, j/5)$, $(j = 1, \cdots, 4)$ for $50.00 \leq t \leq 60.00$. Figure 7 is the Dirichlet L-function $L(1/2 + it, \chi)$, where χ is the Dirichlet character mod 5 determined by $\chi(2) = \exp(2\pi i \frac{3}{4})$. It is interesting that the Hurwitz zeta function moves rather randomly but their sum, the Dirichlet L-function, moves as the Riemann zeta function. We calculated their values by the Euler-Maclaurin summation formula.

For L-functions associated to elliptic curves, we study four examples with different Mordell-Weil ranks. Let

$$
\begin{aligned}
E_{11} : y^2 + y &= x^3 - x^2 - 10x - 20, & N &= 11, & r &= 0, \\
E_{37} : y^2 + y &= x^3 - x, & N &= 37, & r &= 1, \\
E_{446} : y^2 + xy &= x^3 - x^2 - 4x + 4, & N &= 446, & r &= 2, \\
E_{5077} : y^2 + y &= x^3 - 7x + 6, & N &= 5077, & r &= 3,
\end{aligned}
$$

where N and r represent the conductor and the Mordell-Weil rank of E, respectively. (See Figures 8, 9, 10, 11, 13, 14, 16, 18, 17 and 19.) In the following subsections, we show the graphs of $L(s, E_N)$ and some other data. For these, we used the method of Sections 2 and 4.

Figures 12, 15 and 20 are the graphs of $L(1 + it, E_N)$ when t is close to 0. By Tables 1, 2, and 3, up to the numerical precision of the calculation, we can see that $L(s, E_{37}), L(s, E_{446})$ and $L(s, E_{5077})$ have zeros of order 1, 2 and 3 at $s = 1$, respectively. This observation is compatible with the Birch and Swinnerton-Dyer conjecture.

One can ask whether all non-trivial zeros lie on the critical line $\mathrm{Re}(s) = 1$ or not. The generalized Riemann hypothesis insists that this is the case. As is stated in Section 1, Yoshida and Fermigier showed that it holds true in the range of their calculation. In our cases, we have

Theorem 4. *The generalized Riemann Hypothesis is true for the above four elliptic curves in the range* $\mathrm{Im}(s) \leq 400$. *Moreover, all non-trivial zeros except the one at* $s = 1$ *are simple.*

Theorem 4 can be shown by the classical method of Backlund (cf. Edwards [3], Yoshida [24]). The zeros in this range are listed in Tables 4, 5 and 6.

Figures 21, 22 and 23 are the graphs of $L(\sigma + it, E_{5077})$ $17.7 \leq t \leq 20.5$ for $\sigma = 0.98, 1.00, 1.02$, respectively. It is interesting to see how $L(s, E)$ takes the value 0 on the critical line $\mathrm{Re}(s) = 1$.

Figures 24, 25, 26 and 27 are numerical experiments of Conjecture 1 in Section 1. In these figures, the horizontal axis and vertical axis represent K and $\sqrt{K} D_K^{(ST)}(x_n)$, respectively.

We used Kida's UBASIC86 on 80486-66Mhz first and Pentium-133Mhz later. We also used PARI for the calculations of a_n and Mathematica for drawing the graphs. According to Theorem 1, time evaluation for the calculation of $L(1 + it, E_N)$ is $O(t\sqrt{N})$ if we ignore the incomplete gamma function. But the bottleneck is the calculation of the incomplete gamma function. It takes an enormous amount of time at the present stage. For example, on 80486-66Mhz, it takes about 150 hours for $L(1 + it, E_{11})$ and 900 hours for $L(1 + it, E_{5077})$ with sufficiently many division points in the range $0 \leq t \leq 400$. So we did not aim to make a complete list here, leaving it as a future problem. But for a single t, even if it is large, we can calculate the value of $L(s, E)$ in the critical strip. For example,

$$
\begin{aligned}
L(1 + 1000i, E_{5077}) &= 0.97714 - 0.79882i, \\
L(1 + 3000i, E_{5077}) &= 1.87754 - 1.90091i.
\end{aligned}
$$

The correctness of our calculation is checked by comparison with Yoshida's data for $L(s, E_{11})$. Moreover, for the above four curves, we checked the invariance of the data with $\delta(t) = 1/(1 + t)$ and $\delta(t) = 1/(1 + \log^2 t)$. (It seems to be a convenient way to check the correctness.)

One can find our programs on `ftp://ftp.math.metro-u.ac.jp/tnt`.

5.1. The Riemann zeta function: $\zeta(1/2 + it)$.
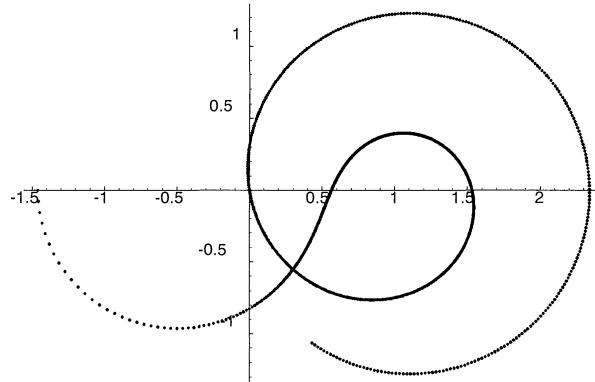
FIGURE 1. $0 \leq t \leq 20$
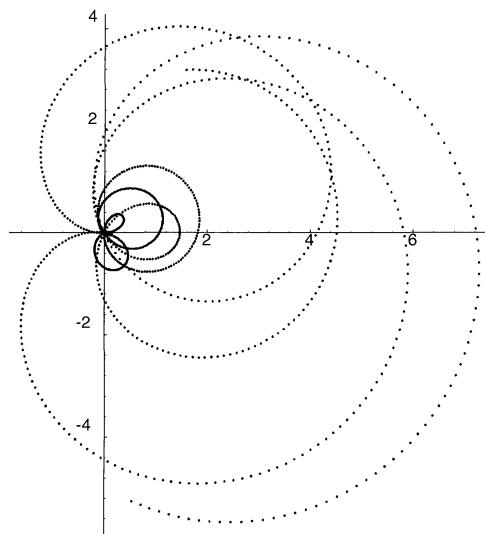
FIGURE 2. $3000 \leq t \leq 3010$

(The horizontal and vertical axes indicate the real and imaginary parts of the Riemann zeta function.)

The Hurwitz zeta function: $\zeta(1/2 + it, \alpha)$.
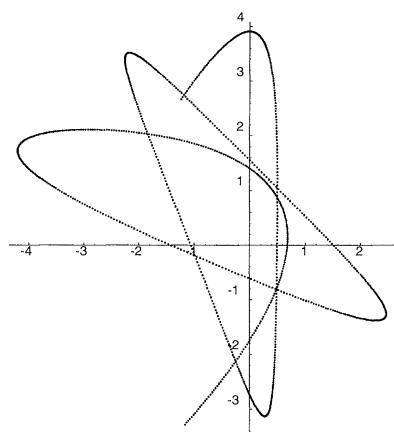
FIGURE 3. $\alpha = 1/5$ for $50 \leq t \leq 60$
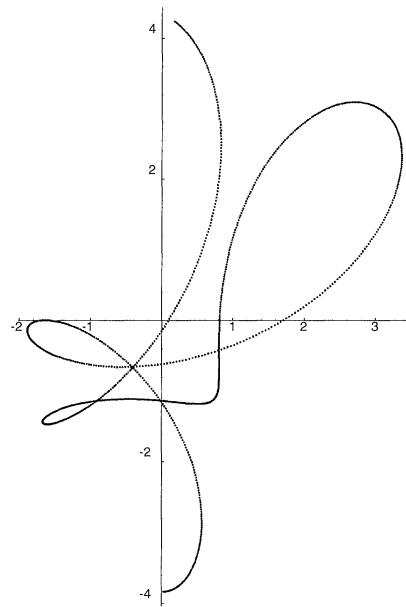
FIGURE 4. $\alpha = 2/5$ for $50 \leq t \leq 60$
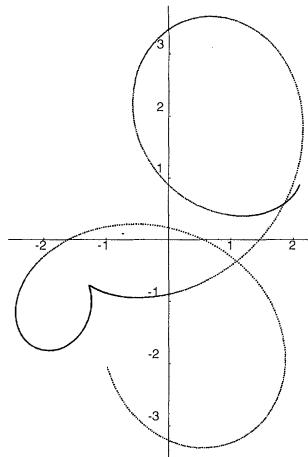
FIGURE 5. $\alpha = 3/5$ for $50 \leq t \leq 60$
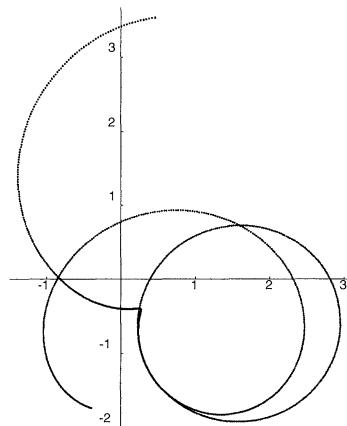
FIGURE 6. $\alpha = 4/5$ for $50 \leq t \leq 60$

(The horizontal and vertical axes indicate the real and imaginary parts of the Hurwitz zeta function.)

The Dirichlet L-function: $L(1/2 + it, \chi)$.
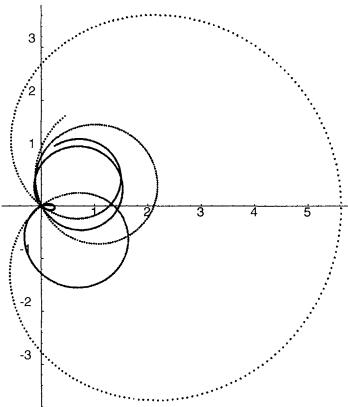
Here χ is the Dirichlet character mod 5 determined by $\chi(2) = \exp(2\pi i \frac{3}{4})$. Hence, we have

$$L(s, \chi) = 5^{-s} \sum_{j=1}^{4} \chi(j) \zeta(s, j/5).$$

FIGURE 7. $50.00 \leq t \leq 60$

5.2. *L*-function of E_{11}: $L(1 + it, E_{11})$
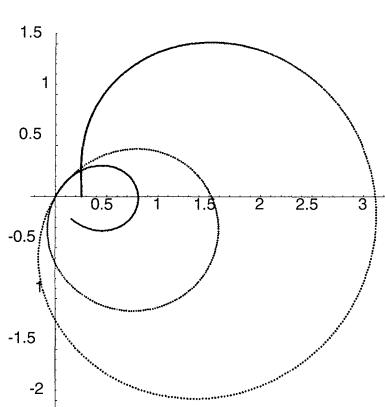
FIGURE 8. $0.00 \leq t \leq 9.89$
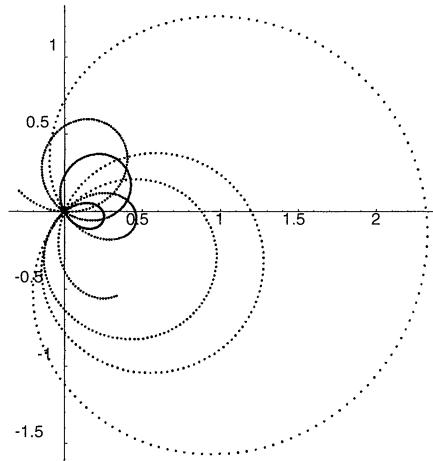
FIGURE 9. $397.00 \leq t \leq 401.00$

(The horizontal and vertical axes indicate the real and imaginary parts of the $L(1/2 + it, \chi)$ and $(1 + it, E_{11})$.)

5.3. *L*-function of E_{37}: $L(1 + it, E_{37})$.
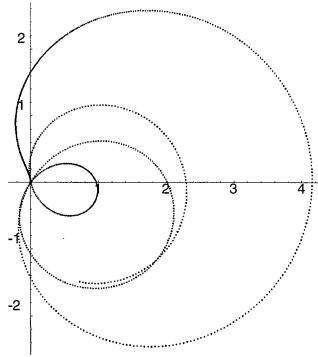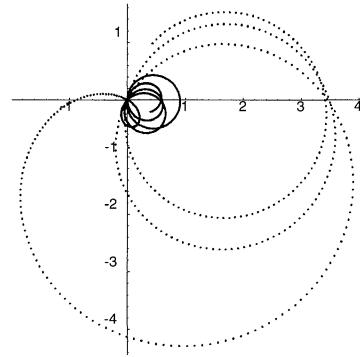
FIGURE 10. $0.00 \le t \le 9.39$ FIGURE 11. $396.03 \le t \le 400.00$
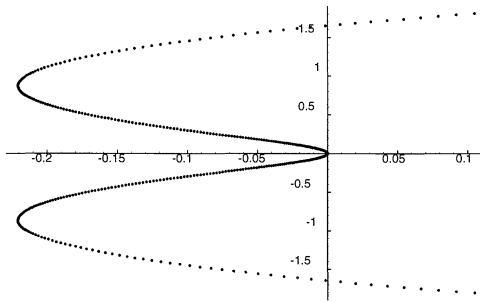
FIGURE 12. $-1.99 \le t \le 1.99$

TABLE 1

t	$L(1 + it, E_{37})$
0.000	0.000000000000000
0.001	-0.000000186547781+0.000305999910626 i
0.002	-0.000000746190931+0.000612000642000 i
0.003	-0.000001678928871+0.000918003014876 i
0.004	-0.000002984760633+0.001224007850009 i
0.005	-0.000004663684865+0.0015300159 68161 i
0.006	-0.000006715699827+0.001836028190105 i
0.007	-0.000009140803393+0.002142045336623 i
0.008	-0.000011938993049+0.002448068228511 i
0.009	-0.000015110265897+0.002754097686581 i
0.010	-0.000018654618649+0.003060134531663 i
0.011	-0.000022572047631+0.003366179584607 i
0.012	-0.000026862548784+0.003672233666284 i
0.013	-0.000031526117659+0.003978297597591 i
0.014	-0.000036562749423+0.004284372199452 i
0.015	-0.000041972438852+0.004590458292819 i

(The horizontal and vertical axes indicate the real and imaginary parts of the $L(1 + it, E_{37})$.)

5.4. *L*-function of E_{446}: $L(1 + it, E_{446})$.
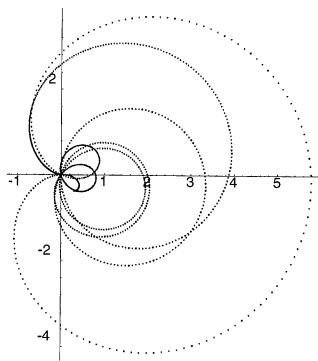
FIGURE 13. $0.00 \leq t \leq 9.39$
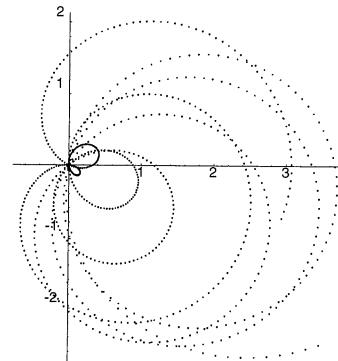
FIGURE 14. $396.03 \leq t \leq 400.00$
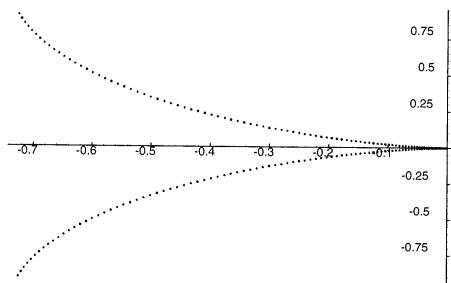
FIGURE 15. $-1.99 \leq t \leq 1.99$

TABLE 2

t	$L(1 + it, E_{446})$
0.000	0.000000000000000
0.001	-0.000000940282805+0.000000000597143i
0.002	-0.000003761133629+0.000000004777156i
0.003	-0.000008462559692+0.000000016122982i
0.004	-0.000015044573030+0.000000038217700i
0.005	-0.000023507190494+0.000000074644603i
0.006	-0.000033850433742+0.000000128987263i
0.007	-0.000046074329248+0.000000204829601i
0.008	-0.000060178908289+0.000000305755963i
0.009	-0.000076164206951+0.000000435351183i
0.010	-0.000094030266122+0.000000597200654i
0.011	-0.000113777131493+0.000000794890405i
0.012	-0.000135404853548+0.000001032007162i
0.013	-0.000158913487567+0.000001312138424i
0.014	-0.000184303093618+0.000001638872532i
0.015	-0.000211573736555+0.000002015798735i

(The horizontal and vertical axes indicate the real and imaginary parts of the $L(1 + it, E_{446})$.)

5.5. *L*-**function of** E_{5077}: $L(1 + it, E_{5077})$.
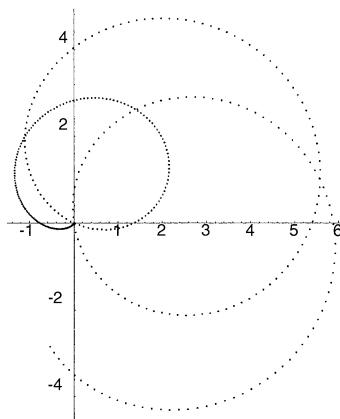

FIGURE 16. $0 \leq t \leq 4.19$
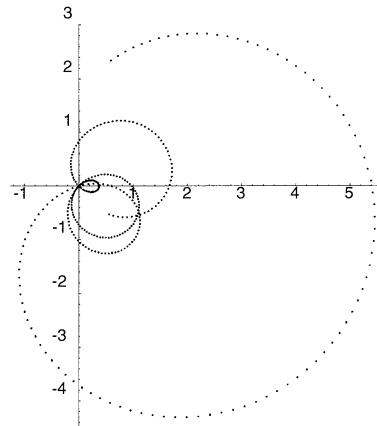
FIGURE 17. $5.00 \leq t \leq 8.39$
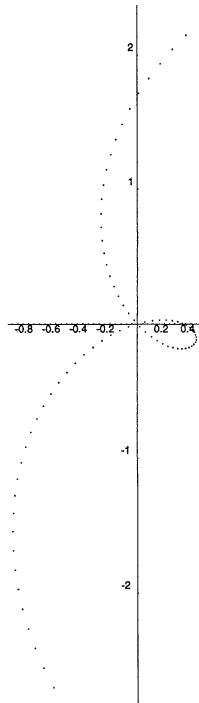
FIGURE 18. $4.20 \leq t \leq 4.99$

(The horizontal and vertical axes indicate the real and imaginary parts of the $L(1 + it, E_{5077})$.)
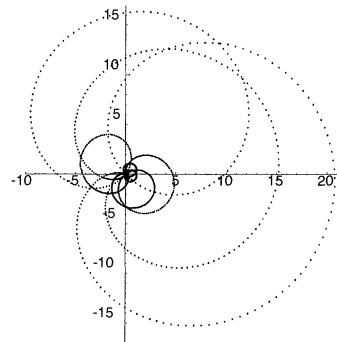
FIGURE 19. $390.00 \leq t \leq 396.00$
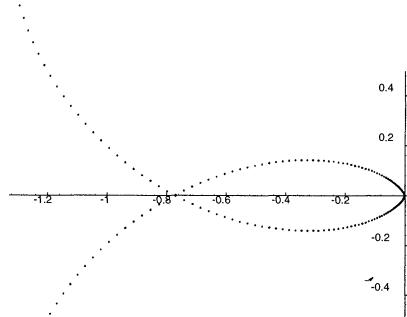
FIGURE 20. $-0.99 \leq t \leq 0.99$

TABLE 3

t	$L(1+it, E_{5077})$
0.000	0.000000000000000
0.001	-0.000000000003206-0.000000001731847 i
0.002	-0.000000000051294-0.000000013854710 i
0.003	-0.000000000259678-0.000000046759267 i
0.004	-0.000000000820708-0.000000110835526 i
0.005	-0.000000002003678-0.000000216472487 i
0.006	-0.000000004154815-0.000000374057805 i
0.007	-0.000000007697282-0.000000593977454 i
0.008	-0.000000013131172-0.000000886615393 i
0.009	-0.000000021033507-0.000001262353229 i
0.010	-0.000000032058228-0.000001731569878 i
0.011	-0.000000046936197-0.000002304641235 i
0.012	-0.000000066475186-0.000002991939830 i
0.013	-0.000000091559873-0.000003803834498 i
0.014	-0.000000123151835-0.000004750690038 i
0.015	-0.000000162289539-0.000005842866880 i

(The horizontal and vertical axes indicate the real and imaginary parts of the $L(1 + it, E_{5077})$.)

Graphs of $L(\sigma + it, E_{5077})$ for various σ, and $17.7 \le t \le 20.045$.
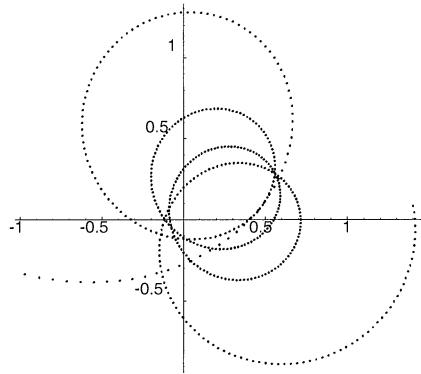
FIGURE 21. $\sigma = 0.98$
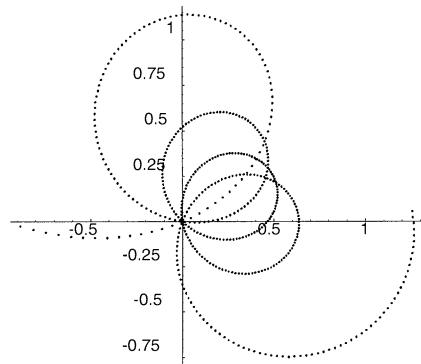
FIGURE 22. $\sigma = 1.00$
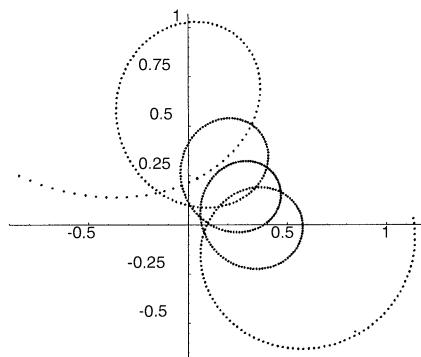
FIGURE 23. $\sigma = 1.02$

(The horizontal and vertical axes indicate the real and imaginary parts of the $L(\sigma + it, E_{5077})$.)

5.6. Numerical experiment of Conjecture 1.
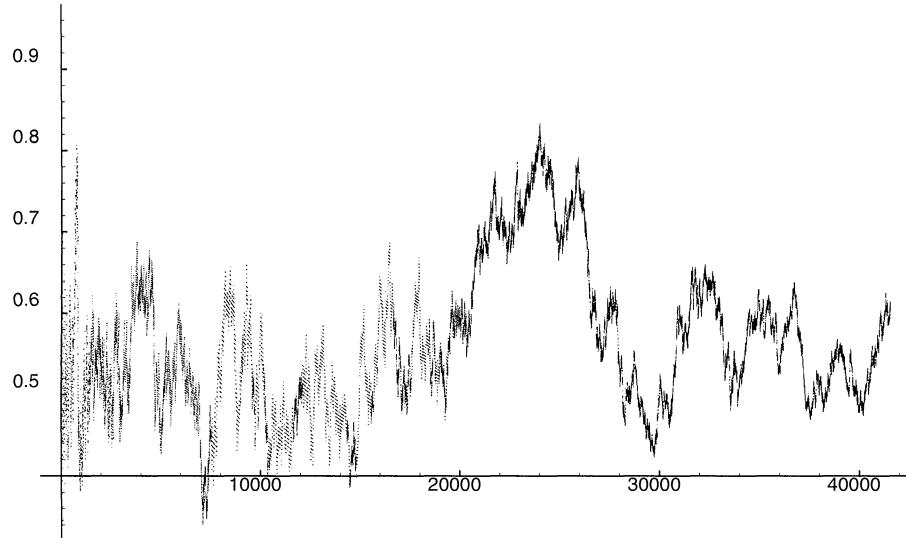
FIGURE 24. Discrepancy for E_{11}
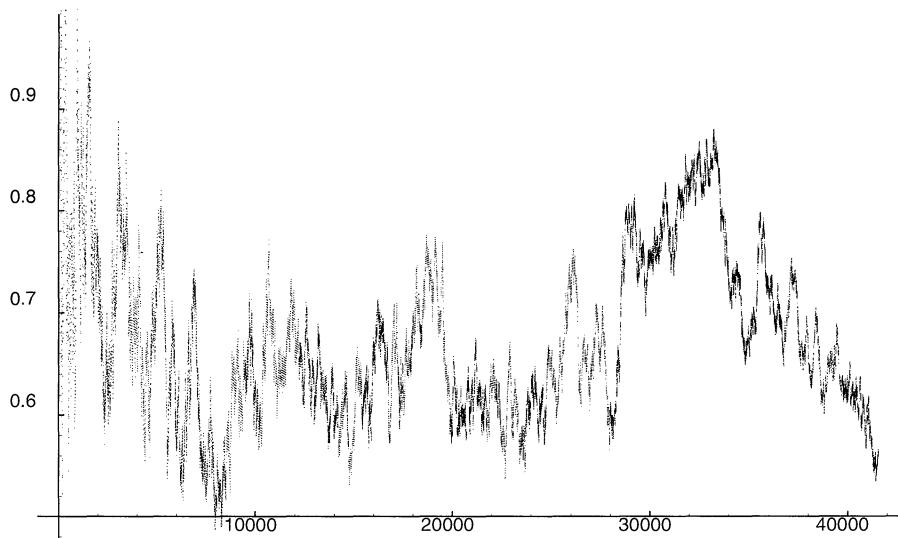
FIGURE 25. Discrepancy for E_{37}

(The horizontal and vertical axes indicate K and $\sqrt{K}D_K^{(ST)}(x_n)$, respectively.)
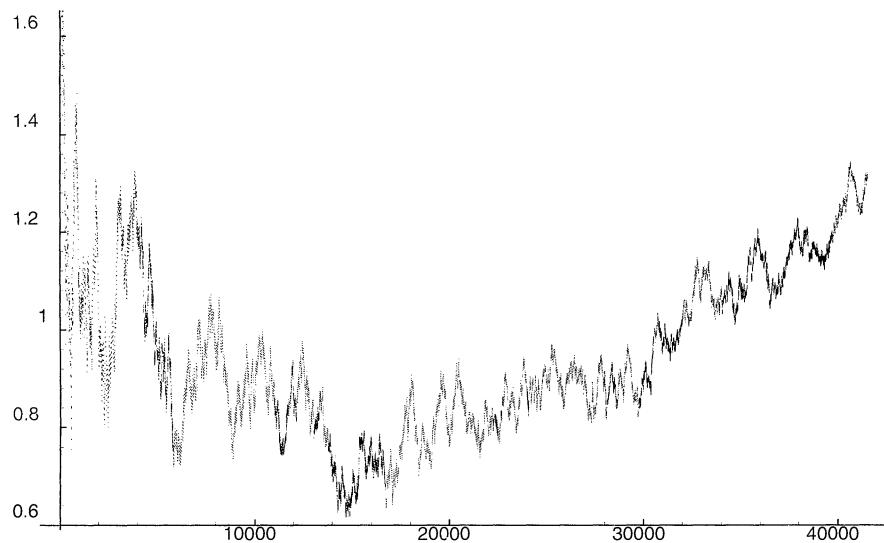
FIGURE 26. Discrepancy for E_{446}
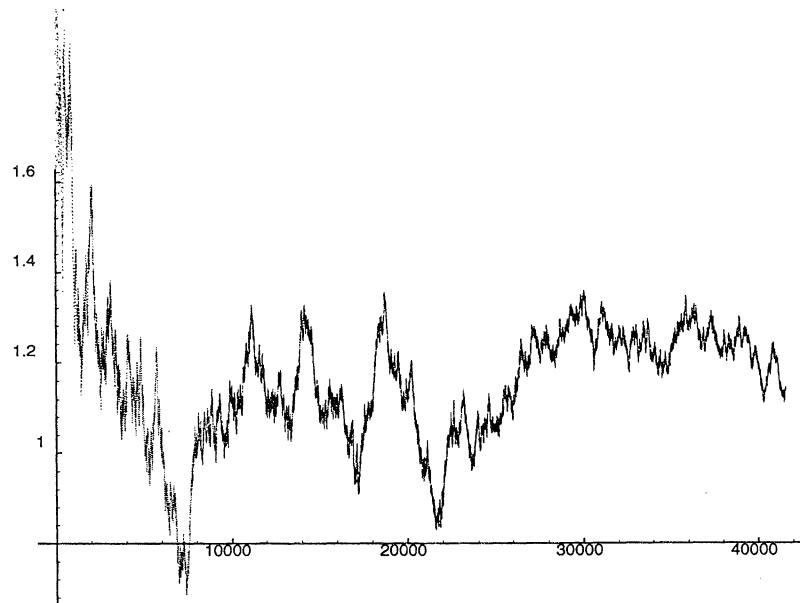
FIGURE 27. Discrepancy for E_{5077}

(The horizontal and vertical axes indicate K and $\sqrt{K}D_K^{(ST)}(x_n)$, respectively.)

5.7. Zeros of *L*-functions. In Tables 4, 5, 6 and 7, we list the imaginary parts of zeros of *L*-functions. All zeros we have found are of the form $s = 1 + it$.

TABLE 4. Zeros of $L(s, E_{11})$, $\text{Im}(s) \leq 400$

6.363	8.604	10.036	11.451	13.569	15.914	17.034	17.941	19.186	20.379	22.172	23.301
25.210	25.876	27.068	28.450	28.684	29.975	31.664	33.083	34.113	35.236	35.723	37.036
37.762	38.833	40.147	41.038	42.512	43.429	44.746	45.159	45.910	46.758	47.972	48.808
50.244	51.371	52.256	52.913	54.007	54.763	55.571	56.480	57.332	58.184	59.560	60.612
61.614	62.339	63.027	63.898	64.641	65.172	66.399	67.682	68.283	69.304	70.384	71.178
71.997	72.547	73.573	74.416	74.610	75.881	77.126	77.677	79.277	79.804	80.363	81.052
81.952	82.649	83.562	84.219	85.170	86.242	86.984	88.016	88.825	89.627	90.451	91.004
91.798	92.166	93.192	94.124	95.081	96.169	96.972	97.590	98.382	99.228	100.001	100.211
101.499	101.876	103.108	103.524	104.404	105.672	106.654	106.950	107.962	108.631	109.096	109.742
110.664	111.261	112.151	113.388	113.845	114.928	115.543	116.533	116.765	117.807	118.547	119.221
119.692	120.301	121.362	122.279	123.094	123.865	125.017	125.438	126.322	126.484	127.243	128.283
128.728	129.392	130.603	131.181	131.940	132.780	133.698	134.444	135.084	135.657	136.471	136.865
137.917	138.129	138.968	140.052	140.755	141.956	142.459	143.163	143.672	144.393	145.416	145.631
146.295	147.141	147.971	148.674	149.226	150.385	150.945	151.967	152.453	153.389	153.789	154.548
154.977	155.556	156.251	157.304	158.095	158.615	159.689	160.455	161.024	161.720	162.380	162.890
163.756	164.288	165.056	165.721	166.001	167.246	167.835	168.738	169.575	170.518	170.768	171.497
172.075	172.658	173.366	174.063	174.591	175.288	176.486	177.163	177.632	178.309	179.345	180.014
180.442	181.166	181.883	182.372	182.884	183.763	184.128	185.318	185.726	186.875	187.530	188.262
188.836	189.521	189.899	190.576	191.277	191.878	192.667	193.241	194.057	194.839	195.177	196.549
197.063	197.557	198.194	199.113	199.693	200.046	200.679	201.147	201.954	202.731	203.577	204.312
205.048	205.930	206.366	206.952	207.572	208.391	208.892	209.464	210.047	210.746	211.462	212.097
212.989	213.353	214.545	215.005	216.049	216.537	216.870	217.357	218.192	218.672	219.471	219.866
220.645	221.604	222.302	222.842	223.875	224.136	224.926	225.744	226.176	226.657	227.431	228.142
228.726	228.980	229.690	230.691	231.619	231.993	232.967	233.571	234.291	234.654	235.340	235.952
236.316	237.245	237.422	238.656	239.001	239.950	240.399	241.116	242.130	242.474	243.362	243.919
244.304	245.206	245.646	246.011	246.672	247.319	248.419	248.562	249.516	250.453	251.260	251.676
252.118	252.717	253.475	253.922	254.733	255.270	255.667	256.383	257.179	257.834	258.538	259.009
260.175	260.574	261.296	261.834	262.543	262.967	263.642	263.911	264.567	265.492	265.796	266.832
267.653	268.157	268.731	269.448	270.264	270.785	271.250	271.812	272.363	273.203	273.644	274.235
274.866	275.384	276.128	277.160	277.574	278.463	279.161	279.508	280.315	280.749	281.008	281.837
282.504	283.091	283.469	284.454	285.030	285.922	286.279	287.196	287.694	288.325	288.912	289.696
290.187	290.747	291.166	291.673	292.513	292.756	293.849	294.579	294.918	295.864	296.789	297.206
297.693	298.017	298.941	299.301	300.097	300.437	300.970	301.894	302.584	303.021	303.633	304.370
305.131	306.159	306.224	307.010	307.590	308.110	308.663	309.302	309.788	310.280	310.734	311.764
312.508	312.987	313.999	314.324	315.066	315.528	316.147	316.900	317.246	317.831	318.434	319.163
319.683	319.978	320.955	321.521	322.328	322.847	323.603	324.388	324.818	325.616	325.943	326.435
326.786	327.497	328.204	328.850	329.335	330.090	330.765	331.553	332.136	332.587	333.417	334.111
334.255	335.285	335.388	336.428	336.688	337.285	337.711	338.455	339.211	339.730	340.503	341.470
341.986	342.320	343.144	343.526	344.008	344.786	345.194	345.720	346.427	346.700	347.696	348.539
348.927	349.323	350.129	350.993	351.603	352.161	352.449	353.350	353.787	354.256	354.785	355.358
355.706	356.666	357.238	357.980	358.538	359.377	359.895	360.728	361.104	361.401	362.042	362.744
363.335	363.719	364.561	364.900	365.578	366.135	366.788	367.525	368.325	368.920	369.210	370.302
370.662	371.193	371.603	372.228	372.667	373.188	373.980	374.290	375.158	375.916	376.578	377.062
377.894	378.239	378.832	379.564	380.099	380.775	381.112	381.422	382.397	382.805	383.355	384.039
384.412	385.424	385.847	386.651	387.565	387.816	388.488	389.051	389.116	390.115	390.646	390.870
391.624	392.369	392.650	393.716	394.259	394.543	395.565	395.959	396.828	397.176	397.836	398.169
399.005	399.619	399.952	400.447								

TABLE 5. Zeros of $L(s, E_{37})$, $\mathrm{Im}(s) \leq 400$

0.000	5.003	6.870	8.014	9.933	10.775	11.757	12.958	15.604	16.192	17.142	18.064
18.787	19.815	21.323	22.620	23.328	24.169	25.657	26.814	27.339	28.190	29.030	29.282
30.896	32.042	33.441	34.363	34.636	35.462	36.164	37.084	38.468	39.002	39.604	40.649
41.652	42.575	44.034	44.236	45.393	45.565	46.653	46.957	47.574	49.342	50.219	51.003
51.719	52.411	53.001	54.039	54.977	55.291	56.257	56.957	57.715	58.399	59.096	60.668
61.445	62.295	62.640	63.039	63.884	64.344	65.379	66.069	67.215	68.084	68.400	69.527
70.195	70.817	71.458	72.700	73.278	73.751	74.154	74.865	75.524	76.441	77.756	78.530
79.152	79.788	80.613	81.143	81.338	82.167	83.093	83.860	84.574	85.464	86.193	86.405
87.315	88.478	89.271	89.926	90.347	91.074	91.548	92.031	92.847	93.343	94.325	95.292
96.414	96.816	97.553	97.887	98.417	99.431	99.767	100.922	101.328	102.155	102.361	103.247
104.001	104.763	105.428	106.597	107.276	107.716	108.302	108.887	109.600	109.764	110.248	111.228
112.600	113.141	113.336	114.440	115.059	115.577	116.225	117.031	117.610	118.304	118.869	119.594
120.065	120.692	120.963	122.027	122.793	123.911	124.659	125.243	125.546	126.049	126.558	127.302
127.863	128.624	129.122	130.023	130.774	131.434	131.976	132.572	133.443	133.706	134.912	135.569
136.038	136.615	136.990	137.368	138.141	138.778	139.303	140.614	141.088	142.143	142.236	143.248
143.677	144.192	144.596	145.085	146.040	146.993	147.038	148.074	148.519	149.116	149.630	150.269
151.676	151.976	152.555	153.273	153.824	154.363	154.749	155.311	155.770	156.405	157.161	158.099
159.089	159.477	159.994	160.681	161.112	161.793	162.273	163.139	163.709	164.556	164.686	165.223
165.844	166.576	167.484	167.572	168.489	169.646	170.273	170.781	171.249	171.611	172.207	172.752
173.007	173.852	174.598	175.479	176.071	176.403	177.350	177.831	178.397	179.151	179.542	180.471
181.002	181.432	182.128	182.732	183.240	183.522	183.937	184.654	185.838	186.363	187.466	187.685
188.320	188.875	189.247	189.749	190.615	190.930	191.439	192.110	193.059	193.492	194.152	194.651
195.226	195.832	196.503	197.216	198.152	198.689	199.121	199.582	199.785	200.710	200.905	201.572
202.108	203.030	203.789	204.439	204.993	205.837	206.347	206.717	207.145	207.560	208.285	209.111
209.863	210.121	210.634	211.245	211.788	212.374	212.985	213.588	214.687	215.172	215.778	216.286
216.930	217.385	217.767	218.192	218.732	219.296	219.859	220.649	221.769	222.047	222.796	223.118
223.446	224.654	224.775	225.570	226.087	226.776	227.292	227.788	228.337	228.644	229.310	230.060
230.396	230.945	232.085	232.933	233.396	233.866	234.215	234.941	235.236	235.679	236.203	236.795
237.783	238.209	238.596	239.445	239.951	240.518	241.230	241.689	242.159	243.070	243.534	244.215
244.808	245.322	245.676	246.074	246.516	247.061	247.497	248.742	249.134	249.970	250.460	251.307
251.563	252.056	252.605	253.079	253.693	254.176	254.696	255.383	256.050	256.706	256.939	257.723
258.044	258.680	259.347	260.102	261.078	261.595	261.831	262.329	262.684	263.300	264.037	264.206
264.829	265.211	266.026	267.107	267.568	267.828	268.962	269.124	269.737	270.019	270.629	271.312
272.064	272.567	273.118	273.412	274.105	274.426	275.000	275.574	276.211	277.070	277.546	278.322
278.988	279.494	279.970	280.464	280.827	281.255	281.510	282.433	282.735	283.694	284.286	284.947
285.457	285.815	286.322	287.215	287.585	288.276	288.827	289.264	290.080	290.524	290.931	291.307
291.934	292.342	292.826	293.546	293.852	294.933	295.932	296.241	296.669	297.274	297.515	298.289
298.553	299.036	299.760	300.211	301.093	301.385	301.667	302.689	303.193	303.750	304.138	304.487
305.436	306.123	306.915	307.058	307.777	308.404	308.587	309.004	309.527	309.883	310.458	311.311
311.926	312.715	313.148	313.751	314.493	314.811	315.250	316.062	316.333	316.576	317.367	318.090
318.542	319.328	319.600	320.074	320.263	321.233	321.639	322.234	323.096	323.824	324.410	324.775
325.353	325.639	326.172	326.694	327.211	327.655	327.939	328.566	329.599	330.032	330.826	331.190
331.676	332.453	332.695	333.218	333.655	334.681	335.205	335.550	336.041	336.441	336.958	337.507
338.046	338.370	338.990	339.839	340.228	341.101	341.749	342.527	342.751	343.094	343.651	344.059
344.256	345.182	345.592	346.062	346.861	347.454	347.958	348.473	348.820	349.481	350.076	350.771
351.253	351.818	352.270	353.191	353.480	353.761	354.461	354.843	355.246	355.678	356.013	356.798
357.650	358.585	358.883	359.311	360.013	360.361	360.959	361.539	361.676	362.417	362.756	363.656
363.966	364.569	365.018	365.505	366.214	366.831	367.069	367.428	368.306	369.180	369.813	370.189
370.735	371.137	371.547	372.077	372.341	372.672	373.567	373.924	374.689	375.120	375.896	376.501
377.159	377.534	377.954	378.607	378.926	379.507	380.025	380.557	381.328	381.866	382.267	382.519
383.045	383.417	384.167	384.626	385.331	385.798	386.948	387.176	387.809	388.159	388.577	389.313
389.611	390.000	390.539	390.799	391.339	392.262	392.802	393.534	393.905	394.349	394.915	395.328
395.975	396.528	397.099	397.901	398.204	398.688	399.126	399.684	400.123			

TABLE 6. Zeros of $L(s, E_{446})$, $\mathrm{Im}(s) \le 400$

0.000	3.017	4.440	5.480	6.649	7.361	8.060	9.759	10.227	11.122	11.943	12.387
13.046	15.042	15.644	16.306	16.548	17.516	18.552	18.851	19.701	20.271	21.663	22.239
23.002	23.421	24.000	24.720	26.117	26.489	27.238	27.793	28.535	28.872	29.524	30.167
31.285	31.977	32.376	33.782	34.081	34.548	35.207	35.576	36.140	36.906	38.247	38.776
39.038	39.528	39.852	41.164	41.769	42.180	42.820	43.986	44.247	44.865	45.598	46.041
46.609	46.980	47.504	48.280	49.272	50.095	50.573	51.286	51.564	52.062	52.433	53.520
54.306	54.540	55.299	55.708	56.325	57.211	57.686	58.071	58.601	58.893	60.128	60.999
61.663	61.838	62.382	63.011	63.506	63.653	64.581	65.183	65.873	66.383	67.125	67.722
68.238	68.682	69.396	70.124	70.515	70.897	71.288	72.419	73.320	73.609	73.856	74.267
74.736	75.360	75.878	76.665	77.697	78.243	78.484	78.891	79.765	80.376	80.885	81.112
81.525	82.057	83.000	83.498	84.108	84.646	85.267	85.897	86.233	86.560	87.159	87.894
88.583	89.525	89.729	90.363	90.593	91.084	91.631	92.301	92.917	93.255	93.752	94.229
94.992	96.097	96.529	96.891	97.287	97.757	98.087	98.549	99.383	100.178	100.405	101.101
101.891	101.926	102.705	102.874	103.480	104.234	104.849	105.253	105.912	106.550	107.074	107.716
108.071	108.589	109.210	109.556	109.963	110.240	110.508	112.142	112.497	112.948	113.138	113.952
114.170	115.042	115.542	115.778	116.471	116.992	117.542	118.078	118.512	119.389	119.674	120.309
120.435	120.751	121.532	122.076	122.648	123.415	124.087	124.807	125.081	125.404	125.710	126.342
126.535	127.407	128.002	128.385	129.085	129.345	129.973	130.454	131.161	131.943	132.186	132.445
132.891	133.481	133.984	134.905	135.504	136.014	136.279	136.665	137.073	137.462	138.161	138.747
138.847	139.801	140.127	141.134	141.612	141.970	142.190	142.881	143.630	143.975	144.198	144.848
144.972	145.659	146.543	147.226	147.354	148.059	148.570	148.704	149.310	149.445	150.304	151.299
151.727	152.164	152.349	152.846	153.774	154.025	154.615	154.876	155.296	155.703	156.480	156.596
157.310	158.014	158.867	159.184	159.537	160.258	160.378	160.850	161.364	161.999	162.148	163.073
163.864	164.094	164.549	164.888	165.142	165.814	166.567	166.788	167.508	167.885	168.128	168.887
169.641	170.319	170.812	171.066	171.451	171.768	172.183	172.757	172.995	173.934	174.232	174.940
175.515	175.908	176.242	176.949	177.164	178.097	178.335	178.636	179.111	179.927	180.331	180.870
181.220	181.699	182.474	182.886	183.290	183.692	183.792	184.289	184.613	185.826	186.438	186.859
187.313	187.797	188.231	188.428	188.953	189.456	190.235	190.521	191.010	191.118	191.844	192.408
193.239	193.552	194.038	194.270	194.679	195.434	195.889	196.273	196.499	197.417	198.075	198.704
198.856	199.355	199.605	200.287	200.455	200.852	201.570	201.944	202.694	202.978	203.736	203.898
204.596	205.276	205.795	206.317	206.478	206.752	207.372	207.490	208.148	208.983	209.716	210.007
210.361	210.652	211.193	211.508	212.143	212.719	212.839	213.678	214.111	214.902	215.196	215.682
216.106	216.627	217.043	217.608	217.883	218.415	218.654	219.022	219.816	220.139	220.549	221.596
222.163	222.497	222.623	223.157	223.388	223.809	224.742	225.309	225.492	226.032	226.522	227.010
227.543	227.852	228.487	228.673	229.418	229.784	230.025	230.520	230.790	231.783	232.560	233.019
233.096	233.772	234.092	234.633	234.986	235.387	235.709	235.968	236.848	237.387	238.027	238.454
238.709	238.971	239.579	240.300	240.608	241.182	241.754	241.924	242.379	242.882	243.263	244.352
244.619	244.979	245.539	245.718	246.117	246.464	246.842	247.363	247.996	248.697	249.092	249.687
250.109	250.525	251.083	251.573	251.908	252.226	252.674	253.168	253.660	254.278	254.563	254.877
255.585	256.225	256.646	257.177	257.305	257.709	258.252	258.658	259.063	259.662	260.489	261.039
261.400	261.938	262.114	262.374	262.593	263.529	264.110	264.363	264.766	265.089	265.288	266.213
266.869	267.226	267.805	268.251	268.760	269.100	269.423	269.909	270.119	270.710	271.200	272.151
272.250	272.746	273.207	273.869	274.120	274.376	274.612	275.259	275.774	276.332	277.048	277.404
277.640	278.286	278.847	279.188	280.007	280.267	280.603	280.764	281.348	281.558	281.969	282.626
283.320	284.048	284.337	284.562	285.055	285.589	285.973	286.394	286.943	287.377	287.867	288.282
288.820	289.172	289.766	290.283	290.798	290.910	291.593	291.734	292.332	292.901	293.089	293.552
293.919	294.407	295.623	296.005	296.385	296.654	296.904	297.250	297.765	298.347	298.562	299.258
299.486	300.404	300.639	300.930	301.373	301.937	302.263	303.202	303.378	303.806	304.103	304.411
304.899	305.608	306.331	306.553	307.106	307.490	308.126	308.414	308.709	309.246	309.298	309.787
310.410	310.796	311.566	311.809	312.559	312.989	313.302	313.758	314.266	314.579	315.269	315.685
316.012	316.279	316.649	317.117	317.597	318.617	318.823	319.414	319.536	320.020	320.350	320.628
321.085	321.586	322.482	322.847	323.145	323.878	324.154	324.841	325.011	325.384	325.905	326.423
326.655	327.058	327.640	328.008	328.315	328.778	329.536	329.961	330.513	330.959	331.296	331.882
332.348	332.561	332.798	333.169	333.697	334.721	335.154	335.492	335.821	336.149	336.347	336.954
337.569	337.950	338.389	338.676	339.068	339.597	340.246	340.590	341.106	341.937	342.269	342.699
343.057	343.200	343.525	344.030	344.585	344.942	345.483	345.810	346.491	346.973	347.521	347.834
348.233	348.619	348.793	349.416	350.115	350.613	350.849	351.228	351.898	352.181	352.755	353.028
353.795	354.220	354.483	354.845	355.307	355.611	355.711	356.279	356.815	357.774	358.184	358.714
359.068	359.290	359.734	360.189	360.777	360.906	361.587	361.979	362.408	362.504	363.219	364.006
364.125	364.634	365.087	365.390	365.852	366.552	366.820	367.148	367.459	367.942	368.455	369.234
369.774	370.147	370.530	371.021	371.242	371.599	372.100	372.181	372.723	373.270	374.079	374.334
374.681	375.028	375.422	376.094	376.921	377.229	377.460	377.788	378.306	378.576	379.068	379.389
380.004	380.432	380.972	381.715	381.946	382.322	382.525	382.951	383.435	383.735	384.378	384.654
385.078	385.696	386.514	386.893	387.202	387.503	387.916	388.292	388.993	389.310	389.802	390.029
390.521	390.676	391.073	391.595	392.438	392.870	393.393	393.858	394.090	394.327	394.872	395.187
395.675	396.257	396.707	397.181	397.670	398.083	398.408	399.082	399.269	399.566	400.335	

TABLE 7. Zeros of $L(s, E_{5077})$, $\mathrm{Im}(s) \le 400$

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  0.000   2.052   3.262   4.471   4.754   6.012   6.623   7.343   7.707   8.477   9.382  10.203
 10.496  11.033  11.687  12.287  12.973  13.152  14.942  15.515  15.895  16.440  16.643  17.412
 18.073  18.560  19.031  19.497  19.975  20.660  21.759  22.216  22.735  23.224  23.721  24.030
 24.827  25.822  26.322  26.557  27.205  27.741  28.215  28.455  29.030  29.581  29.778  30.873
 31.675  31.827  32.338  33.462  33.975  34.394  34.547  35.200  35.648  35.817  36.427  36.913
 38.058  38.411  38.720  39.262  39.513  39.965  40.721  41.575  41.816  42.524  42.670  43.755
 44.262  44.430  44.988  45.323  46.028  46.336  46.743  46.999  47.413  48.029  48.932  49.488
 50.192  50.521  51.132  51.475  51.811  52.379  52.575  53.526  53.922  54.317  55.018  55.261
 55.633  55.837  56.859  57.229  57.777  58.096  58.459  58.689  59.695  60.127  61.152  61.534
 61.905  62.192  62.406  63.165  63.346  63.757  64.193  64.662  65.336  65.719  66.272  66.712
 67.463  67.915  68.274  68.546  69.163  70.023  70.303  70.709  70.817  71.315  72.024  72.925
 73.211  73.440  73.865  74.169  74.580  74.871  75.306  76.001  76.623  77.210  77.917  78.264
 78.690  78.916  79.613  80.315  80.428  80.958  81.212  81.487  82.004  82.312  83.197  83.604
 83.863  84.439  85.000  85.603  85.895  86.312  86.683  86.858  87.702  88.242  88.910  89.422
 89.845  90.003  90.431  90.865  91.209  91.622  91.973  92.666  92.964  93.422  93.821  94.148
 94.877  95.529  96.267  96.530  97.149  97.315  97.453  98.168  98.340  98.664  99.591  99.956
100.227 100.712 101.209 101.820 102.029 102.397 102.809 103.016 104.066 104.317 104.576 105.075
105.909 106.246 106.775 107.046 107.746 107.949 108.480 108.791 109.194 109.586 109.955 110.005
110.573 110.800 112.206 112.571 112.769 113.289 113.424 113.846 114.532 115.173 115.528 115.714
116.101 116.859 117.188 117.459 117.943 118.313 118.935 119.341 119.877 120.060 120.262 120.665
121.314 121.663 122.256 122.542 123.319 123.959 124.304 124.921 125.218 125.431 125.866 126.075
126.354 126.873 127.424 128.158 128.229 128.689 128.945 129.619 130.022 130.454 130.997 131.656
131.910 132.311 132.659 132.919 133.418 134.046 134.378 135.155 135.578 135.944 136.262 136.437
136.829 137.173 137.574 138.015 138.603 138.816 139.227 139.946 140.441 140.905 141.518 141.909
142.128 142.407 143.306 143.556 143.755 144.400 144.527 144.799 145.156 145.729 146.338 146.871
147.167 147.617 148.002 148.598 148.719 149.207 149.387 149.737 150.582 151.388 151.791 151.908
152.326 152.650 153.221 153.616 154.018 154.446 154.834 155.050 155.433 155.626 156.252 156.597
157.083 157.577 158.386 158.597 159.178 159.669 159.987 160.194 160.600 161.046 161.592 161.892
162.114 162.658 163.234 163.941 164.161 164.405 164.701 164.882 165.364 165.993 166.435 166.793
167.368 167.745 167.985 168.470 168.741 169.848 170.197 170.655 170.884 171.167 171.392 171.859
172.308 172.433 172.971 173.168 173.785 174.332 174.782 175.244 175.632 175.962 176.465 177.004
177.184 177.656 178.226 178.588 178.986 179.219 179.859 180.300 180.791 180.871 181.448 181.754
182.232 182.811 183.189 183.320 183.664 183.940 184.187 184.626 185.572 186.153 186.572 186.762
187.514 187.856 188.046 188.455 188.745 188.983 189.792 190.190 190.318 190.805 191.032 191.426
191.855 192.384 192.806 193.462 193.821 194.175 194.394 194.662 195.328 195.818 196.172 196.282
197.012 197.475 198.146 198.425 198.928 199.145 199.285 199.812 200.118 200.474 200.702 201.141
201.725 202.128 202.388 203.120 203.529 203.852 204.192 204.803 205.161 205.949 206.227 206.414
206.826 206.966 207.274 207.661 208.111 208.886 209.253 209.539 210.235 210.355 210.612 210.946
211.167 212.089 212.232 212.534 213.149 213.441 213.915 214.610 215.133 215.478 215.513 216.092
216.615 216.870 217.362 217.590 217.945 218.270 218.591 218.922 219.522 219.779 220.064 220.546
221.591 221.965 222.283 222.423 222.713 223.229 223.475 223.910 224.439 224.997 225.465 225.786
225.973 226.403 226.884 227.110 227.754 228.052 228.322 228.615 229.076 229.555 230.011 230.307
230.497 230.896 231.502 232.491 232.749 233.016 233.325 233.756 234.327 234.439 234.962 235.085
235.447 235.805 236.065 236.668 237.234 237.463 238.124 238.458 238.792 239.005 239.283 240.119
240.379 240.818 241.335 241.579 242.039 242.254 242.760 243.172 243.570 244.180 244.630 244.964
245.240 245.588 245.813 246.271 246.351 246.715 247.106 247.710 248.354 248.659 249.266 249.564
249.891 250.440 250.830 251.326 251.586 252.007 252.306 252.540 252.896 253.589 253.753 254.192
254.588 254.742 255.273 255.833 256.273 256.594 257.017 257.395 257.720 258.061 258.296 258.877
259.111 259.704 260.432 260.841 261.353 261.577 261.876 262.164 262.356 262.683 262.993 263.869
264.196 264.321 264.691 264.969 265.128 265.774 266.521 266.905 267.224 267.754 268.134 268.553
268.941 269.225 269.540 269.892 270.442 270.628 270.847 271.497 272.284 272.451 272.594 272.978
273.562 273.850 274.098 274.475 274.813 275.086 275.613 276.095 276.804 277.024 277.348 277.881
278.107 278.678 279.175 279.736 279.791 280.348 280.527 280.871 281.055 281.502 281.773 282.029
282.475 283.278 283.760 284.011 284.302 284.855 285.243 285.548 285.887 286.141 286.673 287.232
287.573 288.035 288.339 288.676 288.898 289.623 290.017 290.263 290.679 290.961 291.201 291.586
292.098 292.555 292.694 292.988 293.497 293.849 294.378 294.632 295.692 296.016 296.412 296.600
296.915 297.071 297.553 297.752 298.503 298.718 298.970 299.198 299.992 300.401 300.612 300.846
301.336 301.635 301.961 302.625 303.284 303.471 303.696 304.071 304.243 304.672 305.255 305.916
306.157 306.733 307.014 307.296 307.703 308.258 308.369 308.823 308.947 309.367 309.644 309.898
310.377 310.902 311.403 311.783 312.428 312.615 313.090 313.425 313.756 314.178 314.690 315.116
315.399 315.795 316.065 316.461 316.651 317.033 317.291 317.994 318.463 318.996 319.224 319.571
319.816 319.997 320.451 320.796 321.144 321.775 322.156 322.688 322.918 323.548 324.028 324.328
324.933 324.933 325.288 325.806 326.033 326.303 326.798 327.008 327.329 327.861 328.116 328.456
328.750 329.297 330.083 330.397 330.579 331.146 331.257 331.940 332.422 332.640 332.782 332.916
333.357 334.158 334.745 335.068 335.361 335.634 335.973 336.108 336.548 337.020 337.289 337.713
338.159 338.441 338.962 339.289 339.416 340.136 340.403 341.006 341.403 342.013 342.409 342.694
342.890 343.285 343.566 343.734 344.013 344.569 345.079 345.287 345.710 346.105 346.404 347.001
347.496 347.810 348.217 348.386 348.620 349.102 349.527 350.088 350.586 350.806 351.294 351.424
352.093 352.384 352.680 352.964 353.505 354.149 354.338 354.574 354.651 355.160 355.485 355.893
356.012 356.325 357.067 357.678 358.415 358.496 358.848 359.130 359.402 359.997 360.357 360.668
360.984 361.427 361.779 362.120 362.403 362.584 363.219 363.633 364.101 364.331 364.658 364.997
365.380 365.886 366.234 366.771 366.955 367.423 367.752 368.694 368.962 369.577 369.934
370.381 370.581 370.877 371.351 371.485 371.795 372.083 372.342 372.914 373.123 373.737 374.151
374.619 374.783 375.090 375.483 376.069 376.667 377.161 377.312 377.560 378.033 378.266 378.744
378.973 379.460 379.626 380.186 380.536 381.002 381.563 381.801 382.154 382.347 382.730 382.838
383.391 383.810 384.118 384.519 384.938 385.287 385.853 386.429 386.729 387.163 387.567 387.837
388.007 388.538 388.674 389.421 389.730 389.948 390.196 390.574 390.797 390.886 391.495 392.077
392.781 392.896 393.393 393.899 394.049 394.480 394.658 395.074 395.232 396.055 396.243 396.980
397.112 397.387 397.861 398.324 398.687 398.998 399.135 399.527 399.926 400.512
```

6. Appendix: Type sequence and Rosser's law

In the history of calculation of zeros of the Riemann zeta function, two laws, which are not exact laws, have appeared that play an important role. They are Gram's and Rosser's laws and formulated as follows. First we recall that the argument of $\zeta(1/2 + it)$ is given by $-\vartheta(t)$, where

$$\vartheta(t) = \text{Im}(\log \Gamma(\frac{1}{4} + i\frac{t}{2})) - \frac{1}{2}t \log \pi.$$

Let g_k be the positive real number which satisfies

$$\vartheta(g_k) = k\pi, \qquad k = -1, 0, 1, 2, \cdots.$$

These numbers g_k are called (k-th) Gram points. Gram's law is stated as follows.

Gram's law: There exists exactly one zero in the interval $(1/2 + ig_k, 1/2 + ig_{k+1})$.

This statement is slightly stronger than usual in the sense that it asserts "exactly one" instead of "at least one". To simplify matters, we assume that $\zeta(1/2 + ig_k) \neq 0$ for any Gram point g_k. If $\zeta(1/2 + ig_k) > 0$ (resp. < 0), then g_k is called a good (resp. bad) Gram point. If all the Gram points are good then Gram's law is true. But this is not the case. This law fails for the first time at $g_{126} = 282.454\cdots$. In fact, $\zeta(1/2 + it) \neq 0$ for $g_{125} < t < g_{126}$, and g_{126} is a bad Gram point.

Define a Gram block of length k by the set of consecutive Gram points $B_n = \{g_n, g_{n+1}, \ldots, g_{n+k}\}$, where g_n and g_{n+k} are good and g_{n+j} ($j = 1, 2, \ldots, k-1$) are bad. Then Rosser's law is stated as follows.

Rosser's law: Let $B_n = \{g_n, g_{n+1}, \ldots, g_{n+k}\}$ be a Gram block of length k, then $\zeta(s)$ has at least k zeros in the interval $(1/2 + ig_n, 1/2 + ig_{n+k})$.

This law fails for the first time at $B_{13999525}$.

Now we classify all the non-trivial zeros of $\zeta(s)$ into five classes.

Definition 1. Let $u = 1/2 + it_0$ ($t > 0$) be a zero of $\zeta(s)$. Define the zero u to be of type j ($j = 1, 2, 3, 4$) when $-i\zeta'(u)$ belongs to the j-th quadrant R_j. If u is not in the above cases, we say that u is of type 0.

It seems that type 0 does not occur for the Riemann zeta function. But for the case of the L-function associated to an elliptic curve E of Mordell-Weil rank greater than one, it actually happens on the critical line $\text{Re}(s) = 1$, because $L'(1, E) = 0$ (under the Birch and Swinnerton-Dyer conjecture for this curve). Let ε be a small positive number. When $u = 1/2 + it_0$ is of type j, then the orbit of $\zeta(1/2 + it)$ for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ starts from R_j and passes through the origin.

We define the type sequence by the sequence of types of zeros and Gram points (represented by G or B according to a good or a bad Gram point) arranged in increasing order. For example, in the middle of Table 8, one can find the sequence "G B24 G4". The meaning of this sequence is as follows. As t is increasing, the **G**ood Gram point($=g_{125}$), the **B**ad Gram point ($=g_{126}$), zero of type **2**, zero of type **4**, the **G**ood Gram point($=g_{127}$) and zero of type **4** occur in this order. Hence we see that $B_{125} = \{g_{125}, g_{126}, g_{127}\}$ is a Gram block of length 2 and contains 2 zeros. We mentioned above that Rosser's law fails at $B_{13999525}$. The type sequence about this point is "B$\underline{\text{GBG}}$424G". The Table 8 suggests that:

1. Gram's law seems to be true in a certain average sense,
2. A bad Gram point occurs when the order of a zero and a Gram point is reversed.

Let us consider the first point. Let $S(r) = \pi^{-1} \arg \zeta(1/2+ir)$ where the branch of $i \arg \zeta(s) = \mathrm{Im}(\log(\zeta(s)))$ is taken along the lines joining $2, 2+ir$ and $2+ir, 1/2+ir$. It is known that the number of zeros in the rectangle $\{z \in \mathbf{C} : 0 < \mathrm{Re}(z) < 1, 0 < \mathrm{Im}(z) \leq r\}$ is equal to

$$\frac{r}{2\pi} \log(\frac{r}{2\pi}) - \frac{r}{2\pi} + S(r) + O(1/r).$$

We can easily see that

$$\frac{g_k}{2\pi} \log(\frac{g_k}{2\pi}) - \frac{g_k}{2\pi} = k + O(1/g_k).$$

Hence, it would be plausible that

Conjecture 2. $S(g_k) = 0$ *for all the good Gram points g_k except the ones with zero density.*

Conjecture 2 has another heuristic explanation. Let g_k be a "generic" good Gram point. Then we may assume that the value of $\zeta(1/2 + ig_k)$ is greater than 1 (see Theorem 10.6 of [20]). Let $\sigma > 1/2$ and δ be a small positive number. We compare the graphs of $\zeta(1/2 + ir)$ and $\zeta(\sigma + ir)$ in the Gaussian plain for $r \in [g_k - \delta, g_k + \delta]$. When σ is increasing, the orbit of the graph of $\zeta(\sigma + ir)$ moves to the outer normal direction by the Cauchy-Riemann relation. Note that the argument of $\zeta(1/2 + ir)$ always decreases. Thus when σ gets larger, we may expect that the graph of $\zeta(\sigma + ir)$ will naturally approach a single value

$$\lim_{\sigma \to \infty} \zeta(\sigma + ir) = 1.$$

In this process, the graph would not approach the origin, as $\zeta(1/2 + ig_k) > 1$. This suggests Conjecture 2. Of course, in the above argument, we assume that there are no non-trivial zeros in $\{z \in \mathbf{C} : 1/2 \leq \mathrm{Re}(z), |\mathrm{Im}(z) - g_k| \leq \delta\}$.

If Conjecture 2 is true, we could formulate Rosser's law in a strengthened way:

Each Gram block of length k has exactly k zeros.

In fact, if a Gram block of length k has more than k zeros, then the front or rear Gram block may have length more than the number of zeros in it.

Now we will treat the second point. Consider relatively simple cases when a zero and its adjacent Gram point are reversed. When the graph of $\zeta(1/2 + ir)$ moves a little along the real axis, then the changes

(29) $3G \to B2, \qquad G4 \to 1B$

occur in the type sequence. On the other hand, when the graph moves along the imaginary axis, the changes

(30) $1B \to 2B, \qquad B2 \to B1$

occur. For example, $GB24G4$ or $G31BG4$ in Table 8 are obtained from the normal sequence $G3G4G4$ by the replacements in (29).

Let T be the type sequence provided with Rosser's law (we treat here the "abstract" type sequence), and let T' be the type sequence generated from T by the successive replacements of (29) and (30). Then it is easily seen that T' also satisfies Rosser's law. Thus Rosser's law, somewhat complicated at first glance, can be grasped in the context of *Gram's law in the "average" sense.*

Our argument is naturally extended to L-functions associated to elliptic curves. Tables 9 and 10 are the type sequence of the L-functions associated to E_{11} and E_{5077}.

TABLE 8. Type sequence of the Riemann zeta function   Read G4G3G4G3···

G4	G3	G4	G3	G4	G3	G3	G4	G3	G4	G4	G3	G3	G4	G3
G4	G4	G4	G3	G4	G4	G3	G4	G3	G4	G3	G3	G4	G4	G3
G3	G4	G4	G3	G4	G4	G4	G3	G3	G4	G4	G3	G3	G4	G3
G4	G3	G3	G4	G4	G3	G4	G3	G4	G4	G3	G3	G4	G4	G4
G3	G3	G3	G4	G3	G3	G4	G4	G4	G3	G3	G4	G4	G4	G3
G4	G3	G4	G4	G3	G3	G3	G4	G3	G3	G3	G3	G4	G4	G3
G3	G4	G3	G4	G3	G3	G3	G4	G4	G3	G3	G3	G4	G4	G4
G3	G3	G4	G4	G4	G3	G4	G3	G4	G4	G3	G3	G4	G4	G4
G4	G3	G3	G4	G4	G4	G	B24	G4	G3	G4	G4	G3	G3	G31
B	G4	G3	G3	G4	G3	G4	G4	G3	G3	G4	G4	G4	G3	G4
G3	G4	G4	G4	G3	G3	G4	G4	G3	G4	G3	G3	G3	G4	G4
G3	G3	G4	G3	G4	G4	G4	G3	G3	G4	G4	G4	G3	G3	G3
G3	G4	G4	G3	G3	G3	G4	G4	G4	G4	G3	G3	G4	G4	G4
G	B23	G4	G3	G4	G4	G3	G3	G3	G4	G4	G3	G3	G3	G3
G4	G31	B	G3	G3	G4	G4	G4	G4	G3	G4	G3	G3	G4	G4

TABLE 9. Type sequence for $L(s, E_{11})$ Read G3G3G4G4···

G3	G3	G4	G4	G4	G3	G3	G4	G4	G4	G3	G4	G3	G3	G3
G31	B	G4	G3	G3	G3	G3	G4	G3	G4	G4	G4	G4	G3	G3
G	B24	G4	G4	G4	G4	G3	G3	G3	G4	G3	G4	G4	G4	G4
G4	G3	G3	G3	G3	G4	G4	G41	B	G4	G3	G4	G3	G3	G3
G3	G4	G3	G31	B	G3	G3	G4	G	B23	G3	G4	G4	G4	G4
G4	G4	G3	G3	G3	G3	G3	G3	G4	G41	B	G4	G4	G4	G3
G3	G3	G3	G3	G31	B	G3	G4	G3	G4	G4	G3	G	B23	G3
G3	G4	G4	G4	G4	G4	G3	G4	G3	G3	G3	G4	G3	G3	G4
G41	B	G4	G3	G3	G3	G	B23	G3	G4	G4	G3	G4	G4	G3
G4	G4	G3	G3	G3	G3	G3	G3	G4	G31	B	G4	G3	G4	G
B23	G3	G4	G4	G3	G4	G4	G4	G4	G4	G4	G3	G3	G3	G3
G3	G3	G3	G4	G41	B	G4	G3	G4	G3	G3	G3	G3	G3	G4
G3	G4	G4	G41	B	G3	G4	G3	G3	G	B23	G3	G4	G4	G4
G4	G4	G4	G3	G3	G4	G4	G3	G3	G3	G3	G3	G4	G4	G41
B	G3	G4	G3	G3	G3	G3	G3	G4	G4	G4	G4	G4	G4	G4

TABLE 10. Type sequence of $L(s, E_{5077})$ Read 0BG4G4···

0	B	G4	G4	G3	G4	G3	G3	G3	G4	G4	G3	G3	G4	G4
G4	G4	G41	B	G	B2	B23	G3	G4	G4	G3	G4	G4	G41	B
G4	G3	G3	G3	G3	G41	B	G4	G	B2	B23	G3	G3	G3	G4
G4	G42	B	G4	G3	G4	G4	G	B2	B2	B24	G3	G31	B1	B1
B	G	B23	G4	G41	B1	B	G4	G3	G4	G3	G4	G	B2	B23
G3	G4	G3	G4	G41	B1	B1	B	G3	G4	G	B23	G	B23	G4
G31	B	G3	G3	G3	G3	G4	G41	B	G3	G4	G3	G4	G41	B
G3	G4	G	B1	B2	B23	G4	G3	G4	G41	B1	B	G4	G4	G4
G4	G3	G3	G3	G4	G4	G	B23	G31	B	G4	G3	G	B2	B23
G4	G4	G41	B1	B	G4	G4	G3	G	B23	G3	G4	G3	G	B23
G3	G41	B	G41	B	G3	G3	G4	G4	G3	G3	G3	G4	G41	B
G3	G3	G	B2	B2	B23	G3	G4	G4	G41	B	G4	G4	G4	G41
B	G4	G3	G	B2	B2	B23	G4	G3	G41	B	G3	G3	G4	G4
G3	G3	G4	G4	G41	B	G3	G4	G4	G4	G	B23	G	B23	G
B23	G3	G3	G3	G4	G42	B1	B2	B	G	B2	B23	G3	G4	G4

In contrast with the Riemann zeta function, the type sequence is pretty complicated for $L(s, E_{5077})$ but does not violate Rosser's law in this range. There exist many reverses of $3G \rightarrow B2$ and $G4 \rightarrow 1B$. Furthermore, one can find some reverses coming from (30). For example, in Table 10, $G42BG$ is obtained as

$$G4G4G \rightarrow G41BG \rightarrow G42BG.$$

This observation shows why counterexamples of Rosser's law cannot be found in the short range. However, if we perform the calculation to a wider range, an arbitrary

complicated change of type sequences would happen, and might violate Rosser's law. We can actually observe such a change at the breaking point of Rosser's law.

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