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IMPROVING THE CONVERGENCE OF NON-INTERIOR POINT ALGORITHMS FOR NONLINEAR COMPLEMENTARITY PROBLEMS

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ABSTRACT. Recently, based upon the Chen-Harker-Kanzow-Smale smoothing function and the trajectory and the neighbourhood techniques, Hotta and Yoshise proposed a noninterior point algorithm for solving the nonlinear complementarity problem. Their algorithm is globally convergent under a relatively mild condition. In this paper, we modify their algorithm and combine it with the superlinear convergence theory for nonlinear equations. We provide a globally linearly convergent result for a slightly updated version of the Hotta-Yoshise algorithm and show that a further modified Hotta-Yoshise algorithm is globally and superlinearly convergent, with a convergence Q-order $1+t$, under suitable conditions, where $t \in (0,1)$ is an additional parameter.

1. INTRODUCTION

Consider the nonlinear complementarity problem (NCP): Find an $(x, y) \in \Re^n \times \Re^n$ such that

$$y - f(x) = 0, \quad x \geq 0, \quad y \geq 0, \quad x^T y = 0, \tag{1}$$

where $f : \Re^n \to \Re^n$ is a continuously differentiable function. The NCP has received a lot of attention due to its various applications in operations research, economic equilibrium, and engineering design [18, 25, 16].

It is easy to see (e.g., see [18]) that finding a solution of (1) is equivalent to finding a root of the following equation:

$$H(x, y) := \begin{bmatrix} 2\min\{x, y\} \\ y - f(x) \end{bmatrix} = 0. \tag{2}$$

By combining the form of H with the so-called Chen-Harker-Kanzow-Smale smoothing technique we get the following approximation mapping $F : \Re^n_+ \times \Re^{2n} \to \Re^n_+ \times \Re^{2n}$:

$$F(u, x, y) := \begin{bmatrix} u \\ \Phi(u, x, y) \\ y - f(x) \end{bmatrix}, \tag{3}$$

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where

$$\Phi(u,x,y) := \begin{bmatrix} \phi(u_1,x_1,y_1) \\ \cdots \\ \phi(u_n,x_n,y_n) \end{bmatrix} \tag{4}$$

and $\phi : \Re^3 \to \Re$ is the Chen-Harker-Kanzow-Smale smoothing function [6, 20, 30]:

$$\phi(\mu,a,b) = a + b - \sqrt{(a-b)^2 + 4\mu^2}. \tag{5}$$

For $\mu > 0$, the following property holds:

$$\phi(\mu,a,b) = 0 \Longleftrightarrow a > 0, \quad b > 0, \quad ab = \mu^2. \tag{6}$$

By letting $u = 0$ in (3) we get

$$F(0,x,y) = \begin{bmatrix} 0 \\ H(x,y) \end{bmatrix}.$$

Lemma 1 ([19], Lemma 1.4). *For every nonnegative number $\mu \geq 0$, a triple $(a,b,c) \in \Re^3$ satisfies $\phi(\mu,a,b) = c$ if and only if $((a-c/2),(b-c/2)) \geq 0$ and $(a-c/2)(b-c/2) = \mu^2$.*

Throughout this paper we let $\|\cdot\|$ denote the l_2-norm of $\Re^n$ and its induced matrix norm.

Lemma 2. *For any $z = (\mu,a,b) \in \Re^3$ and $z^1 = (\mu^1,a^1,b^1) \in \Re^3$ with $\mu,\mu^1 > 0$ we have*

$$\|\phi''(z)\| \leq \frac{4}{\sqrt{(a-b)^2+4\mu^2}}, \tag{7}$$

and for any $\alpha \in [0,1)$,

$$|\phi(z+\alpha(z^1-z)) - \phi(z) - \alpha\phi'(z)(z^1-z)| \leq \frac{\alpha^2}{1-\alpha}\mu^{-1}\|z^1-z\|^2. \tag{8}$$

Proof. After simple computations, we have

$$\nabla\phi(z) = \begin{pmatrix} \dfrac{-4\mu}{\sqrt{(a-b)^2+4\mu^2}} \\ 1 - \dfrac{a-b}{\sqrt{(a-b)^2+4\mu^2}} \\ 1 - \dfrac{b-a}{\sqrt{(a-b)^2+4\mu^2}} \end{pmatrix}$$

and

$$\phi''(z) = \frac{4}{(\sqrt{(a-b)^2+4\mu^2})^3} \begin{pmatrix} -(a-b)^2 & (a-b)\mu & (b-a)\mu \\ (a-b)\mu & -\mu^2 & -\mu^2 \\ (b-a)\mu & -\mu^2 & -\mu^2 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \|\phi''(z)\| &\leq \frac{4}{(\sqrt{(a-b)^2+4\mu^2})^3}\sqrt{(a-b)^4+4(a-b)^2\mu^2+4\mu^4} \\ &= \frac{4}{(\sqrt{(a-b)^2+4\mu^2})^3}((a-b)^2+2\mu^2) \\ &\leq \frac{4}{\sqrt{(a-b)^2+4\mu^2}}. \end{aligned}$$

This proves (7). It then follows from (7) that $\|\phi''(z)\| \le 2\mu^{-1}$. Then for any $\alpha \in [0, 1)$, we have

$$
\begin{aligned}
&|\phi(z+\alpha(z^1-z)) - \phi(z) - \alpha\phi'(z)(z^1-z)| \\
&= |\alpha \int_0^1 [\phi'(z+\alpha\theta(z^1-z)) - \phi'(z)](z^1-z)d\theta| \\
&= \alpha^2 | \int_0^1 \theta \int_0^1 (z^1-z)^T \phi''(z+\alpha\theta s(z^1-z))(z^1-z) ds d\theta| \\
&\le \alpha^2 \int_0^1 \theta \int_0^1 \frac{2}{\mu+\alpha\theta s(\mu^1-\mu)} ds d\theta \|z^1-z\|^2 \\
&= \alpha^2 \int_0^1 \theta \int_0^1 \frac{2}{(1-\alpha\theta s)\mu+\alpha\theta s\mu^1} ds d\theta \|z^1-z\|^2 \\
&\le \alpha^2 \int_0^1 \theta \int_0^1 \frac{2}{(1-\alpha\theta s)\mu} ds d\theta \|z^1-z\|^2 \\
&\le \alpha^2 \int_0^1 \theta \int_0^1 \frac{2}{(1-\alpha)\mu} ds d\theta \|z^1-z\|^2 \\
&= \frac{\alpha^2}{1-\alpha} \mu^{-1} \|z^1-z\|^2.
\end{aligned}
$$

This proves (8), and completes the proof of this lemma. □

Recently, based on F defined by (3) (the only difference is that instead of using (5) the definition $\phi(\mu, a, b) = a + b - \sqrt{(a-b)^2 + 4\mu}$ was used in [19]) and the trajectory and the neighbourhood techniques, Hotta and Yoshise proposed a globally convergent noninterior point method for solving the NCP [19]. Their method does not require the initial point $(x^1, y^1) \in \Re^n \times \Re^n$ to be in the positive orthant. This is quite different from (infeasible) interior point methods, where a positive initial point is always required (e.g., see [31, 33, 34]). Given initial point $\bar{z}$ and $\bar{w} = F(\bar{z}) \in \Re^n_{++} \times \Re^n_{--} \times \Re^n_{++}$, Hotta and Yoshise's neighborhood is defined in terms of the vector $\bar{w}$ and contains the initial point $\bar{z}$ in its interior. Another type of neighborhood has been studied in [1, 4, 9, 35, 36] where the neighborhoods are prespecified. Algorithms based on these neighborhoods require choosing an initial point in the prespecified neighborhood. In many cases, this requirement does not impose much restriction. For example, such initial points are easily obtained for the $P_0 + R_0$ problem [1, 4, 9, 35, 36]. Compared to the existing noninterior point methods or related smoothing methods [1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 17, 21, 27, 32, 35, 36], the most outstanding feature of the Hotta-Yoshise algorithm is that their algorithm can keep the iteration sequence in a bounded neighbourhood without requiring the initial point to start from a bounded level set or its variants. This feature is very favourable for those functions which cannot guarantee the boundedness of every level set. However, unlike other noninterior point methods [1, 4, 9, 12, 13, 27, 32, 35, 36], there is no convergence rate provided in [19]. In this paper we will modify the Hotta-Yoshise algorithm and discuss its convergence rate.

When we were finalizing our paper, we received a new report by Chen and Chen [5] that describes a noninterior point algorithm which is related to the Hotta-Yoshise algorithm. They provided a local superlinear convergence result. Their result is quite different from ours because during the process they update a sequence of neighbourhoods associated with the smoothing paths dynamically while we only use one neighbourhood by introducing the smoothing parameter u in the set of variable parameters. When this paper was under review, two reports by Burke and Xu [2, 3] were released. Based on their previous work on $P_0 + R_0$ linear complementarity problems (LCPs), Burke and Xu [2, 3] refined their neighborhood, which differs markedly from that used in this paper, to allow them to present a predictor-corrector noninterior path following algorithm for monotone and non-monotone LCPs.

Our modified version of the Hotta-Yoshise algorithm is specified in Section 2. The global and monotone convergence result is proved in Section 3. In Section 4 we discuss a global linear convergence result. The superlinear convergence result with a Q-order $1+t, t \in (0,1)$ is established in Section 5.

2. THE MODIFIED VERSION OF THE HOTTA-YOSHISE ALGORITHM

Let $v, r : \Re^n_+ \times \Re^{2n} \to \Re^n$ be defined as

$$v_i(u,x,y) = \phi(u_i, x_i, y_i), \ i = 1, 2, ..., n$$

and

$$r(u,x,y) = y - f(x),$$

where $u \in \Re^n_+$. Then

$$F(u,x,y) = \begin{pmatrix} u \\ v(u,x,y) \\ r(u,x,y) \end{pmatrix}.$$

Let $V(u,x,y) := \begin{pmatrix} v(u,x,y) \\ r(u,x,y) \end{pmatrix}$ and $N := \{1, 2, ..., n\}$ and denote $z := \begin{pmatrix} u \\ x \\ y \end{pmatrix}$ and $w := \begin{pmatrix} u \\ v(u,x,y) \\ r(u,x,y) \end{pmatrix}$.

Let $\bar{z} \in \Re^n_{++} \times \Re^{2n}$ be such that $\bar{w} := F(\bar{z}) \in \Re^n_{++} \times \Re^n_{--} \times \Re^n_{++}$. Such a point $\bar{z}$ can be chosen easily. In fact, Hotta and Yoshise [19] used the following simple method to choose $\bar{z}$. Let $\tilde{z} = (\tilde{u}, \tilde{x}, \tilde{y})$ be an arbitrary point of $\Re^n_+ \times \Re^{2n}$. Even if $F(\tilde{z}) \notin \Re^n_{++} \times \Re^n_{--} \times \Re^n_{++}$, we may choose a $(dv, dr) \in \Re^{2n}$ so that

$$(\tilde{x}_i - (\tilde{v}_i + dv_i)/2, \tilde{y}_i + dr_i - (\tilde{v}_i + dv_i)/2) > 0, \quad i \in N,$$

$$\tilde{y} + dr = f(\tilde{x}) + (\tilde{r} + dr),$$

$$\tilde{v} + dv < 0, \ \tilde{r} + dr > 0.$$

By setting

$$\bar{u}_i := \{[\tilde{x}_i - (\tilde{v}_i + dv_i)/2][\tilde{y}_i + dr_i - (\tilde{v}_i + dv_i)/2]\}^{1/2} > 0, \quad i \in N,$$
$$\bar{x} := \tilde{x},$$
$$\bar{y} := \tilde{y} + dr,$$

we obtain a point $\bar{z}$ which satisfies $F(\bar{z}) \in \Re^n_{++} \times \Re^n_{--} \times \Re^n_{++}$. Then let τ be a constant satisfying

$$0 < \tau < \min\{|\bar{w}_i| : \ i = 1, 2, ..., 3n\}$$

and define

$$C := \{w \in \Re^{3n} : \ \|w - (\bar{w}^T w/\|\bar{w}\|^2)\bar{w}\| \le \tau(\bar{w}^T w/\|\bar{w}\|^2)\},$$
$$H_{\bar{w}} := \{w \in \Re^{3n} : \ \bar{w}^T w \le \|\bar{w}\|^2\},$$

and

$$\Omega =: C \cap H_{\bar{w}}.$$

Then it is easy to see that Ω is a compact set and $\Omega \subset \Re^n_+ \times \Re^n_- \times \Re^n_+$. Let $\rho : \Re^2 \to \Re$ be defined by

$$\rho(\alpha, \beta) := 1 - \alpha(1 - \beta)/2,$$

and the merit function $\psi : \Re^{3n} \to \Re$ be defined by

$$\psi(z) := \bar{w}^T F(z)/\|\bar{w}\|^2.$$

Before describing the modified version of the Hotta-Yoshise algorithm, we will list several conditions used in the following discussion and give some lemmas related to these conditions.

Assumption 1.
(i) *The mapping f is monotone, i.e.,*

$$(x^1 - x^2)^T(f(x^1) - f(x^2)) \ge 0$$

for every $x^1, x^2 \in \Re^n$.
(ii) *There exists a feasible interior-point (x, y) of the NCP, i.e.,*

$$(x, y) > 0 \quad \textit{and} \quad y = f(x).$$

Assumption 2.
(i) *The mapping f is a P_0-function, i.e., for every $x^1, x^2 \in \Re^n$ with $x^1 \ne x^2$ there exists an index $i \in N$ such that*

$$x_i^1 \ne x_i^2 \quad \textit{and} \quad (x_i^1 - x_i^2)(f_i(x^1) - f_i(x^2)) \ge 0.$$

(ii) *There exists a feasible interior-point (x, y) of the NCP, i.e.,*

$$(x, y) > 0 \quad \textit{and} \quad y = f(x).$$

(iii) $F^{-1}(D) := \{(u, x, y) \in \Re^n_+ \times \Re^{2n} : \ F(u, x, y) \in D\}$ *is bounded for every compact subset D of $\Re^n_+ \times V(\Re^n_{++} \times \Re^{2n})$.*

Notice that Assumptions 1 and 2 are Conditions 1.3 and 2.2 in [19], respectively.

Lemma 3. *If Assumption 1 holds so does Assumption 2.*

Proof. The proof of this lemma is similar to that of Lemma 2.3 in [19] despite that the definition of $\phi(\mu, a, b)$ used in [19] is equivalent to $\phi(\sqrt{\mu}, a, b)$ here. □

Lemma 4 ([19], Lemma 2.1). (i) *$V(\Re^n_{++} \times \Re^{2n})$ is an open subset of $\Re^{2n}$.*
(ii) *If $(\bar{v}, \bar{r}) \in V(\Re^n_{++} \times \Re^{2n})$, then*

$$(\bar{v} + \Re^n_-) \times (\bar{r} + \Re^n_+) \subset V(\Re^n_{++} \times \Re^{2n}).$$

(iii) *Specially, if* $(0,0) \in V(\Re^n_{++} \times \Re^{2n})$, *which is equivalent to saying that the NCP has a feasible interior-point, then*

$$\Re^n_- \times \Re^n_+ \subset V(\Re^n_{++} \times \Re^{2n}).$$

By noting Lemma 3 and (iii) of Lemma 4, we have the following useful lemma.

Lemma 5 ([19], Lemma 2.7). *If Assumption 2 holds, then*

$$F^{-1}(D) := \{(u,x,y) \in \Re^n_+ \times \Re^{2n} : F(u,x,y) \in D\}$$

is bounded for every bounded subset D *of* $\Re^n_+ \times \Re^n_- \times \Re^n_+$.

Lemma 6. *Suppose that condition* (i) *of Assumption 2 is satisfied, i.e.,* f *is a* P_0*-function. Then*

(i) *The Jacobian matrix* $f'(x)$ *is a* P_0*-matrix at every* $x \in \Re^n$.

(ii) *The Jacobian matrix* $F'(u,x,y)$ *is given by*

$$F'(u,x,y) = \begin{pmatrix} I & 0 & 0 \\ -4\tilde{D} & I-(X-Y)D & I+(X-Y)D \\ 0 & -f'(x) & I \end{pmatrix},$$

where $X = \mathrm{diag}\{x_i(i \in N)\}$, $Y = \mathrm{diag}\{y_i(i \in N)\}$, $D = \mathrm{diag}\{d_i(i \in N)\}$, $\tilde{D} = \mathrm{diag}\{\tilde{d}_i(i \in N)\}$, *and*

$$d_i = 1/\sqrt{(x_i - y_i)^2 + 4u_i^2}, \quad \tilde{d}_i = u_i d_i, \qquad i \in N$$

for every $(u,x,y) \in \Re^n_{++} \times \Re^{2n}$.

(iii)

$$0 < 1 - (x_i - y_i)d_i < 2, \quad 0 < 1 + (x_i - y_i)d_i < 2,$$

and $I-(X-Y)D$ *and* $I+(X-Y)D$ *are positive diagonal matrices for every* $z \in \Re^n_{++} \times \Re^{2n}$.

(iv) $F'(u,x,y)$ *is a* $3n \times 3n$ *nonsingular matrix for every* $(u,x,y) \in \Re^n_{++} \times \Re^{2n}$.

Proof. (i) has been proved in Lemma 5.4 of [22]. By a direct computation, we have (ii) and (iii). By noting that $f'(x)$ is a P_0-matrix and that (iii) holds, we can deduce that the matrix

$$\begin{pmatrix} I-(X-Y)D & I+(X-Y)D \\ -f'(x) & I \end{pmatrix}$$

is nonsingular for every $z \in \Re^n_{++} \times \Re^{2n}$ (see, e.g., Lemma 4.1 of [23]). Thus, by (ii), the matrix $F'(u,x,y)$ is nonsingular for every $z \in \Re^n_{++} \times \Re^{2n}$. So, (iv) is also proved. □

Now we can describe our modified version of the Hotta-Yoshise algorithm.

Algorithm 1. *Step* 0. *Choose constants* $\delta, \gamma \in (0,1)$, *and* $t \in [0,1)$. *Let* $z^1 := \bar{z}$, $\psi_1 := \psi(z^1)$, *and* $k := 1$.

Step 1. *If* $F(z^k) = 0$, *then stop. Otherwise, let* $z := z^k$, $\psi := \psi_k$, *and* $\beta := \beta_k = \min\{\gamma, \psi^t\}$.

Step 2. *Compute* Δz *by*

$$F'(z)\Delta z = -F(z) + \beta\psi(z)\bar{w}. \tag{9}$$

Step 3. *Let* l_k *be the smallest nonnegative integer* l *satisfying*

$$F(z + \delta^l \Delta z) \in \Omega \tag{10}$$

and

$$\psi(z+\delta^l \Delta z) \leq \rho(\delta^l, \beta)\psi. \tag{11}$$

Here δ^l is the lth power of δ. Define $z^{k+1} := z + \delta^{l_k}\Delta z$ and $\psi_{k+1} := \psi(z^{k+1})$.
Step 4. Replace k by $k+1$ and go to Step 1.

Remark 1. (i) If $t = 0$, then we have a slightly updated version of the Hotta-Yoshise algorithm. In [19] the definition of $\phi(\mu, a, b)$ is equivalent to $\phi(\sqrt{\mu}, a, b)$ here. Our modification does not affect the global convergence property of the Hotta-Yoshise algorithm but allows us to prove a global linear result. The reason is that the variables μ, a, b in $\phi(\mu, a, b)$ have the same growth rate and such defined ϕ is locally Lipshitz continuous in $\Re^3$. The latter property allows us to prove that Assumption 3, which is essential for the global linear convergence of our algorithm, can be satisfied under a regularity condition (see Section 4). The same conclusion does not go to $\phi(\sqrt{\mu}, a, b)$. By choosing $t \in (0,1)$, we will prove a superlinear convergent result with Q-order $1+t$ in Section 5.

(ii) In [19], the vector $F(z+\delta^l\Delta z)$ in Step 3 is required to stay in the interior of Ω. Here we only require that it stays in Ω.

Proposition 1. *If f is a P_0-function, then Algorithm 1 is well defined.*

Proof. The proof of this lemma is largely based on that of Lemma 6.2 of [19]. To make the material provided here complete and explicit, we give the proof. It is obvious that we only need to verify that Steps 2 and 3 of Algorithm 1 are well defined. By Lemma 6, for $z = z^k \in \Re^n_{++} \times \Re^{2n}$ the matrix $F'(u,x,y)$ is nonsingular. So, Step 2 is well defined. Next, we prove that Step 3 is also well defined. First, from (ii) of Lemma 6 and (9) of Algorithm 1, for $z = z^k \in \Re^n_{++} \times \Re^{2n}$ and $\beta = \beta_k$ we have

$$\Delta u = -u + \beta\psi(z)\bar{u}. \tag{12}$$

Then for $z = z^k \in \Re^n_{++} \times \Re^{2n}$ and any $\alpha \in [0,1]$, it follows from (12) that

$$u + \alpha\Delta u = (1-\alpha)u + \alpha\beta\psi(z)\bar{u} \in \Re^n_{++},$$

and so,

$$z + \alpha\Delta z \in \Re^n_{++} \times \Re^{2n}.$$

For $z = z^k$ and $\alpha \in [0,1]$, define

$$g(\alpha) = F(z+\alpha\Delta z) - F(z) - \alpha F'(z)\Delta z. \tag{13}$$

Since F is continuously differentiable at $z = z^k$,

$$g(\alpha) = o(\alpha). \tag{14}$$

Combining (9) with (13), for $z = z^k$, $\beta = \beta_k$, and any $\alpha \in [0,1]$, we have

$$F(z+\alpha\Delta z) = (1-\alpha)F(z) + \alpha[\beta\psi(z)\bar{w} + g(\alpha)/\alpha] \tag{15}$$

and

$$\begin{aligned}\psi(z+\alpha\Delta z) &= (1-\alpha)\psi(z) + \alpha[\beta\psi(z) + \bar{w}^T g(\alpha)/(\alpha\|\bar{w}\|^2)] \\ &\leq (1-\alpha)\psi(z) + \alpha\left[\beta\psi(z) + \frac{\|g(\alpha)\|}{\alpha\|\bar{w}\|}\right].\end{aligned} \tag{16}$$

Define

$$\alpha_\psi := \sup\{\alpha' \in (0,1] : \|g(\alpha)\|/\alpha \le (1-\beta)\psi(z)\|\bar{w}\|/2 \quad \forall \alpha \in (0,\alpha']\} \tag{17}$$

and

$$\alpha_1 := \sup\{\alpha' \in (0,1] : (2+\tau/\|\bar{w}\|)\|g(\alpha)\|/\alpha \le \tau\beta\psi(z) \quad \forall \alpha \in (0,\alpha']\}. \tag{18}$$

Then, by using (14), the constants α_ψ and α_1 are positive and well defined by (17) and (18), respectively. It then follows from (16), (17), (15), and (18) that for all $\alpha \in (0, \alpha_\psi]$,

$$\begin{aligned} \psi(z+\alpha\Delta z) &\le \{(1-\alpha)+\alpha[\beta+(1-\beta)/2]\}\psi(z) \\ &= [1-\alpha(1-\beta)/2]\psi(z) \\ &= \rho(\alpha,\beta)\psi(z), \end{aligned} \tag{19}$$

and for all $\alpha \in (0, \alpha_1]$,

$$\begin{aligned} &\left\| [\beta\psi(z)\bar{w}+g(\alpha)/\alpha] - \frac{\bar{w}^T[\beta\psi(z)\bar{w}+g(\alpha)/\alpha]}{\|\bar{w}\|^2}\bar{w} \right\| - \tau\frac{\bar{w}^T[\beta\psi(z)\bar{w}+g(\alpha)/\alpha]}{\|\bar{w}\|^2} \\ &= \left\| g(\alpha)/\alpha - \frac{\bar{w}^T g(\alpha)/\alpha}{\|\bar{w}\|^2}\bar{w} \right\| - \tau\left(\beta\psi(z) + \frac{\bar{w}^T g(\alpha)/\alpha}{\|\bar{w}\|^2}\right) \\ &\le \|g(\alpha)\|/\alpha + \|g(\alpha)\|/\alpha - \tau\beta\psi(z) + \tau\frac{\|g(\alpha)\|/\alpha}{\|\bar{w}\|} \\ &\le (2+\tau/\|\bar{w}\|)\|g(\alpha)\|/\alpha - \tau\beta\psi(z) \\ &\le 0. \end{aligned} \tag{20}$$

Hence

$$\beta\psi(z)\bar{w} + g(\alpha)/\alpha \in C.$$

Then from $F(z) \in C$, the definition of C, and (15) that for all $\alpha \in (0, \alpha_1]$, we have

$$F(z+\alpha\Delta z) = (1-\alpha)F(z) + \alpha[\beta\psi(z)\bar{w} + g(\alpha)/\alpha] \in C. \tag{21}$$

Also, since (19) holds for all $\alpha \in (0, \alpha_\psi]$, it follows from the fact $F(z) \in H_{\bar{w}}$ that for these α's we have

$$\bar{w}^T F(z+\alpha\Delta z) = \psi(z+\alpha\Delta z)\|\bar{w}\|^2 \le \psi(z)\|\bar{w}\|^2 = \bar{w}^T F(z) \le \|\bar{w}\|^2. \tag{22}$$

Then for all $\alpha \in (0, \min\{\alpha_\psi, \alpha_1\}]$, we have from (21), (22), and (19) that

$$F(z+\alpha\Delta z) \in \Omega \quad \text{and} \quad \psi(z+\alpha\Delta z) \le \rho(\alpha,\beta)\psi(z).$$

This shows that in Step 3 l_k is well defined and finite, i.e., $\delta^{l_k} > 0$ and Step 3 is well defined. $\square$

3. Global and monotone convergence

Theorem 1. *Suppose that Assumption 2 holds. Let* $\{(z^k, \psi_k) \subseteq \Omega \times [0,1]\}$ *be a sequence generated by Algorithm 1. Then*

(i) *The sequence* $\{z^k = (u^k, x^k, y^k)\}$ *is bounded.*
(ii) *The sequence* $\{\psi_k\}$ *is monotonically decreasing and converges to* 0 *as* $k \to \infty$.
(iii) $\lim_{k\to\infty} u^k = 0$ *and every accumulation point of* $\{(x^k, y^k)\}$ *is a solution of the NCP.*

Proof. (i) Since Ω is compact and $\Omega \subset \Re^n_+ \times \Re^n_- \times \Re^n_+$, from Lemma 5 we know that $F^{-1}(\Omega)$ is bounded. It then follows from $F(z^k) \in \Omega$ that the sequence $\{z^k\}$ is bounded.

(ii) From Algorithm 1 and Proposition 1 we can see that $\psi_k > \psi_{k+1}$ $(k = 1, 2, ...)$. Hence the sequence $\{\psi_k\}$ is monotonically decreasing. Since $\psi_k \geq 0$ $(k = 1, 2, ...)$, there exists a $\tilde{\psi} \geq 0$ such that $\psi_k \to \tilde{\psi}$ as $k \to \infty$. If $\tilde{\psi} = 0$, then we obtain the desired result. Suppose that $\tilde{\psi} > 0$. Since, by (i), the sequence $\{z^k\}$ is bounded, by taking a subsequence if necessary, we may assume that $\{z^k\}$ converges to some point $\tilde{z}$. It is easy to see that $\tilde{\psi} = \bar{w}^T F(\tilde{z})/\|\bar{w}\|^2 = \psi(\tilde{z})$ and $F(\tilde{z}) \in \Omega$. Thus, from $\psi(\tilde{z}) > 0$ and $F(\tilde{z}) \in C$, we can see that $F(\tilde{z}) \in \Re^n_{++} \times \Re^n_{--} \times \Re^n_{++}$. Hence $\tilde{z} \in \Re^n_{++} \times \Re^{2n}$ because $\tilde{u}_i = F_i(\tilde{z})$, $i \in N$. Since for all k, $\psi_k \geq \tilde{\psi} > 0$, there exists a positive number $\tilde{\beta}$ such that $\beta_k \to \tilde{\beta}$. Let $z \in \Re^n_{++} \times \Re^{2n}$ and $\beta(z) = \min\{\gamma, \psi(z)^t\}$. Then from Lemma 6, $F'(z)$ is nonsingular. Let Δz be the unique solution of the following linear system of the equations

$$F'(z)\Delta z = -F(z) + \beta(z)\psi(z)\bar{w}.$$

For $\alpha \in [0,1]$, define

$$g_z(\alpha) = F(z + \alpha\Delta z) - F(z) - \alpha F'(z)\Delta z.$$

Then from the Mean Value Theorem [24],

$$g_z(\alpha) = \alpha \int_0^1 [F'(z + \theta\alpha\Delta z) - F'(z)]\Delta z d\theta.$$

From (ii) of Lemma 6 we can easily see that $F'(\cdot)$ exists and is continuous in a neighbourhood of $\tilde{z}$, and so, it is uniformly continuous in this neighbourhood. Furthermore, since $\Delta z \to \Delta\tilde{z}$ as $z \to \tilde{z}$, for any given $\varepsilon > 0$ there exists a neighbourhood $N(\tilde{z})$ of $\tilde{z}$ such that for all $z \in N(\tilde{z})$, $\|g_z(\alpha)\|/\alpha \leq \varepsilon$. Hence, since

$$[1 - \beta(z)]\psi(z)\|\bar{w}\|/2 \to [1 - \beta(\tilde{z})]\psi(\tilde{z})\|\bar{w}\|/2 > 0$$

and

$$\beta(z)\psi(z)/(2 + \tau/\|\bar{w}\|) \to \beta(\tilde{z})\psi(\tilde{z})/(2 + \tau/\|\bar{w}\|) > 0$$

as $z \to \tilde{z}$, there exist a positive number $\tilde{\alpha} > 0$ and a neighbourhood $N(\tilde{z})$ of $\tilde{z}$ such that for all $\alpha \in (0, \tilde{\alpha}]$,

$$\|g_z(\alpha)\|/\alpha \leq [1 - \beta(z)]\psi(z)\|\bar{w}\|/2$$

and

$$(2 + \tau/\|\bar{w}\|)\|g_z(\alpha)\|/\alpha \leq \tau\beta(z)\psi(z).$$

Then by examining the proof of Proposition 1, we can see that for any $\alpha \in (0, \tilde{\alpha}]$ and all $z \in N(\tilde{z})$ such that $F(z) \in \Omega$, we have

$$F(z + \alpha\Delta z) \in \Omega \quad \text{and} \quad \psi(z + \alpha\Delta z) \leq \rho(\alpha, \beta(z))\psi(z).$$

Therefore, for a nonnegative integer l such that $\delta^l \in (0, \tilde{\alpha}]$, we have

$$F(z^k + \delta^l \Delta z^k) \in \Omega \quad \text{and} \quad \psi(z^k + \delta^l \Delta z^k) \le \rho(\delta^l, \beta_k)\psi(z^k)$$

for all sufficiently large k. Then, for every sufficiently large k, we see that $l^k \le l$ and hence $\delta^{l_k} \ge \delta^l$. Then

$$\psi_{k+1} \le \rho(\delta^{l_k}, \beta_k)\psi_k \le \rho(\delta^l, \beta_k)\psi_k \le \rho(\delta^l, \beta_1)\psi_k$$

for all sufficiently large k. This contradicts the fact that the sequence $\{\psi_k\}$ converges to $\tilde{\psi} > 0$.

(iii) From the design of Algorithm 1, $F(z^k) \in C$, i.e.,

$$\|F(z^k) - \psi(z^k)\bar{w}\| \le \tau\psi(z^k).$$

By assertion (ii) above, we have $\lim_{k\to\infty} \psi(z^k) = 0$. Then by taking limits on both sides of the above inequality, we obtain $\lim_{k\to\infty} F(z^k) = 0$. Hence, $\lim_{k\to\infty} u^k = 0$. Suppose that $(\bar{x}, \bar{y})$ is an arbitrary accumulation point of $\{(x^k, y^k)\}$. Then $(0, \bar{x}, \bar{y}) \in \Re^{3n}$ is an accumulation point of $\{z^k\}$. By the continuity of F, we have $F(0, \bar{x}, \bar{y}) = 0$, i.e,

$$H(\bar{x}, \bar{y}) = 0.$$

Thus $(\bar{x}, \bar{y})$ is a solution of the NCP. □

4. A GLOBAL LINEAR CONVERGENCE RESULT

In this section we will provide a global linear convergence result. The most distinctive feature of our result is that we do not require the initial point to stay in a specified bounded level set or its variants, which may not be easy to know. There are some global linear convergence results for noninterior point algorithms or smoothing methods, as in [1, 4, 9, 35, 36], but they need this requirement. We avoid this requirement by using a neighbourhood different from those of [1, 4, 9, 35, 36]. This requirement was also avoided in three recent reports [4, 2, 3] by refining a neighborhood or its variants as studied in [1, 4, 9, 35, 36].

Assumption 3. *There exists a constant $c_0 > 0$ such that for all $k \ge 1$,*

$$\|F'(z^k)^{-1}\| \le c_0.$$

Let (x^*, y^*) be a solution of the NCP, and define

$$I(x^*, y^*) = \{i \in N : \ x_i^* > 0, \ y_i^* = 0\},$$

$$J(x^*, y^*) = \{i \in N : \ x_i^* = 0, \ y_i^* = 0\},$$

and

$$K(x^*, y^*) = \{i \in N : \ x_i^* = 0, \ y_i^* > 0\}.$$

We say that the R-regularity condition holds at (x^*, y^*) if M_{II} is nonsingular and the matrix

$$M_{JJ} - M_{JI}M_{II}^{-1}M_{IJ}$$

is a P-matrix, where $M := f'(x^*)$ and I, J, and K are abbreviations of $I(x^*, y^*)$, $J(x^*, y^*)$, and $K(x^*, y^*)$, respectively [29].

Proposition 2. *Suppose that Assumption 2 is satisfied and the sequence $\{z^k\}$ is generated by Algorithm 1. If the R-regularity condition holds at all $(x^*, y^*) \in \Re^{2n}$ with $(0, x^*, y^*)$ being an accumulation point of $\{z^k\}$, then Assumption 3 holds.*

Proof. First, according to Theorem 1, the sequence $\{z^k\}$ generated by Algorithm 1 is bounded and each accumulation point (x^*, y^*) of $\{(x^k, y^k)\}$ is a solution of the NCP. Then, that the R-regularity condition holds at (x^*, y^*) is meaningful. It is easy to verify that $F(\cdot)$ is locally Lipschitz continuous. Let $\partial F(z)$ be the generalized Jacobian of F at z, as defined in [14]. Then, by Lemma 6, after a simple computation, we have

$$\partial F(0, x^*, y^*) \subset \left\{ \begin{pmatrix} I & 0 & 0 \\ -4D^* & V^* & W^* \\ 0 & -f'(x^*) & I \end{pmatrix} \right\},$$

where $D^* = \mathrm{diag}\{d_i^*(i \in N)\}$, $d_i^* \in [-1/2, 1/2]$, and $V^*, W^* \in \Re^{n\times n}$ satisfying

$$\begin{pmatrix} V^* & W^* \\ -f'(x^*) & I \end{pmatrix} \in \partial H(x^*, y^*).$$

Since the R-regularity condition holds at (x^*, y^*), all the matrices $T \in \partial H(x^*, y^*)$ are nonsingular (e.g., see Proposition 4 of [4]). This further ensures that all the matrices $S \in \partial F(0, x^*, y^*)$ are nonsingular. Then by Proposition 2.5 of [26] we know that $(0, x^*, y^*)$ is an isolated solution of $F(z) = 0$, i.e., (x^*, y^*) is an isolated solution of the NCP. This means that the sequence $\{z^k\}$ has only finitely many accumulation points; otherwise, there must exist an accumulation point of $\{z^k\}$, which is not an isolated solution of $F(z) = 0$. Then by Proposition 3.1 of [28] and the fact that $\partial F(z^k) = \{F'(z^k)\}$ since $F(\cdot)$ is continuously differentiable at z^k for any $k \geq 1$, we can find a constant $c_0 > 0$ such that Assumption 3 holds. This completes the proof. □

Theorem 2. *Suppose that Assumptions 2 and 3 are satisfied and in Algorithm 1 the constant t is set to be 0, i.e., $\beta_k \equiv \gamma$ for all $k \geq 1$. Then there exists a constant $c \in (0,1)$ such that for all $k \geq 1$,*

$$\psi(z^{k+1}) \leq c\psi(z^k). \tag{23}$$

Moreover, if γ satisfies

$$\gamma\bar{u}_i/(\bar{u}_i - \tau) < 1, \quad i \in N, \tag{24}$$

then there exists another constant $\bar{c} \in (0,1)$ such that for all $k \geq 1$,

$$u_i^{k+1} \leq \bar{c}u_i^k, \quad i \in N. \tag{25}$$

Proof. First, from $F(z^k) \in C$ and $F_i(z^k) = u_i^k, i \in N$, we get

$$\|F(z^k)\| \leq (\tau + \|\bar{w}\|)\psi(z^k) \tag{26}$$

and

$$|u_i^k - \psi(z^k)\bar{u}_i| \leq \tau\psi(z^k), \quad i \in N.$$

Hence, from the definition of τ,

$$0 < (\bar{u}_i - \tau)\psi(z^k) \leq u_i^k \leq (\bar{u}_i + \tau)\psi(z^k), \quad i \in N. \tag{27}$$

Then, by (9), Assumption 3, (26), and the fact that $\beta_k = \gamma$, we get

$$\begin{aligned} \|\Delta z^k\| &\le c_0\| - F(z^k) + \beta_k \psi(z^k)\bar{w}\| \\ &\le c_0[(\tau + \|\bar{w}\|)\psi(z^k) + \gamma\|\bar{w}\|\psi(z^k)] \qquad (28) \\ &= c_1\psi(z^k), \end{aligned}$$

where $c_1 := c_0[\tau + (1+\gamma)\|\bar{w}\|]$. Let

$$g^k(\alpha) := F(z^k + \alpha\Delta z^k) - F(z^k) - \alpha F'(z^k)\Delta z^k$$

and

$$\sigma^k(\alpha) = \alpha \int_0^1 [f'(x^k + \alpha\theta\Delta x^k) - f'(x^k)]\Delta x^k d\theta.$$

By using Lemma 2 and the structure of F, for any $\alpha \in [0,1)$ and $i \in N$ we have

$$\begin{aligned} &|g^k_{n+i}(\alpha)| \\ &= |F_{n+i}(z^k + \alpha\Delta z^k) - F_{n+i}(z^k) - \alpha F'_{n+i}(z^k)\Delta z^k| \\ &= |\phi(u_i^k + \alpha\Delta u_i^k, x_i^k + \alpha\Delta x_i^k, y_i^k + \alpha\Delta y_i^k) - \phi(u_i^k, x_i^k, y_i^k) \\ &\qquad - \alpha\phi'(u_i^k, x_i^k, y_i^k)(\Delta u_i^k, \Delta x_i^k, \Delta y_i^k)| \\ &\le \frac{\alpha^2}{1-\alpha}(u_i^k)^{-1}\|(\Delta u_i^k, \Delta x_i^k, \Delta y_i^k)\|^2. \qquad (29) \end{aligned}$$

From Theorem 1 we know that $\{z^k\}$ is bounded and $\{\psi(z^k)\} \to 0$ as $k \to \infty$, and so from (28) $\{\|\Delta z^k\|\}$ also converges to 0. Since $f'(\cdot)$ is continuous, it is uniformly continuous on every compact set. Let

$$\varepsilon := \min\left\{\frac{(1-\gamma)\|\bar{w}\|}{4c_1}, \frac{\tau\gamma}{2(2+\tau/\|\bar{w}\|)c_1}\right\}. \qquad (30)$$

Then there exists a positive number $\tilde{\alpha} \in (0,1]$ such that for any $\alpha \in [0,\tilde{\alpha}]$, any $\theta \in [0,1]$, and any $k \ge 1$,

$$\|f'(x^k + \alpha\theta\Delta x^k) - f'(x^k)\| \le \varepsilon.$$

Hence for any $\alpha \in [0,\tilde{\alpha}]$ and any $k \ge 1$,

$$\|\sigma^k(\alpha)\| \le \alpha\varepsilon\|\Delta x^k\| \le \alpha\varepsilon\|\Delta z^k\|. \qquad (31)$$

By noting that $g_i^k(\alpha) = 0$ for all $i \in N$ we have

$$\begin{aligned}
\|g^k(\alpha)\| &= \left[\sum_{i=1}^{n}(|g_i^k(\alpha)|^2 + |g_{n+i}^k(\alpha)|^2 + |g_{2n+i}^k(\alpha)|^2)\right]^{1/2} \\
&= \left[\sum_{i=1}^{n}(|g_{n+i}^k(\alpha)|^2 + |g_{2n+i}^k(\alpha)|^2)\right]^{1/2} \\
&\leq \left[\sum_{i=1}^{n} |g_{n+i}^k(\alpha)|^2\right]^{1/2} + \left[\sum_{i=1}^{n} |g_{2n+i}^k(\alpha)|^2\right]^{1/2} \\
&\leq \sum_{i=1}^{n} |g_{n+i}^k(\alpha)| + \|\sigma^k(\alpha)\|. \qquad (32)
\end{aligned}$$

Let $c_2 := (\min_{i \in N} \bar{u}_i - \tau)^{-1} c_1^2$. Then, from (32), (29), (31), (27), and (28), for any $\alpha \in [0, \tilde{\alpha})$ (note that $\tilde{\alpha} \leq 1$) we have

$$\begin{aligned}
\|g^k(\alpha)\| &\leq \frac{\alpha^2}{1-\alpha}(\min_{i \in N} u_i^k)^{-1}\|\Delta z^k\|^2 + \alpha\varepsilon\|\Delta z^k\| \\
&\leq \frac{\alpha^2}{1-\alpha}(\min_{i \in N} \bar{u}_i - \tau)^{-1}\psi(z^k)^{-1}\|\Delta z^k\|^2 + \alpha\varepsilon\|\Delta z^k\| \\
&\leq \frac{\alpha^2}{1-\alpha}(\min_{i \in N} \bar{u}_i - \tau)^{-1}\psi(z^k)^{-1}c_1^2\psi(z^k)^2 + \alpha\varepsilon c_1\psi(z^k) \\
&= \alpha\left(\frac{\alpha}{1-\alpha}c_2 + c_1\varepsilon\right)\psi(z^k). \qquad (33)
\end{aligned}$$

Define $\bar{\alpha}$ as

$$\bar{\alpha} := \min\left\{\tilde{\alpha}, \frac{(1-\gamma)\|\bar{w}\|}{8c_2}, \frac{\tau\gamma}{4(2+\tau/\|\bar{w}\|)c_2}, \frac{1}{2}\right\}. \qquad (34)$$

Then from (33) for all $\alpha \in (0, \bar{\alpha}]$ we have

(35)
$$
\begin{aligned}
&\psi(z^k + \alpha \Delta z^k) - \rho(\alpha, \beta_k)\psi(z^k) \\
&\quad = \psi(z^k + \alpha \Delta z^k) - \rho(\alpha, \gamma)\psi(z^k) \\
&\quad = \bar{w} F(z^k + \alpha \Delta z^k)/\|\bar{w}\|^2 - [1 - \alpha(1-\gamma)/2]\psi(z^k) \\
&\quad \le \bar{w}^T [F(z^k) + \alpha F'(z^k)\Delta z^k]/\|\bar{w}\|^2 + \|g^k(\alpha)\|/\|\bar{w}\| - [1 - \alpha(1-\gamma)/2]\psi(z^k) \\
&\quad = \bar{w}^T F(z^k)/\|\bar{w}\|^2 + \alpha \bar{w}^T[-F(z^k) + \gamma\psi(z^k)\bar{w}]/\|\bar{w}\|^2 \\
&\quad\quad -[1 - \alpha(1-\gamma)/2]\psi(z^k) + \|g^k(\alpha)\|/\|\bar{w}\| \\
&\quad = \psi(z^k) - \alpha\psi(z^k) + \alpha\gamma\psi(z^k) - [1 - \alpha(1-\gamma)/2]\psi(z^k) + \|g^k(\alpha)\|/\|\bar{w}\| \\
&\quad = [-\alpha(1-\gamma)/2]\psi(z^k) + \|g^k(\alpha)\|/\|\bar{w}\| \\
&\quad \le [-\alpha(1-\gamma)/2]\psi(z^k) + \alpha\left(\frac{\alpha}{1-\alpha}c_2 + c_1\varepsilon\right)\psi(z^k)/\|\bar{w}\|
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\| [\gamma\psi(z^k)\bar{w} + g^k(\alpha)/\alpha] - \frac{\bar{w}^T[\gamma\psi(z^k)\bar{w} + g^k(\alpha)/\alpha]}{\|\bar{w}\|^2}\bar{w} \right\| \\
&\quad -\tau \frac{\bar{w}^T[\gamma\psi(z^k)\bar{w} + g^k(\alpha)/\alpha]}{\|\bar{w}\|^2} \\
&= \left\| g^k(\alpha)/\alpha - \frac{\bar{w}^T g^k(\alpha)/\alpha}{\|\bar{w}\|^2}\bar{w} \right\| - \tau\left(\gamma\psi(z^k) + \frac{\bar{w}^T g^k(\alpha)/\alpha}{\|\bar{w}\|^2}\right) \\
&\le \|g^k(\alpha)\|/\alpha + \|g^k(\alpha)\|/\alpha - \tau\gamma\psi(z^k) + \tau\frac{\|g^k(\alpha)\|/\alpha}{\|\bar{w}\|} \\
&\le (2 + \tau/\|\bar{w}\|)\|g^k(\alpha)\|/\alpha - \tau\gamma\psi(z^k) \\
&\le (2 + \tau/\|\bar{w}\|)\left(\frac{\alpha}{1-\alpha}c_2 + c_1\varepsilon\right)\psi(z^k) - \tau\gamma\psi(z^k).
\end{aligned}
\tag{36}
$$

By considering (30), (34), (35), and (36) we have for all $\alpha \in (0, \bar{\alpha}]$ that

(37)

$$\begin{aligned}
&\psi(z^k + \alpha\Delta z^k) - \rho(\alpha, \beta_k)\psi(z^k) \\
&\le [-\alpha(1-\gamma)/2]\psi(z^k) + \alpha(2\alpha c_2 + c_1\varepsilon)\psi(z^k)/\|\bar{w}\| \\
&= [-\alpha(1-\gamma)/4 + \alpha c_1\varepsilon/\|\bar{w}\|]\psi(z^k) + [-\alpha(1-\gamma)/4 + 2\alpha^2 c_2/\|\bar{w}\|]\psi(z^k) \\
&\le [-\alpha(1-\gamma)/4 + \alpha(1-\gamma)/4]\psi(z^k) + [-\alpha(1-\gamma)/4 + 2\alpha(1-\gamma)/8]\psi(z^k) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
&\left\| [\gamma\psi(z^k)\bar{w} + g^k(\alpha)/\alpha] - \frac{\bar{w}^T[\gamma\psi(z^k)\bar{w} + g^k(\alpha)/\alpha]}{\|\bar{w}\|^2}\bar{w} \right\| \\
&\quad -\tau\frac{\bar{w}^T[\gamma\psi(z^k)\bar{w} + g^k(\alpha)/\alpha]}{\|\bar{w}\|^2} \\
&\quad \le (2 + \tau/\|\bar{w}\|)(2\alpha c_2 + c_1\varepsilon)\psi(z^k) - \tau\gamma\psi(z^k) \\
&\quad = [2(2 + \tau/\|\bar{w}\|)\alpha c_2 - \tau\gamma/2]\psi(z^k) + [(2 + \tau/\|\bar{w}\|)c_1\varepsilon - \tau\gamma/2]\psi(z^k)
\end{aligned}$$

$$\le 0 + 0. \tag{38}$$

Hence from the inequality (38) for all $\alpha \in (0, \bar{\alpha}]$,

$$\gamma\psi(z^k)\bar{w} + g^k(\alpha)/\alpha \in C.$$

Then from $F(z^k) \in C$, the definition of C, and the fact $F(z^k + \alpha\Delta z^k) = (1-\alpha)F(z^k) + \alpha[\gamma\psi(z^k)\bar{w} + g^k(\alpha)/\alpha]$ for all $\alpha \in (0, \bar{\alpha}]$, we have

$$F(z^k + \alpha\Delta z^k) \in C. \tag{39}$$

Also, from (37), for all $\alpha \in (0, \bar{\alpha}]$,

$$\bar{w}^T F(z^k + \alpha\Delta z^k) = \psi(z^k + \alpha\Delta z^k)\|\bar{w}\|^2 \le \psi(z^k)\|\bar{w}\|^2 = \bar{w}^T F(z^k) \le \|w\|^2. \tag{40}$$

Then, from (39), (40), and (37), for all $\alpha \in (0, \bar{\alpha}]$ we have

$$F(z^k + \alpha\Delta z) \in \Omega \quad \text{and} \quad \psi(z^k + \alpha\Delta z^k) \le \rho(\alpha, \gamma)\psi(z^k).$$

Let l be the smallest nonnegative number such that $\delta^l \le \bar{\alpha}$. Then $\alpha_k \ge \delta^l$. Let $c := \rho(\delta^l, \gamma)$, then

$$\psi(z^{k+1}) \le \rho(\alpha_k, \gamma)\psi(z^k) \le \rho(\delta^l, \gamma)\psi(z^k) = c\psi(z^k).$$

This proves (23).

Next, we prove (25) under the assumptions. From (9), we have

$$\Delta u^k = -u^k + \gamma\psi(z^k)\bar{u}.$$

Then,

$$u_i^{k+1} = (1-\alpha_k)u_i^k + \alpha_k\gamma\psi(z^k)\bar{u}_i, \quad i \in N,$$

which, together with (27), gives

(41)
$$u_i^{k+1} \leq [1 - \alpha_k + \alpha_k \gamma \bar{u}_i/(\bar{u}_i - \tau)]u_i^k = \{1 - [1 - \gamma \bar{u}_i/(\bar{u}_i - \tau)]\alpha_k\}u_i^k, \quad i \in N.$$

Let

$$\bar{c} := 1 - \{1 - \gamma \max_{i \in N}[\bar{u}_i/(\bar{u}_i - \tau)]\}\delta^l.$$

Then, since γ satisfies (24) and $\delta^l \in (0,1]$, $\bar{c} \in (0,1)$. Hence, from (41) and the fact $\alpha_k \geq \delta^l$, we get

$$u_i^{k+1} \leq \bar{c} u_i^k, \quad i = N,$$

which completes the proof. □

Remark 2. (i) The results in Theorem 2 do not hold for the original version of the Hotta-Yoshise algorithm, where the definition of $\phi(\mu, a, b)$ is $\phi(\mu, a, b) = a + b - \sqrt{(a-b)^2 + 4\mu}$.

(ii) In [4, 9, 35], the authors provide a global linear convergence theorem similar to Theorem 2 under the additional assumption that $f'(\cdot)$ is Lipschitz continuous. Here we do not make such an assumption.

5. Superlinear convergence

In this section we will discuss superlinear convergence of the algorithm by setting $t \in (0,1)$ in Algorithm 1. Suppose $z^* = (0, x^*, y^*)$ is an accumulation point of the sequence $\{z^k\}$ generated by the algorithm. Then under the assumptions made in Theorem 1, z^* is a solution of $F(z) = 0$ and (x^*, y^*) is a solution of the NCP. We make the following assumptions at z^*.

Assumption 4. *$F'(z^*)$ exists and is nonsingular.*

Assumption 5. *There exist positive constants L and ε such that for all $z, z' \in B(z^*, \varepsilon) := \{z \in \Re^{3n} : \|z - z^*\| \leq \varepsilon\}$,*

$$\|F(z') - F(z) - F'(z)(z' - z)\| \leq L\|z' - z\|^2. \tag{42}$$

Proposition 3. *Suppose that z^* satisfies*

$$x^* + f(x^*) > 0$$

and $f'(\cdot)$ is Lipschitz continuous around x^*. If $f'(x^*)_{II}$ is nonsingular, then Assumptions 4 and 5 are satisfied, where

$$I := \{i : x_i^* > 0\}.$$

Proof. First, it is easy to verify that $F'(z^*)$ exists under the assumption that $x^* + f(x^*) > 0$. Moreover,

$$F'(z^*) = \begin{pmatrix} I & 0 & 0 \\ 0 & V^* & W^* \\ 0 & -f'(x^*) & I \end{pmatrix},$$

where $V^*, W^* \in \Re^{n \times n}$ satisfying

$$H'(x^*, y^*) = \begin{pmatrix} V^* & W^* \\ -f'(x^*) & I \end{pmatrix}.$$

Then $F'(z^*)$ is nonsingular because $H'(x^*, y^*)$ is nonsingular under the assumptions that $x^* + f(x^*) > 0$ and $f'(x^*)_{II}$ is nonsingular. This verifies Assumption 4.

To verify Assumption 5 we only need to prove that $\Phi(\cdot)$ is continuously differentiable in a neighbourhood of $(0, x^*, y^*)$ and its derivative is Lipschitz continuous because $F_i(z) = u_i, i \in N$, all $F_i(\cdot), i \in \{2n+1, 2n+2, ..., 3n\}$ are continuously differentiable on $\Re^{3n}$, and their derivatives are Lipschitz continuous under the assumptions. However, since $x^* + f(x^*) > 0$, it is easy to see that $\Phi(\cdot)$ is twice continuously differentiable in a neighbourhood of z^*. Then Assumption 5 is verified. $\square$

Theorem 3. *Suppose that Assumption 2 is satisfied and z^* is an accumulation point of $\{z^k\}$. If t is set to be in $(0,1)$ and Assumptions 4 and 5 are satisfied at z^*, then the whole sequence $\{z^k\}$ converges to z^* with Q-order $1+t$, i.e.,*

$$\|z^{k+1} - z^*\| = O(\|z^k - z^*\|^{1+t}). \tag{43}$$

Moreover,

$$\psi(z^{k+1}) = O(\psi(z^k)^{1+t}) \tag{44}$$

and

$$u_i^{k+1} = O((u_i^k)^{1+t}), \quad i \in N. \tag{45}$$

Proof. By Theorem 1, z^* is a solution of $F(z) = 0$ and (x^*, y^*) is a solution of the NCP. Also, from Theorem 1, we have that

$$F(z^k) \to 0 \quad \text{and} \quad \psi(z^k) \to 0 \tag{46}$$

as $k \to \infty$. If z^k is very near z^*, then, from (9), (46), and Assumptions 4 and 5, Δz^k is very near zero. Thus, from Assumption 5, there exist positive numbers L and ε such that for all $z^k \in B(z^*, \varepsilon)$,

$$\|F(z^k + \Delta z^k) - F(z^k) - F'(z^k)\Delta z^k\| \le L\|\Delta z^k\|^2. \tag{47}$$

Suppose that ε is small enough such that for any $z \in B(z^*, \varepsilon)$, $F'(z)$ exists and is invertible. Let

$$L_1 := \max_{z \in B(z^*, \varepsilon)} \{\|F'(z)^{-1}\|\} \quad \text{and} \quad L_2 := L_1(2\|\bar{w}\| + \tau).$$

Then for all $z^k \in B(z^*, \varepsilon)$,

$$\|\Delta z^k\| \le L_1\| - F(z^k) + \beta_k \psi(z^k)\bar{w}\| \le L_1[\|F(z^k)\| + \beta_k\|\bar{w}\|\psi(z^k)]. \tag{48}$$

Since $F(z^k) \in C$, we have

$$\|F(z^k) - \psi(z^k)\bar{w}\| \le \tau\psi(z^k).$$

This implies that

$$\|F(z^k)\| \le (\|\bar{w}\| + \tau)\psi(z^k). \tag{49}$$

By combining (48) and (49) and using the fact $\beta_k < 1$, for all $z^k \in B(z^*, \varepsilon)$ we have

$$\|\Delta z^k\| \le L_1(\|\bar{w}\| + \tau + \beta_k\|\bar{w}\|)\psi(z^k) \le L_2\psi(z^k). \tag{50}$$

Then, from (47), (9), and (50), for all $z^k \in B(z^*, \varepsilon)$ we have

$$\begin{aligned}
&|\psi(z^k + \Delta z^k) - \beta_k \psi(z^k)| \\
&\quad = |\bar{w}^T F(z^k + \Delta z^k)/\|\bar{w}\|^2 - \beta_k \psi(z^k)| \\
&\quad \le |\bar{w}^T [F(z^k) + F'(z^k)\Delta z^k]/\|\bar{w}\|^2 - \beta_k \psi(z^k)| + L\|\Delta z^k\|^2/\|\bar{w}\| \\
&\quad = |\bar{w}^T [\beta_k \psi(z^k)\bar{w}]/\|\bar{w}\|^2 - \beta_k \psi(z^k)| + L\|\Delta z^k\|^2/\|\bar{w}\| \\
&\quad = L\|\Delta z^k\|^2/\|\bar{w}\| \\
&\quad \le L(L_2)^2 \psi(z^k)^2/\|\bar{w}\|.
\end{aligned}$$

Then, by letting $L_3 := L(L_2)^2/\|\bar{w}\|$, for all $z^k \in B(z^*, \varepsilon)$ we have

$$|\psi(z^k + \Delta z^k) - \beta_k \psi(z^k)| \le L_3 \psi(z^k)^2. \tag{51}$$

According to our algorithm and Theorem 1, when k is sufficiently large, $\beta_k = \psi(x^k)^t$. So, when z^k is sufficiently close to z^*,

$$\beta_k + L_3 \psi(z^k) \le \frac{1}{2} + \frac{\beta_k}{2} = \rho(1, \beta_k). \tag{52}$$

Then from (51) and (52), when z^k is sufficiently close to z^*,

$$\psi(z^k + \Delta z^k) \le \beta_k \psi(k) + L_3 \psi(z^k)^2 \le \rho(1, \beta_k)\psi(z^k). \tag{53}$$

On the other hand, since $\psi(z^k + \Delta z^k) = \bar{w}^T F(z^k + \Delta z^k)/\|\bar{w}\|^2$, from (53) and the fact $F(z^k) \in H_{\bar{w}}$, we get

$$\begin{aligned}
\bar{w}^T F(z^k + \Delta z^k) &= \|\bar{w}\|^2 \psi(z^k + \Delta z^k) \\
&\le \|\bar{w}\|^2 (\tfrac{1}{2} + \tfrac{\beta_k}{2}) \psi(z^k) \\
&= (\tfrac{1}{2} + \tfrac{\beta_k}{2}) \bar{w}^T F(z^k) \\
&\le (\tfrac{1}{2} + \tfrac{\beta_k}{2}) \|\bar{w}\|^2 \\
&< \|\bar{w}\|^2.
\end{aligned}$$

So,

$$F(z^k + \Delta z^k) \in H_{\bar{w}}. \tag{54}$$

Meanwhile, from (51), (9), (47), and (50), for all z^k sufficiently close to z^* we have

$$\begin{aligned}
&\|F(z^k+\Delta z^k) - [\bar{w}^T F(z^k+\Delta z^k)/\|\bar{w}\|^2]\bar{w}\| \\
&= \|F(z^k+\Delta z^k) - \psi(z^k+\Delta z^k)\bar{w}\| \\
&\le \|F(z^k+\Delta z^k) - \beta_k\psi(z^k)\bar{w}\| + L_3\|\bar{w}\|\psi(z^k)^2 \\
&= \|F(z^k+\Delta z^k) - F(z^k) - F'(z^k)\Delta z^k\| + L_3\|\bar{w}\|\psi(z^k)^2 \\
&\le L\|\Delta z^k\|^2 + L_3\|\bar{w}\|\psi(z^k)^2 \\
&\le L(L_2)^2\psi(z^k)^2 + L_3\|\bar{w}\|\psi(z^k)^2.
\end{aligned}$$

By letting $L_4 := L(L_2)^2 + L_3\|\bar{w}\|$, for all z^k sufficiently close to z^* we have

$$\|F(z^k+\Delta z^k) - [\bar{w}^T F(z^k+\Delta z^k)/\|\bar{w}\|^2]\bar{w}\| \le L_4\psi(z^k)^2. \tag{55}$$

Suppose that z^k is sufficiently close to z^* such that

$$\beta_k - L_3\psi(z^k) = \psi(z^k)^t - L_3\psi(z^k) \ge \frac{2}{\tau}L_4\psi(z^k). \tag{56}$$

Then, from (55), (56), and (51), for all z^k sufficiently close to z^* we have

$$\begin{aligned}
&\|F(z^k+\Delta z^k) - [\bar{w}^T F(z^k+\Delta z^k)/\|\bar{w}\|^2]\bar{w}\| \\
&\le L_4\psi(z^k)^2 \\
&\le \tfrac{\tau}{2}[\beta_k - L_3\psi(z^k)]\psi(z^k) \\
&\le \tfrac{\tau}{2}\psi(z^k+\Delta z^k). \qquad (57)
\end{aligned}$$

Thus, from (53), (54), and (57) we have in fact proved that for all z^k sufficiently close to z^*,

$$z^{k+1} = z^k + \Delta z^k, \tag{58}$$

i.e., $l_k = 0$. Again, from (9), for all z^k sufficiently close to z^*,

$$\begin{aligned}
\|z^k + \Delta z^k - z^*\| &= \|z^k + F'(z^k)^{-1}[-F(z^k) + \beta_k\psi(z^k)\bar{w}] - z^*\| \\
&= O[\|F(z^k) - F(z^*) - F'(z^k)(z^k - z^*)\| + \psi(z^k)^{1+t}\|\bar{w}\|] \\
&= O(\|z^k - z^*\|^2) + O(\|F(z^k)\|^{1+t}) \qquad (59) \\
&= O(\|z^k - z^*\|^2) + O(\|z^k - z^*\|^{1+t}) \\
&= O(\|z^k - z^*\|^{1+t}).
\end{aligned}$$

Then, by combining (59) with (58), we know that when k is sufficiently large we have

$$z^{k+1} = z^k + \Delta z^k$$

and

$$\|z^{k+1} - z^*\| = O(\|z^k - z^*\|^{1+t}).$$

Hence the whole sequence $\{z^k\}$ converges to z^* with Q-order $1+t$. Then (43) is proved. Since the whole sequence $\{z^k\}$ converges to z^*, from (51) and $\beta_k = \psi(z^k)^t$ for all k sufficiently large we have

$$\psi(z^{k+1}) = O(\psi(z^k)^{1+t}).$$

This proves (44). Furthermore, from (9), when $z^{k+1} = z^k + \Delta z^k$,

$$u^{k+1} = u^k + \Delta u^k = u^k + [-u^k + \beta_k \psi(z^k)\bar{u}] = \beta_k \psi(z^k)\bar{u}.$$

Then, because when k is sufficiently large, $z^{k+1} = z^k + \Delta z^k$, for all k sufficiently large we have

$$u^{k+1} = \beta_k \psi(z^k)\bar{u}. \tag{60}$$

It follows from $F(z^k) \in C$ and $F_i(z^k) = u_i^k, i \in N$ that

$$|u_i^k - \psi(z^k)\bar{u}_i| \le \tau\psi(z^k).$$

But, since $0 < \tau < \min_{i \in N}\{\bar{u}_i\}$, we have $\psi(z^k) = O(u_i^k), i \in N$. Hence from (60) we have

$$u_i^{k+1} = O((u_i^k)^{1+t}), \quad i \in N.$$

This is (45). So, we complete the proof of this theorem. □

For different choices of a parameter $t \in [0, 1)$, the algorithm introduced in this paper is shown to be either globally linearly convergent (when $t = 0$) or globally and locally superlinearly convergent (when $t \in (0, 1)$). It was pointed out by the referee that the predictor-corrector strategy may be useful to get an algorithm with both global linear convergence and local superlinear convergence properties. By using a different neighborhood, Burke and Xu [2, 3] provided such results for monotone and nonmonotone linear complementarity problems.

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