

NEW MAXIMAL PRIME GAPS AND FIRST OCCURRENCES

THOMAS R. NICELY

ABSTRACT. The search for first occurrences of prime gaps and maximal prime gaps is extended to 10^{15} . New maximal prime gaps of 806 and 906 are found, and sixty-two previously unpublished first occurrences are found for gaps varying from 676 to 906.

1. INTRODUCTION

The study of the distribution of the prime numbers among the positive integers occupies a central place in number theory. This distribution may be specified by the differences $G_k = p_{k+1} - p_k$ between successive primes; G_k is also referred to as the (prime) gap following the k th prime p_k , and the unmodified symbol G will be used to refer to a specific gap as well as the set of all prime gaps of a specified magnitude. A gap G contains $G - 1$ consecutive composite integers. All gaps are even positive integers except for $G_1 = 1$; it is an open question whether or not gaps of magnitude $2n$ exist corresponding to each and every positive integer n . The “first occurrence” of a gap G is defined by the smallest prime p_k preceding (immediately followed by) such a gap. A (first occurrence of a) gap G is said to be “maximal” if all preceding gaps (between smaller consecutive primes) are strictly less than G . Thus the first occurrence of a gap of 10 follows the prime 139, but this is not a maximal gap, since an equal or larger gap (a maximal gap of 14, following 113) appears earlier in the sequence of positive integers. Note that some authors (e.g., Riesel [14, p. 80]) specify first occurrences by the terminating prime p_{k+1} , while others specify a gap by the parameter $r = G/2$ (e.g., Brent [2, 3]).

No general method more sophisticated than an exhaustive search is known for the determination of first occurrences and maximal prime gaps. As in the present study, this is most efficiently done by sieving successive blocks of positive integers for primes, recording the successive differences, and thus determining directly the first occurrences and maximal gaps. This technique has been used by Shanks [15], Lander and Parkin [10], Brent [2, 3], and Young and Potler [17] to extend the search through all primes $< 7.263512 \times 10^{13}$. Thus all first occurrences of gaps through 674, as well as scattered first occurrences for gaps through 778, were tabulated, and all maximal prime gaps through 778 were located. See Young and Potler [17] for an exhaustive listing of these previous results. In addition, Young and Potler continued their calculations to an unpublished higher level; Ribenboim [13, p. 142]

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credits them with the discovery of an additional maximal prime gap of 804 following the prime 90874329411493, and this was confirmed by Young in [18].

Isolated occurrences of much larger prime gaps have been found. It is well known that arbitrarily large gaps exist, for the positive integer $(n! + 1)$ must be followed by at least $(n - 1)$ consecutive composite integers; but no instance of this formula beyond $n = 5$ (first occurrence and maximal gap of 14 following 113) is known to yield a first occurrence. Weintraub [16] has discovered a gap of 864 following 6505941701960039, and Baugh and O'Hara [1] discovered a gap of 4248 following $10^{314} - 1929$, but these are not known or believed to be maximal gaps or even first occurrences; the present work demonstrates conclusively that Weintraub's gap is not maximal. In conjunction with the search for seven consecutive primes in arithmetic progression [9], Dubner [7] has discovered a gap of 1092 following the prime 409534375009657239721; this is the first known occurrence of a gap of 1000 or greater, but again it is not known to be maximal or a first occurrence. Dubner in [8] also reports a gap of 12540 following the 385-digit prime:

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1028115851618596629291338345969573325611755920349536050557212232499_
6950065379512197585317961759000690328913319244717897688019822063737812_
5686339726137874956095491930654497693978715833794999935477468391789508_
3444495414063479003554272907008549459458538251939796513140998638325548_
2457633841427250249367844894786016514356294279402896163593801089250404_
09462881632270278716570882306451587569.
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2. COMPUTATIONAL TECHNIQUE

The search for prime gaps is being carried out as part of a larger program [11], which includes the enumeration of the primes, twin primes, prime triplets, and prime quadruplets for $0(10^9)10^{14}$ and for $10^{14}(10^{10})10^{15}$ and beyond. Also computed are the floating point (64-bit mantissa) sums and ultraprecision (53 decimal places) sums of the reciprocals of the twins, triplets, and quadruplets, in order to extrapolate estimates for the Brun's constants (limits of the sums of the reciprocals) for each constellation. All computations are being executed during slack hours on personal computers belonging to the author or assigned to the Department of Mathematics at Lynchburg College. The number of systems (most of them Pentiums) in use has averaged about fifteen since the present program began in 1993. The source code is written in C and compiled using Borland C++4.52; it is written to run under Borland's 32-bit DOS extender so that all available extended memory can be used for the integer arrays. Earlier versions ran successively under DOS, Windows 3.x, and Win 32s, but operation under 32-bit extended DOS proved most effective, eliminating much of the irrelevant overhead imposed by the Windows environment. Typical throughput is about 10^{11} integers per day on a 60 MHz Pentium. The computations are distributed independently across the systems, currently in runs of 2×10^{12} ; each interval is run in duplicate on two systems to guard against machine errors. In the event that two runs of the same interval disagree, additional runs are carried out to resolve the discrepancy. Thus far, errors have been detected in nearly two dozen instances, including the Pentium FDIV flaw affair [12] in the fall of 1994; faults in memory chips appear to be the most frequent culprit, although it appears impossible to completely rule out errors in either system or application software. The most significant independent check available is the value of $\pi(x)$, the count

of primes, which has been carried out by indirect means to 10^{20} by Deleglise and Rivat [6, 4]; the direct counts obtained from the present calculations agree through 10^{15} with the values published by Riesel [14, p. 34 and pp. 380–383]. The first occurrences listed have also been checked directly by means of the Derive software program for DOS and Windows.

It is of interest to note that two of the Pentiums in service are P5-60 systems with (FDIV) flawed CPUs; the flawed floating point divisions and remainders are being detected and corrected in real time, using a combination of the -fp switch in Borland C++4.52 and a custom procedure (C function) which traps suspect divisors in all fmod and fmodl remaindering calls. With these errors trapped and corrected, and their results checked against runs on CPUs free of the flaw, these two systems have remained error free for more than a year.

3. COMPUTATIONAL RESULTS

Table 1 lists the first occurrences of prime gaps found in the present study, now complete to 10^{15} . The new maximal gaps are marked with an asterisk (*). As was pointed out above, the first occurrence and maximal gap of 804 following the prime 90874329411493 is actually due to Young and Potler [13, p. 142] and is confirmed by the present work. Presumably, the other first occurrences between 7.2×10^{13} and 9.1×10^{13} (for the gaps of 676, 680, 686, and 718) were also known to Young and Potler, but were never published.

These results supplement those previously known and herein omitted for brevity; an exhaustive listing of previously known gaps was given by Young and Potler [17]. The smallest gap whose first occurrence is still unaccounted for is the gap of 796. First occurrences of all gaps greater than 796, not listed in Table 1, also remain to be discovered. Discovery of the new maximal gap of 906 brings us closer to the goal alluded to by Weintraub [16], that of finding the first occurrence of a gap of 1000 or greater. Motivated by a result obtained by Cramér [5], Shanks [15] conjectured that a maximal gap of magnitude M could be expected to appear at approximately $e^{\sqrt{M}}$; Riesel [14, p. 80] measures the success of this conjecture by the ratio $R = \ln(p_{k+1})/\sqrt{M}$, with R expected to approach 1 as M and p_{k+1} increase without bound. For the largest known maximal gaps, R has remained near 1.13, although for the new maximal gap of 906 it attains a value of 1.0969, its absolute minimum to date. Thus, assuming that the first gap of 1000 or greater will actually be about 1050, a reasonable estimate for the location of the first occurrence of a gap of 1000 or greater would be $e^{1.13\sqrt{1050}} \approx 7.98 \times 10^{15}$. The present program would not attain that level for several more years. However, this is little more than an order of magnitude estimate, since an argument could also be made for a much smaller value of $e^{1.0969\sqrt{1000}} \approx 1.16 \times 10^{15}$. The discovery of the first “kilogap” thus remains difficult to anticipate.

TABLE 1. First occurrences of prime gaps in $7.2 \times 10^{13} < p < 10^{15}$

Gap	Following the prime	Gap	Following the prime
676	78610833115261	782	726507223559111
680	82385435331119	784	497687231721157
686	74014757794301	786	554544106989673
688	110526670235599	788	96949415903999
704	97731545943599	790	678106044936511
708	143679495784681	792	244668132223727
710	138965383978937	794	673252372176533
712	106749746034601	798	309715100117419
718	82342388119111	800	486258341004083
720	111113196467011	802	913982990753641
722	218356872845927	804*	90874329411493
726	156100489308167	806*	171231342420521
732	140085225001801	808	546609721879171
734	154312610974979	810	518557948410967
736	161443383249583	814	827873854500949
738	143282994823909	816	632213931500513
742	189442329715069	818	860149012919321
746	184219698008123	820	497067290087413
748	172373989611793	822	799615339016671
750	145508250945419	826	407835172832953
752	255294593822687	828	807201813046091
754	219831875554399	830	507747400047473
760	98103148488133	832	243212983783999
762	144895907074481	834	743844653663833
764	323811481625339	836	880772773476623
768	423683030575549	840	670250273356109
770	214198375528463	844	782685877447783
772	186129514280467	860	844893392671019
774	469789142849483	862	425746080787897
776	187865909338091	872	455780714877767
780	471911699384963	880	277900416100927
		906*	218209405436543

*Maximal gap.

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DEPARTMENT OF MATHEMATICS, LYNCHBURG COLLEGE, 1501 LAKESIDE DRIVE, LYNCHBURG, VIRGINIA 24501-3199

E-mail address: nicely@acavax.lynchburg.edu

URL: <http://www.lynchburg.edu/public/academic/math/nicely/index.htm>