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# 7 373 170 279 850

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ABSTRACT. We conjecture that 7,373,170,279,850 is the largest integer which cannot be expressed as the sum of four nonnegative integral cubes.

## INTRODUCTION

We consider one aspect of Waring's problem for cubes, namely the representation of nonnegative integers as sums of nonnegative integral cubes. Dickson [6] showed in 1939 that every positive integer is the sum of 8 nonnegative cubes, with the only exceptions being 23 and 239. Linnik [9] proved that every sufficiently large integer is a sum of 7 cubes; Watson [12] simplified the proof and McCurley [3] gave an effective and explicit proof of this result.

Using the traditional symbol $G(3)$ to denote the smallest $n$ such that every sufficiently large integer is a sum of $n$ nonnegative cubes, Linnik's result may be reformulated as $G(3) \leq 7$. On the other hand, it is easy to see that $G(3) \geq 4$: indeed, cubes are congruent to $0, 1$ or $-1$ modulo 9, so that integers which are congruent to 4 or 5 modulo 9 require at least 4 cubes. Moreover, Davenport [4] has shown that up to $x$, every integer is a sum of 4 nonnegative cubes with the exception of at most $o(x)$ terms.

We say that an integer is $C_k$ if it can be represented as a sum of $k$ nonnegative cubes. Western [13] gave heuristic support to the conjecture $G(3) = 4$ and also conjectured that the largest integer which is $C_5$ and not $C_4$ is located between $10^{12}$ and $10^{14}$. We present here some support for the following conjecture:

**Conjecture 1.** The integer $N = 7,373,170,279,850$ is the largest integer which cannot be expressed as the sum of four nonnegative integral cubes.

Work of Bohman and Fröberg [2] and Romani [10] suggests that there are exactly 15 integers which are $C_8$ and not $C_7$, the largest of which is 454; exactly 121 integers which are $C_7$ and not $C_6$, the largest being 8,042; and 3,922 integers which are $C_6$ and not $C_5$, the largest being 1,290,740. Bohman and Fröberg also gave some arguments in favour of the conjecture $G(3) = 4$ and proposed the estimate 112 millions for the number of integers which are $C_5$ and not $C_4$. Our computations lead us to the following:

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**Conjecture 2.** There are exactly 113, 936, 676 positive integers which are not the sum of four nonnegative integral cubes.

## 1. The method

The basic principle is to find $N_1$ such that $N_1$ is not $C_4$ but all the integers in the interval $(N_1, \kappa N_1]$, for some "security coefficient" $\kappa$, are $C_4$, and then declare $N_1$ to be the candidate for being the largest integer which is not $C_4$.

Our choice for $\kappa$ was 10; we are thankful to P. Purnaba for having performed many simulations on pseudo-cubes sequences to provide us with a decent expectation that 10 is a secure choice (cf. the Appendix). Another argument in favour of this choice comes from the computations we performed on actual cubes.

However, this principle cannot be implemented in such a direct way: it is easily checked that 7,373,170,279,850 is not $C_4$ (we are thankful to P. Zimmermann, who checked this point independently of us). Thus, implementing our principle would require us to check that all integers between $N+1$ and $7.4 \cdot 10^{13}$ are $C_4$, computations that cannot be performed with current computers and algorithms.

We modify this basic principle by inserting the irregularity of the distribution of cubes in arithmetic progressions, keeping the same security coefficient $\kappa = 10$. As we already noticed, cubes are badly distributed modulo 9: the number $\rho(k, 9)$ is the number of solutions of the congruence

$$k_1^3 + k_2^3 + k_3^3 + k_4^3 \equiv k \mod 9,$$

are given by the following table.

| $k$ | 0 | $\pm 1$ | $\pm 2$ | $\pm 3$ | $\pm 4$ |
|---|---|---|---|---|---|
| $\rho(k, 9)/9^3$ | 19/9 | 16/9 | 10/9 | 4/9 | 1/9 |

This means that it is easier to represent, as $C_4$, integers which are not congruent to $\pm 4$ modulo 9. We give in section 2 the results for those cases; let us simply mention here that we found that the number $M = 75, 377, 772, 852$ is not $C_4$ but that any integer between $M+1$ and $8.8 \cdot 10^{11}$, which is not congruent to $\pm 4$ modulo 9, is $C_4$. We are thus left with checking integers up to $7.4 \cdot 10^{13}$ which are congruent to $\pm 4$ modulo 9: we have thus won a factor 2/9, ... which is still too large.

For the remaining classes modulo 9, we went one step further in the arithmetic and considered classes modulo 7 (since 3 divides $(7-1)$, cubes are badly distributed modulo 7). Here again, some classes could be dealt with in a reasonable time (for classes modulo $\pm 4$ modulo 9 and $0, \pm 1, \pm 2$ modulo 7, the largest non-$C_4$ found turned out to be around $1.4 \cdot 10^{12}$). In the four remaining classes, the largest exceptions turned out to be between $5 \cdot 10^{12}$ and $7.4 \cdot 10^{12}$. This part is presented in section 3.

Section 4 deals with the conjectured number of non-$C_4$ numbers (Conjecture 2). In section 5, we give some heuristic support to our conjectures by using the arithmetic refinement of the Erdős-Rényi model we introduced in [5]. We finally give an application of our computations to the determination of an interval containing only $C_5$ numbers.

We close this section with two remarks concerning the computations. To determine which elements in a given interval are $C_4$, we represent them by their address in a string of bits: initially, we give the value 0 to these bits; we build strings representing $C_2$ numbers and add those strings to give the value 1 to one bit as soon as its address is seen as a sum of two $C_2$ numbers.

At the end of the process, we check which bits are 0 and which are 1: when all are 1, then all the integers in the interval are $C_4$. The reader will easily see how to modify this algorithm to take into account congruence conditions. Computations have been performed on DEC-Alpha or SUN-Sparc stations of different laboratories. We are specially thankful to the Laboratoire de Mathématiques Appliquées de Bordeaux for helping in getting access to a CRAY-t3d computer (CEA Grenoble) as well as to a DEC-Alpha station. The total CPU time involved is around 10,000 hours, and the computations were performed over a full year.

## 2. The high residue classes modulo 9

In what follows, an *exception* will mean a positive integer which is not a sum of four nonnegative cubes.

We first look at the residues classes $0, \pm 1, \pm 2, \pm 3$ modulo 9. In these classes, the likely exception $N_0$ is not too large, and the computations, which consist in checking that any integers between $N_0$ and $10N_0$ are sums of four cubes, are swiftly performed.

The largest exceptions $N_0$ that we obtained and the corresponding sifted intervals are given in Table 1. As expected, these numbers are congruent to $\pm 3$ modulo 7, which as a matter of fact are the lowest classes modulo 7.

Table 1. The last exception in the classes $0, \pm 1, \pm 2, \pm 3$ modulo 9.

| class | $\rho(k,9)/9^3$ | largest non-$C_4$ integer | mod 7 | checking up to |
|---|---|---|---|---|
| 0 | 19/9 | 396 953 532 | 3 | $4,5 \cdot 10^{11}$ |
| 1 | 16/9 | 252 716 950 | 3 | $3 \cdot 10^9$ |
| 2 | 10/9 | 1 761 425 102 | 3 | $3.7 \cdot 10^{10}$ |
| 3 | 4/9 | 44 322 060 990 | 4 | $5.3 \cdot 10^{11}$ |
| 6 | 4/9 | 75 377 772 852 | 4 | $8.8 \cdot 10^{11}$ |
| 7 | 10/9 | 4 045 088 338 | 4 | $4.5 \cdot 10^{10}$ |
| 8 | 16/9 | 505 945 682 | 4 | $5.4 \cdot 10^9$ |

As expected, we notice at once that the size of the largest exception found in a given class $k$ strongly depends on the number $\rho(k, 9)$.

Table 2. The ten largest exceptions in the classes $0, \pm 1, \pm 2, \pm 3$ modulo 9.

| 0 [9] | 1 [9] | 2 [9] | 3 [9] |
|---|---|---|---|
| 109 563 030 | 130 242 934 | 905 760 614 | 30 018 581 436 |
| 114 717 348 | 130 576 555 | 931 528 658 | 30 205 280 802 |
| 133 218 684 | 134 916 274 | 934 479 389 | 30 756 454 158 |
| 133 262 559 | 147 350 458 | 1 017 344 108 | 30 794 631 438 |
| 133 297 182 | 152 177 806 | 1 021 218 446 | 31 702 361 898 |
| 136 987 722 | 171 820 702 | 1 123 934 213 | 33 141 245 610 |
| 146 692 746 | 173 788 444 | 1 155 472 427 | 41 155 522 446 |
| 152 955 828 | 198 367 831 | 1 189 684 226 | 41 319 931 908 |
| 188 204 580 | 204 605 740 | 1 680 416 174 | 41 918 435 499 |
| 396 953 532 | 252 716 950 | 1 761 425 102 | 44 322 060 990 |

TABLE 2 (continued)

| -3 [9] | -2 [9] | -1 [9] |
|---|---|---|
| 39 129 270 513 | 1 043 547 838 | 132 173 261 |
| 40 086 582 225 | 1 049 202 214 | 133 045 622 |
| 40 686 577 404 | 1 072 947 949 | 148 532 723 |
| 43 149 463 206 | 1 090 092 580 | 165 687 092 |
| 43 234 286 343 | 1 135 860 478 | 178 270 145 |
| 45 241 168 038 | 1 146 854 860 | 192 810 230 |
| 48 420 314 610 | 1 148 123 959 | 218 223 134 |
| 57 604 173 756 | 1 216 888 054 | 230 528 546 |
| 66 945 773 058 | 1 312 833 274 | 249 325 766 |
| 75 377 772 852 | 4 045 088 338 | 505 945 682 |

Table 2 gives an idea of the distribution of the exceptions in every class, and in particular shows that two consecutive exceptions in a given class are relatively closed, independently of the class.

## 3. THE LABORIOUS CASES: THE CLASSES 4 AND 5 MODULO 9

For these two classes, our computations means were not adapted to obtain the likely exceptions in the same way. We went around these difficulties by considering the 14 residue classes modulo 63 coming from the classes $\pm 4$ modulo 9, and by applying the same algorithm to each of them.

In Table 3, we show the fourteen classes modulo 63 to be studied.

TABLE 3. The low residues modulo 63.

| mod 9 / mod 7 | 4 | 5 |
|---|---|---|
| 0 | 49 | 14 |
| 1 | 22 | 50 |
| 2 | 58 | 23 |
| 3 | 31 | 59 |
| 4 | 4 | 32 |
| 5 | 40 | 5 |
| 6 | 13 | 41 |

We first calculate the number of solutions to the congruence

$$k_1^3 + k_2^3 + k_3^3 + k_4^3 \equiv k \mod 7,$$

denoted by $\rho(k, 7)$, which is given in the following table.

| $k$ | 0 | $\pm 1$ | $\pm 2$ | $\pm 3$ |
|---|---|---|---|---|
| $\rho(k,7)/7^3$ | 595/343 | 336/343 | 378/343 | 189/343 |

Looking at this table, we may expect that the last exceptions should be congruent to $\pm 3$ modulo 7, and thus belong to classes $\pm 4$ or $\pm 31$ modulo 63, which will be called the low classes modulo 63.

In the following subsections, we collect the largest exceptions in the fourteen classes, setting together classes having the same 4-cubes representation coefficient modulo 63. We shall notice that these exceptions essentially belong to the low classes modulo 13 (that is 1,5,8,12), modulo 19 (that is 2,3,5,14,16,17) and modulo 8 (that is 2,4,6).

3.1. **The residue class 0 modulo 7.** The classes are 14 and 49 modulo 63: the representation ratio is $\mathfrak{s}(k,63) = \rho(k,63)/63^3 = (1/9) \times (595/343) = 0.192\ldots$

TABLE 4. The last exception in the classes 14 and 49 modulo 63.

| class | largest non-$C_4$ integer | mod 13 | checking up to |
|---|---|---|---|
| 14 | 83 593 932 170 | 8 | $1.132 \cdot 10^{12}$ |
| 49 | 96 127 145 590 | 8 | $10^{12}$ |

TABLE 5. The ten largest exceptions in class 14 and 49 modulo 63.

| class 14 | [13] | [8] | [19] | class 49 | [13] | [8] | [19] |
|---|---|---|---|---|---|---|---|
| 69 453 814 262 | 1 | 6 | 10 | 59 419 179 652 | 1 | 4 | 16 |
| 70 862 250 074 | 5 | 2 | 3 | 61 296 801 916 | 12 | 4 | 5 |
| 72 465 894 914 | 10 | 2 | 3 | 65 244 097 030 | 8 | 6 | 13 |
| 74 436 498 878 | 1 | 6 | 5 | 67 505 819 458 | 12 | 2 | 4 |
| 74 928 861 266 | 12 | 2 | 3 | 72 110 490 340 | 5 | 4 | 5 |
| 77 461 820 870 | 5 | 6 | 10 | 73 731 109 018 | 8 | 2 | 3 |
| 78 715 215 194 | 5 | 2 | 16 | 74 583 499 522 | 12 | 2 | 5 |
| 80 564 235 458 | 5 | 2 | 16 | 76 969 316 956 | 12 | 4 | 17 |
| 80 912 821 010 | 8 | 2 | 16 | 85 533 027 412 | 1 | 4 | 16 |
| 83 593 932 170 | 8 | 2 | 10 | 96 127 145 590 | 8 | 6 | 2 |

3.2. **The residue classes 1 and -1 modulo 7.** The classes are 13, 22, 41 and 50 modulo 63: the representation ratio is $\mathfrak{s}(k,63) = \rho(k,63)/63^3 = (1/9)\times(336/343) = 0.108\ldots$

TABLE 6. The last exception in the classes 13,22,41 and 50 modulo 63.

| class | largest non-$C_4$ integer | mod 13 | checking up to |
|---|---|---|---|
| 13 | 907 751 255 494 | 5 | $9.2 \cdot 10^{12}$ |
| 22 | 788 129 237 722 | 8 | $8.6 \cdot 10^{12}$ |
| 41 | 1 427 500 392 170 | 8 | $14.364 \cdot 10^{12}$ |
| 50 | 936 140 172 206 | 5 | $9.41 \cdot 10^{12}$ |

TABLE 7. The ten largest exceptions in the classes 13, 22, 41 and 50 modulo 63.

| class 13 | [13] | [8] | [19] | class 22 | [13] | [8] | [19] |
|---|---|---|---|---|---|---|---|
| 515 415 341 713 | 12 | 1 | 1 | 463 699 078 942 | 8 | 6 | 3 |
| 515 716 372 660 | 8 | 4 | 2 | 466 743 009 244 | 5 | 4 | 3 |
| 517 437 367 537 | 12 | 1 | 16 | 470 521 253 749 | 1 | 5 | 2 |
| 521 809 634 254 | 12 | 6 | 14 | 521 760 696 430 | 12 | 6 | 15 |
| 591 096 733 492 | 5 | 4 | 6 | 526 065 260 638 | 8 | 6 | 14 |
| 632 123 355 982 | 1 | 6 | 16 | 542 997 708 394 | 11 | 2 | 11 |
| 644 670 291 838 | 1 | 6 | 17 | 560 084 910 988 | 5 | 4 | 17 |
| 660 658 929 916 | 11 | 4 | 3 | 576 155 440 372 | 5 | 4 | 17 |
| 663 859 461 082 | 12 | 2 | 17 | 666 721 591 726 | 8 | 6 | 16 |
| 907 751 255 494 | 5 | 6 | 2 | 788 129 237 722 | 8 | 2 | 17 |

TABLE 7 (continued)

| class 41 | [13] | [8] | [19] | class 50 | [13] | [8] | [19] |
|---|---|---|---|---|---|---|---|
| 540 717 468 836 | 12 | 4 | 5 | 505 720 130 378 | 8 | 2 | 5 |
| 541 298 339 798 | 8 | 6 | 3 | 527 295 430 670 | 8 | 6 | 17 |
| 542 474 783 339 | 8 | 3 | 5 | 528 887 658 902 | 5 | 6 | 15 |
| 556 064 232 170 | 5 | 2 | 5 | 546 630 271 790 | 5 | 6 | 14 |
| 559 284 113 660 | 12 | 4 | 2 | 558 710 089 988 | 11 | 4 | 17 |
| 563 987 764 094 | 1 | 6 | 5 | 572 694 654 722 | 8 | 2 | 14 |
| 599 435 462 660 | 8 | 4 | 16 | 588 152 293 646 | 8 | 6 | 2 |
| 767 912 798 498 | 1 | 2 | 16 | 725 000 004 338 | 12 | 2 | 3 |
| 856 645 138 166 | 12 | 6 | 5 | 858 098 874 326 | 5 | 6 | 14 |
| 1 427 500 392 170 | 8 | 2 | 17 | 936 140 172 206 | 5 | 6 | 5 |

3.3. **The residue classes 2 and -2 modulo 7.** The classes are 5, 23, 40 and 58 modulo 63: the representation ratio is $\mathfrak{s}(k,63) = \rho(k,63)/63^3 = (1/9)\times(378/343) = 0.122\ldots$.

TABLE 8. The ten largest exceptions in the classes 13, 22, 41 and 50 modulo 63.

| class | largest non-$C_4$ integer | mod 13 | checking up to |
|---|---|---|---|
| 5 | 706 796 978 900 | 12 | $7.071 \cdot 10^{12}$ |
| 23 | 913 105 904 972 | 1 | $9.141 \cdot 10^{12}$ |
| 40 | 515 338 220 164 | 1 | $5.163 \cdot 10^{12}$ |
| 58 | 647 984 206 102 | 12 | $6.491 \cdot 10^{12}$ |

TABLE 9. The ten largest exceptions in the classes 5, 23, 40 and 58 modulo 63.

| class 5 | [13] | [8] | [19] | class 23 | [13] | [8] | [19] |
|---|---|---|---|---|---|---|---|
| 367 699 306 658 | 8 | 2 | 5 | 408 491 662 658 | 1 | 2 | 10 |
| 368 491 257 545 | 8 | 1 | 17 | 416 597 533 340 | 1 | 4 | 5 |
| 375 244 504 556 | 12 | 4 | 2 | 425 893 618 292 | 12 | 4 | 6 |
| 378 197 041 262 | 1 | 6 | 16 | 443 781 861 791 | 8 | 7 | 5 |
| 384 438 283 052 | 5 | 4 | 16 | 452 739 991 118 | 12 | 6 | 3 |
| 401 492 794 379 | 1 | 3 | 3 | 507 587 062 334 | 1 | 6 | 17 |
| 409 896 282 794 | 11 | 2 | 2 | 519 302 630 660 | 8 | 4 | 3 |
| 455 907 401 906 | 8 | 2 | 2 | 559 222 247 390 | 1 | 6 | 12 |
| 506 893 225 298 | 5 | 2 | 3 | 680 757 914 426 | 5 | 2 | 3 |
| 706 796 978 900 | 12 | 4 | 14 | 913 105 904 972 | 1 | 4 | 16 |

| class 40 | [13] | [8] | [19] | class 58 | [13] | [8] | [19] |
|---|---|---|---|---|---|---|---|
| 336 882 895 060 | 10 | 4 | 7 | 353 850 173 266 | 5 | 2 | 15 |
| 345 086 371 642 | 12 | 2 | 14 | 370 423 987 978 | 5 | 2 | 5 |
| 352 712 730 046 | 1 | 6 | 3 | 372 336 481 876 | 8 | 4 | 16 |
| 382 298 697 373 | 8 | 5 | 5 | 378 914 738 827 | 1 | 3 | 17 |
| 402 682 599 814 | 1 | 6 | 16 | 404 782 452 580 | 12 | 4 | 9 |
| 408 265 707 946 | 5 | 2 | 5 | 415 747 301 566 | 12 | 6 | 10 |
| 410 885 502 214 | 5 | 6 | 2 | 454 167 517 162 | 8 | 2 | 16 |
| 420 483 613 324 | 8 | 4 | 14 | 515 331 316 642 | 12 | 2 | 17 |
| 465 966 625 045 | 12 | 5 | 17 | 586 783 317 388 | 8 | 4 | 2 |
| 515 338 220 164 | 1 | 4 | 3 | 647 984 206 102 | 12 | 6 | 2 |

3.4. **The low classes modulo 63 and the likely largest exception.** We now deal with the four remaining classes, those corresponding to the classes $\pm 3$ modulo 7 and $\pm 4$ modulo 9. As expected the examination of these classes has revealed the probable largest number not representable as a sum of four cubes: 7,373,170,279,850. It was found in the class 32 modulo 63.

The representation ratio for the classes $\pm 4, \pm 31$ modulo is
$\mathfrak{s}(k, 63) = \rho(k, 63)/63^3 = (1/9) \times (189/343) = 3/49 = 0.061\ldots$

TABLE 10. The last exception in the classes 4, 31, 32 and 59 modulo 63.

| class | largest non-$C_4$ integer | classe mod 13 | checking up to |
|---|---|---|---|
| 4 | 6 496 802 093 380 | 1 | $6.5 \cdot 10^{13}$ |
| 31 | 5 284 099 948 018 | 8 | $5.35 \cdot 10^{13}$ |
| 32 | 7 373 170 279 850 | 11 | $7.39 \cdot 10^{13}$ |
| 59 | 6 021 018 973 490 | 1 | $6.3 \cdot 10^{13}$ |

TABLE 11. The ten largest exceptions in the classes 4, 31, 32 and 59 modulo 63.

| class 4 | [13] | [8] | [19] | class 31 | [13] | [8] | [19] |
|---|---|---|---|---|---|---|---|
| 4 101 746 020 978 | 8 | 2 | 3 | 4 097 950 646 674 | 1 | 2 | 16 |
| 4 176 071 432 950 | 1 | 6 | 3 | 4 243 508 161 924 | 5 | 4 | 2 |
| 4 258 171 417 378 | 8 | 2 | 3 | 4 351 566 387 514 | 8 | 2 | 3 |
| 4 260 747 448 381 | 8 | 5 | 5 | 4 455 736 568 986 | 1 | 2 | 5 |
| 4 261 453 542 490 | 1 | 2 | 2 | 4 626 872 001 454 | 1 | 6 | 17 |
| 4 335 278 405 602 | 1 | 2 | 5 | 4 662 046 058 890 | 12 | 2 | 14 |
| 4 960 851 010 042 | 5 | 2 | 3 | 4 799 676 641 980 | 12 | 4 | 4 |
| 5 041 706 085 742 | 1 | 6 | 3 | 4 986 551 506 702 | 12 | 6 | 2 |
| 5 269 052 852 662 | 1 | 6 | 14 | 5 263 158 954 910 | 12 | 6 | 17 |
| 6 496 802 093 380 | 1 | 4 | 3 | 5 284 099 948 018 | 8 | 2 | 17 |
| **class 32** | **[13]** | **[8]** | **[19]** | **class 59** | **[13]** | **[8]** | **[19]** |
| 4 075 773 601 316 | 1 | 4 | 5 | 3 802 208 355 158 | 5 | 6 | 16 |
| 4 086 898 600 082 | 12 | 2 | 2 | 3 825 977 414 234 | 1 | 2 | 2 |
| 4 364 287 298 060 | 1 | 4 | 16 | 3 870 821 254 730 | 1 | 2 | 9 |
| 4 592 346 735 722 | 5 | 2 | 16 | 3 889 185 641 834 | 12 | 2 | 3 |
| 4 639 786 626 164 | 1 | 4 | 17 | 4 058 748 783 302 | 5 | 6 | 17 |
| 4 668 204 750 962 | 12 | 2 | 14 | 4 145 452 151 270 | 1 | 6 | 2 |
| 5 521 284 141 881 | 1 | 1 | 5 | 4 798 029 384 914 | 12 | 2 | 17 |
| 5 676 158 919 722 | 5 | 2 | 3 | 4 798 065 694 586 | 8 | 2 | 5 |
| 6 196 484 961 230 | 5 | 6 | 3 | 5 368 106 543 558 | 5 | 6 | 3 |
| **7** 373 170 279 850 | 11 | 2 | 6 | 6 021 018 973 490 | 1 | 2 | 16 |

## 4. The number of exceptions

In Table 12, we have gathered for each class modulo 63 the probable number of integers which cannot be written as a sum of four nonnegative cubes. This leads to Conjecture 2 stated in the introduction.

Looking at Table 12, we observe that there are more exceptions in the class 5 modulo 9 than in the class 4, for a given class modulo 7. This phenomenon can be

TABLE 12. The likely number of exceptions by class modulo 63.

| non-$C_4$ by class mod 63 | 0 [7] | 1 [7] | 2 [7] | 3 [7] | 4 [7] | 5 [7] | 6 [7] | total by class mod 9 |
|---|---|---|---|---|---|---|---|---|
| 0 [9] | 36<br>0 [63] | 334<br>36 [63] | 212<br>9 [63] | 2565<br>45 [63] | 2634<br>18 [63] | 256<br>54 [63] | 296<br>27 [63] | 6333 |
| 1 [9] | 62<br>28 [63] | 471<br>1 [63] | 382<br>37 [63] | 4119<br>10 [63] | 4006<br>46 [63] | 407<br>19 [63] | 555<br>55 [63] | 10002 |
| 2 [9] | 391<br>56 [63] | 2551<br>29 [63] | 1773<br>2 [63] | 17329<br>38 [63] | 17441<br>11 [63] | 2088<br>47 [63] | 2785<br>20 [63] | 44358 |
| 3 [9] | 8828<br>21 [63] | 52346<br>57 [63] | 35949<br>30 [63] | 307600<br>3 [63] | 308974<br>39 [63] | 37884<br>12 [63] | 53794<br>48 [63] | 805375 |
| 4 [9] | 644283<br>49 [63] | 3747040<br>22 [63] | 2536522<br>58 [63] | 21167119<br>31 [63] | 21169677<br>4 [63] | 2618012<br>40 [63] | 3799510<br>13 [63] | 55682163 |
| 5 [9] | 663407<br>14 [63] | 3870947<br>50 [63] | 2572186<br>23 [63] | 21402004<br>59 [63] | 21403611<br>32 [63] | 2656879<br>5 [63] | 3927902<br>41 [63] | 56496936 |
| 6 [9] | 9197<br>42 [63] | 57399<br>15 [63] | 36666<br>51 [63] | 315565<br>24 [63] | 317700<br>60 [63] | 38543<br>33 [63] | 58398<br>6 [63] | 833468 |
| 7 [9] | 415<br>7 [63] | 3025<br>43 [63] | 1789<br>16 [63] | 18330<br>52 [63] | 18489<br>25 [63] | 2074<br>61 [63] | 3325<br>34 [63] | 47447 |
| 8 [9] | 64<br>35 [63] | 592<br>8 [63] | 368<br>44 [63] | 4295<br>17 [63] | 4238<br>53 [63] | 420<br>26 [63] | 617<br>62 [63] | 10594 |
| total by class mod 7 | 1326683 | 7734705 | 5185847 | 43238926 | 43246770 | 5356563 | 7847182 | 113936676 |

explained by the fact that up to a given bound $x$ there are more cubes of the form $(3k+1)^3$ than of the form $(3k+2)^3$, and thus the number of sums of four cubes in the class 4 modulo 9 is greater than that in the class 5 modulo 9. The numbers of exceptions in the respective classes naturally follow the same pattern. Similar but less pronounced phenomena also appear when comparing classes 3 and 4 modulo 7, classes $\pm 1$ modulo 9, classes $\pm 2$ modulo 9 or as well as classes $\pm 3$ modulo 9. Thus, it is not really an accident that the largest exception has been found in the class 32 modulo 63.

We also may look at the residue modulo 13 of the largest exceptions. The number of solutions of the congruence

$$k_1^3 + k_2^3 + k_3^3 + k_4^3 \equiv k \mod 13,$$

denoted by $\rho(k, 13)$ takes the following values.

| $k$ | 0 | $\pm 1$ | $\pm 2$ | $\pm 3$ | $\pm 4$ | $\pm 5$ | $\pm 6$ |
|---|---|---|---|---|---|---|---|
| $\rho(k, 13)$ | 3133 | 1794 | 2106 | 2106 | 2457 | 1794 | 2457 |
| $\rho(k, 13)/13^3$ | 1.426 | 0.816 | 0.958 | 0.958 | 1.118 | 0.816 | 1.118 |

We observe that all the largest exceptions in the low classes 4, 31, 32, 59 modulo 63 are congruent to 1, 5, 8 or 12 modulo 13, except, unexpectedly, the largest one 7,373,170,279,850: it belongs to the class 11 modulo 13 whose representation coefficient 0.958 is rather close to the minimum one 0.816.

Similar remarks can be done when the modulus is 8.

| $k$ | 0 | $\pm 1$ | $\pm 2$ | $\pm 3$ | 4 |
|---|---|---|---|---|---|
| $\rho(k, 8)$ | 704 | 512 | 448 | 512 | 448 |
| $\rho(k, 8)/8^3$ | 1.375 | 1 | 0.875 | 1 | 0.875 |

As expected, the exceptions in tables 5, 7, 9 and 11 are principally in the low class modulo 8, that is 2, 4 or 8. Such a result is true for modulus 19.

Last but not least, the size of the last exceptions as well as the total number of exceptions are consistent with the expectations of Western [13] and Bohman and Fröberg [2], as mentioned in the introduction.

## 5. Probabilistic study

We propose here some heuristics for supporting the numerical results that we previously obtained. We use an arithmetic refinement to the probabilistic model of Erdős and Rényi [7]; this trick has been introduced as part of the sums of 3 cubes program [5] and fits with the sums of four cubes.

5.1. **The Erdős-Rényi model.** Let $(\Omega, \mathcal{T}, P)$ be a probability space and $(\xi_n)_{n\geq 1}$ be a sequence of independent Bernoulli random variables such that

$$P(\xi_n = 1) = \alpha_n \qquad \text{and} \qquad P(\xi_n = 0) = 1 - \alpha_n,$$

where $\alpha_n = 1/(3n^{2/3})$, $n \geq 1$.

We construct a random sequences of integers $(\nu_l)_{l\geq 1}$ by considering the set of $n$ for which $\xi_n = 1$. We easily check that almost everywhere this sequence has infinitely many elements and satisfies $\nu_l \sim l^3$ when $l$ tends to infinity.

Let $R_n$ be the random variable counting the number of ways to represent $n$ as a sum

$$n = \nu_{l_1} + \nu_{l_2} + \nu_{l_3} + \nu_{l_4} \quad \text{with} \quad 1 \leq l_1 < l_2 < l_3 < l_4.$$

We then have

$$R_n = \sum_{\mathbf{h}=(h_1,\dots,h_4)\in\mathcal{H}} \xi_{h_1} \cdots \xi_{h_4},$$

where

$$\mathcal{H} = \mathcal{H}(n) = \{\mathbf{h} = (h_1, \dots, h_4),\ 1 \leq h_1 < \cdots < h_4 \leq n,\ h_1 + \cdots + h_4 = n\}.$$

Here we deal with the probability

$$P(R_n = 0) = p\left(\bigcap_{\mathbf{h}\in\mathcal{H}} \overline{\{\xi_{\mathbf{h}} = 1\}}\right) \tag{1}$$

that $n$ is not a sum of four distinct elements of $(\nu_l)$.

A way to bound it is to use Janson's inequality [8], leading, under the assumption $p(A) \leq \epsilon$ for any $A \in \mathcal{A}$, to

$$\prod_{A\in\mathcal{A}} p(\overline{A}) \leq p\left(\bigcap_{A\in\mathcal{A}} \overline{A}\right) \leq \prod_{A\in\mathcal{A}} p(\overline{A}) \exp\left(\frac{\delta}{1-\epsilon}\right), \tag{2}$$

where $\delta := \sum_{A_1, A_2 \text{dependent}} p(A_1 \cap A_2)$.

Moreover we have

$$\prod_{A\in\mathcal{A}} p(\overline{A}) = \prod_{\mathbf{h}\in\mathcal{H}} (1 - \alpha_{\mathbf{h}}) \leq \exp\left(-\sum_{\mathbf{h}\in\mathcal{H}} \alpha_{\mathbf{h}}\right) = \exp(-\mu),$$

where $\mu = \mu(n) = ER_n$.

Using Lemma 2.9 of [11], we easily obtain

$$\mu(n) \sim \gamma n^{1/3}, \quad \text{when } n \text{ tends to infinity}, \tag{3}$$

with $\gamma = \frac{\Gamma(\frac{1}{3})^3}{3^3 4!}$.

Unfortunately the term $\delta$ is not negligible, and even has the same order as $\mu$; thus Janson's inequality does not give a satisfactory bound.

Since our aim is only to give arguments in consideration of the results in the previous sections, we identify $P(R_n = 0)$ to $e^{-\mu}$, as it would be the case if the events $A$ were independent.

So if we use the estimate

$$P(R_n = 0) \asymp \exp(-\gamma n^{1/3}),$$

we will obtain the value of $n_0$ beyond which $P(R_n = 0) < 1/n$. This condition holds for $n_0 = 3 \cdot 10^8$, a bad result in view of our computations. We now turn to the arithmetic model, which should give a much better estimate.

5.2. **The arithmetic model.** The modulus $K$ being fixed, we consider for any $k$, $1 \leq k \leq K$, a sequence $(\xi_n^{(k)})_{n\geq 1}$ of independent Bernoulli random variables such that

$$P(\xi_n^{(k)} = 1) = \alpha_n = 1 - P(\xi_n^{(k)} = 0),$$

where $\alpha_n = \frac{1}{3(nK)^{2/3}}$. This gives a family of $K$ random increasing sequences $\left(\nu_l^{(k)}\right)_{l\geq 1}$ by considering for each $k$ the integers $n$ for which $\xi_n^{(k)} = 1$. To each of them, we associate the sequence $\left(\mu_l^{(k)}\right)_{l\geq 1}$ defined by

$$\mu_l^{(k)} = \nu_l^{(k)} K + m(k^3),$$

where $m(k^3)$ is the smallest nonnegative integer congruent to $k^3$ modulo $K$. The sequences $\left(\mu_l^{(k)}\right)_{l\geq 1}$ give a probabilistic model of the cubes in the arithmetic progressions modulo $K$: almost everywhere we have $\mu_l^{(k)} \sim (Kl + k)^3$ when $l$ tends to infinity.

Let $k_0$ be a residue class modulo $K$, denote by $\mathbf{k} = (k_1, k_2, k_3, k_4)$ a solution to the congruence

$$k_1^3 + k_2^3 + k_3^3 + k_4^3 \equiv k_0 \pmod{K}, \tag{4}$$

and let $\mathcal{C}(k_0)$ be the set of solutions to (4), $\rho(k_0, K)$ its cardinality.

For $\mathbf{k} = (k_1, \ldots, k_4) \in \mathcal{C}(k_0)$ and $n$ congruent to $k_0$ modulo $K$, we denote by $R_{\mathbf{k}}(n)$ the number of representations of $n$ as

$$n = \mu_{l_1}^{(k_1)} + \cdots + \mu_{l_4}^{(k_4)}, \tag{5}$$

with

$$\mu_{l_1}^{(k_1)} < \cdots < \mu_{l_4}^{(k_4)}. \tag{6}$$

We finally denote by $R_n$ the total number of representations obtained by summing over all solutions $\mathbf{k} \in \mathcal{C}(k_0)$,

$$R_n = \sum_{\mathbf{k} \in \mathcal{C}(k_0)} R_{\mathbf{k}}(n). \tag{7}$$

The class $k_0$ being fixed, for $n$ large enough, $R_{\mathbf{k}}(n)$ denotes the number of representations of $N_{\mathbf{k}} := (n - m(k_1^3) - \cdots - m(k_4^3))/K$ as

$$N_{\mathbf{k}} = \nu_{l_1}^{(k_1)} + \cdots + \nu_{l_4}^{(k_4)},$$

with $\nu_{l_1}^{(k_1)} < \nu_{l_2}^{(k_2)} < \cdots < \nu_{l_4}^{(k_4)}$.

This implies that

$$R_{\mathbf{k}}(n) = \sum_{\mathbf{h}\in\mathcal{H}(N_{\mathbf{k}})} \xi_{h_1}^{(k_1)} \dots \xi_{h_4}^{(k_4)}, \tag{8}$$

where $\mathcal{H}(N) = \{\mathbf{h} = (h_1, \dots, h_4), 1 \le h_1 < \dots < h_4 \le N, h_1 + \dots + h_4 = N\}$.

We then have

$$R_n = \sum_{\mathbf{k}\in\mathcal{C}(k_0)} \sum_{\mathbf{h}\in\mathcal{H}(N_{\mathbf{k}})} \xi_{h_1}^{(k_1)} \dots \xi_{h_4}^{(k_4)} = \sum_{\mathbf{k}\in\mathcal{C}(k_0),\mathbf{h}\in\mathcal{H}(N_{\mathbf{k}})} \theta_{\mathbf{k},\mathbf{h}}, \tag{9}$$

where $\theta_{\mathbf{k},\mathbf{h}} = \xi_{h_1}^{(k_1)} \dots \xi_{h_4}^{(k_4)}$.

As in the simple model, we shall identify $P(R_n = 0)$ to $e^{-\mu}$, where

$$\mu = \sum_{\mathbf{k},\mathbf{h}} P(\{\theta_{\mathbf{k},\mathbf{h}} = 1\}).$$

The estimate of $\mu(n)$ leads to

$$\mu(n) \sim \gamma\mathfrak{s}(n, K)n^{1/3} \quad \text{when } n \text{ tends to infinity}, \tag{10}$$

where $\mathfrak{s}(n, K)$ denotes the 4-cubes representation coefficient $\rho(n, K)/K^3$.

We thus shall use the following estimate:

$$P(R_n = 0) \asymp \exp\left(-\gamma\mathfrak{s}(n, K)n^{1/3}\right). \tag{11}$$

5.3. **The size of the likely largest exception.** Let $\alpha \ge 1$. We first obtain the following properties of the function $\mathfrak{s}(k, K)$:

If $p \equiv 2 \mod 3$,

$$\min_n \mathfrak{s}(n, p^\alpha) = 1 - \frac{1}{p^3}.$$

If $p \equiv 1 \mod 3$, let us write $4p = a^2 + 27b^2$ with $a \equiv 1 \mod 3$. We then have

$$\min_n \mathfrak{s}(n, p^\alpha) = 1 - \frac{27|b| + 5a + 12}{2p^2}. \tag{12}$$

We now compute, for each residue $k$ modulo 63,

$$\mathfrak{s}_k = \min_{\substack{n\equiv k \mod 63\\ K\ge 1}} \mathfrak{s}(n, K).$$

A way to estimate the probable size of the last exception in each class modulo 63 is to locate it where $P(R_n = 0)$ becomes smaller than $63/n$.

Using estimate (11), we shall just compute the value of $n$ for which $\exp(-\gamma\mathfrak{s}_k n^{1/3})$ and $63/n$ are equal. Let us denote

$$\begin{aligned} \mathfrak{a} &:= \min_{\substack{n\\ (K,21)=1}} \mathfrak{s}(n, K) \\ &= \prod_{p\equiv 2 \mod 3} \left(1 - \frac{1}{p^3}\right) \prod_{\substack{p\equiv 1 \mod 3\\ p>7}} \left(1 - \frac{27|b| + 5a + 12}{2p^2}\right) \\ &= 0.64919\cdots; \end{aligned} \tag{13}$$

then

$$t_k := \min_{\substack{\alpha\ge 1\\ n\equiv k \mod 9}} \mathfrak{s}(n, 3^\alpha),$$

and

$$s_k := \min_{\substack{\alpha\geq 1\\ n\equiv k \mod 7}} \mathfrak{s}(n, 7^\alpha).$$

The value of $t_k$ and $s_k$ are listed below.

| $k$ | 0 | $\pm1$ | $\pm2$ | $\pm3$ | $\pm4$ |
|---|---|---|---|---|---|
| $t_k$ | 2 | 16/9 | 10/9 | 4/9 | 1/9 |

| $k$ | 0 | $\pm1$ | $\pm2$ | $\pm3$ |
|---|---|---|---|---|
| $s_k$ | 594/343 | 48/49 | 54/49 | 27/49 |

This leads to Table 13, which gathers for each class modulo 63 the values of $u_k = t_{k_1}s_{k_2}$, where $|k_1| \leq 4$, $|k_2| \leq 3$, $k \equiv k_1 \mod 9$ and $k \equiv k_2 \mod 7$.

TABLE 13. The values of $u_k = t_{k_1}s_{k_2}$.

| $k_1$ mod 9 / $k_2$ mod 7 | 0 | $\pm1$ | $\pm2$ | $\pm3$ | $\pm4$ |
|---|---|---|---|---|---|
| 0 | 1188/343 | 1056/343 | 660/343 | 264/343 | 66/343 |
| $\pm1$ | 96/49 | 256/147 | 160/147 | 64/147 | 16/147 |
| $\pm2$ | 108/49 | 96/49 | 60/49 | 24/49 | 6/49 |
| $\pm3$ | 54/49 | 48/49 | 30/49 | 12/49 | 3/49 |

To take into account the distribution irregularities of the cubes in arithmetic progressions, we are led to consider

$$\mathfrak{s}_k := \min_{\substack{n\equiv k \mod 63\\ K\geq 1}} \mathfrak{s}(n, K) = \mathfrak{a}u_k,$$

for any $k$. This gives the size of the integer $n_k$ for which $\exp(-\mathfrak{s}_k\gamma {n_k}^{1/3}) \asymp 63/n_k$.

TABLE 14. Size of the last exception $n_k$ given by the arithmetic model.

| $k$ mod 9 / $k$ mod 7 | 0 | $\pm1$ | $\pm2$ | $\pm3$ | $\pm4$ |
|---|---|---|---|---|---|
| 0 | 4.78 e 6 | 7.71 e 6 | 4.9 e 7 | 1.50 e 9 | 2.07 e 11 |
| $\pm1$ | 4.57 e 7 | 7.19 e 7 | 4.21 e 8 | 1.17 e 10 | 1.48 e 12 |
| $\pm2$ | 2.90 e 7 | 4.57 e 7 | 2.72 e 8 | 7.69 e 9 | 9.86 e 11 |
| $\pm3$ | 4.02 e 8 | 6.22 e 8 | 3.45 e 9 | 8.91 e 10 | 1.05 e 13 |

These estimates can be compared with the largest exceptions found in each class modulo 63 given in Table 15.

These results show at least that our probabilistic model fits very well with our computation. However we remark that the last exception found in the class 47 modulo 63 (2 mod 9, -2 mod 7) is greater than the corresponding value given in Table 14. As a matter of fact it is in this class that we observe the largest ratio between two consecutive exceptions.

TABLE 15. The last found exceptions in the residue classes modulo 63

| mod 9 / mod 7 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 587 286 | 1 410 346 | 23 375 702 | 508 517 814 | 96 127 145 590 |
| 1 | 40 748 310 | 15 607 054 | 108 609 194 | 5 134 614 906 | 788 129 237 722 |
| 2 | 7 712 154 | 14 621 266 | 118 905 194 | 5 192 800 356 | 647 984 206 102 |
| 3 | 396 953 532 | 252 716 950 | 1 761 425 102 | 41 918 435 499 | 5 284 099 948 018 |
| −3 | 188 204 580 | 198 367 831 | 1 155 472 427 | 44 322 060 990 | 6 496 802 093 380 |
| −2 | 11 919 591 | 11 066 914 | 326 262 620 | 3 403 279 794 | 515 338 220 164 |
| −1 | 19 913 382 | 43 921 828 | 193 830 356 | 3 806 305 950 | 907 751 255 494 |
| mod 9 / mod 7 | | −1 | −2 | −3 | −4 |
| 0 | | 1 294 370 | 27 750 562 | 694 539 132 | 83 593 932 170 |
| 1 | | 32 898 230 | 127 378 987 | 6 125 088 390 | 936 140 172 206 |
| 2 | | 23 933 051 | 185 805 790 | 3 832 335 222 | 913 105 904 972 |
| 3 | | 230 528 546 | 1 148 123 959 | 66 945 773 058 | 6 021 018 973 490 |
| −3 | | 505 945 682 | 4 045 088 338 | 75 377 772 852 | 7 373 170 279 850 |
| −2 | | 18 900 530 | 92 241 196 | 3 879 539 340 | 706 796 978 900 |
| −1 | | 19 566 665 | 133 245 286 | 5 103 923 460 | 1 427 500 392 170 |

## 6. SUMS OF FIVE CUBES

It is well-known since Linnik [9] (1943) and Watson [12] (1951) that every sufficiently large integer is a sum of 7 positive integral cubes. McCurley [3] in 1984 gave an effective version of this result but his bound is too large, exp(exp(13.97)). F. Bertaut, O. Ramaré and P. Zimmermann [1] improved this result in some particular arithmetic progressions. By combining the greedy ascent method with the previous results for sums of 4 cubes, we easily derive the following theorem.

**Theorem.** *Every integer in the interval* $[1\,290\,741, 10^{16}]$ *is a sum of five nonnegative integral cubes.*

*Proof.* We shall establish the result for each class modulo 9.

In a first step, we simply verify on a computer that every integer in the interval $[1\ 290\ 741, 8\cdot 10^{10}]$ is a $C_5$ integer. This is done in the following way. To test the interval $[a,b[$, we compute two vectors, one with all $C_2$ integers from 0 to $b$, and one with $C_3$ integers from 0 to a variable bound $M(b)$. Indeed, the average number of representations of an integer $n$ as a sum of 5 cubes is about $\gamma n^{2/3}$, so every large integer has a representation as $C_2 + C_3$, the $C_3$ integer being relatively small. We then add the two vectors until the interval is completely represented. This took about 120 hours on a DEC Alpha.

In a second step, we use special ranges of $C_4$ integers in 4 classes modulo 9 observed beyond the last apparent exception found. In the class 0, this range has been enlarged for the special need of this theorem.

We have

- every $n \in [4\cdot 10^8; 4.5\cdot 10^{11}]$ and congruent to $0 \mod 9$ is $C_4$,
- every $n \in [4.5\cdot 10^{10}; 5.3\cdot 10^{11}]$ and congruent to $3 \mod 9$ is $C_4$,
- every $n \in [7.6\cdot 10^{10}; 8.8\cdot 10^{11}]$ and congruent to $6 \mod 9$ is $C_4$,
- every $n \in [4.1\cdot 10^{9}; 4.5\cdot 10^{10}]$ and congruent to $7 \mod 9$ is $C_4$.

By adding successively to these ranges a cube $k^3$ for $1 \le k \le K$, we obtain large ranges of $C_5$ integers. The union of these successives ranges is still an interval when $K$ is not too large. This leads to the following technical lemma.

**Lemma.** *If every integer* $n \in [a,b]$ $(b-a \ge 27)$ *congruent to* $i \mod 9$ *is* $C_4$, *then*

(i) *every* $n \in [a, a + \frac{(b-a)^{3/2}}{27}]$ *congruent to* $i \mod 9$ *is* $C_5$,

(ii) *every* $n \in [a, a + \frac{(b-a)^{3/2}}{27}]$ *congruent to* $i + 1 \mod 9$ *is* $C_5$,

(iii) *every* $n \in [a, a + \frac{(b-a)^{3/2}}{27} - \frac{(b-a)}{3}]$ *congruent to* $i - 1 \mod 9$ *is* $C_5$.

The proof is elementary, and is left to the reader.

When using the previous ranges in the classes 3 and 6 modulo 9, we get

- every $n \in [4.5 \cdot 10^{10}; 1.2 \cdot 10^{16}]$ congruent to 2, 3, 4 mod 9 is $C_5$,
- every $n \in [7.6 \cdot 10^{10}; 2.6 \cdot 10^{16}]$ congruent to 5, 6, 7 mod 9 is $C_5$.

Finally, when using the previous range in the class 0 we obtain

- every $n \in [4 \cdot 10^{8}; 1.1 \cdot 10^{16}]$ congruent to 0, 1, 8 mod 9 is $C_5$.

These results combined with the results of the first step clearly establish the theorem. □

# APPENDIX
## SIMULATIONS FOR SUMS OF FOUR PSEUDO-CUBES

ABSTRACT. We present here some simulations for sequences of pseudo-cubes, i.e. for pseudo-random sequences which mimic the behaviour of sequences of cubes. Our aim is to observe experimentally the distribution of the largest number which is not a sum of four pseudo-cubes. Furthermore, we are interested in the distribution of the gap between the ultimate exception and the previous one.

### A.1. CONSTRUCTION OF PSEUDO-CUBES

We follow the Erdős-Rényi model described in [1]. For modeling sequences of cubes, we have to generate a sequence $(\xi_n)_{n\geq 1}$ of independent Bernoulli random variables such that

$$P(\xi_n = 1) = \alpha_n \quad \text{and} \quad P(\xi_n = 0) = 1 - \alpha_n,$$

where $\alpha_n = 1/(3\,n^{2/3})$. This leads to a sequence of pseudo-cubes denoted by $(\nu_l)_{l\geq 1}$. Almost surely, by the Borel-Cantelli Lemma, only a finite number of integers will not be represented as sums of four terms of pseudo-cubes $(\nu_l)$. For this sequence, we experimentally note that the likely largest number $n_0$ which is not sum of four terms of this sequence $(\nu_l)$ will be very large and requires too much time to be computed. So instead we take $\alpha_n = 2/(3\,n^{2/3})$; this choice makes the number $n_0$ significantly smaller and enables us to perform many trials. Furthermore, it does not affect the fundamental behaviour of our model. Indeed, as it can be seen in subsection 5.1, the increase in $\alpha_n$ only leads to replacing in (3) the factor $\gamma$ by $16\gamma$. The value of $n_0$ is then reduced by a factor of 10000.

The concrete realization of the sequence $(\nu_l)$ of pseudo-cubes is done as follows. For each $n$, we take a random number $x$ between 0 and 1. The value of $\xi_n$ is chosen to be 1 or 0, depending on whether $x$ lies in $[a_n, b_n] \subset [0,1]$ or not.

For constructing the interval $[a_n, b_n]$, we randomly choose a point denoted by $a_1$ in [0,1]. We then compute the value of $\alpha_n$ and set

$$[a_n, b_n] = \begin{cases} [a_1, a_1 + \alpha_n] & \text{if } (a_1 + \alpha_n) \leq 1, \\ \\ [a_1, 1] \cup [0, a_1 + \alpha_n - 1] & \text{if } (a_1 + \alpha_n) > 1. \end{cases}$$

### A.2. DISTRIBUTION OF THE FIVE LAST EXCEPTIONS

We generated 4500 sequences of pseudo-cubes in the interval $[1, N = 10^6]$. For each sequence, we computed all sums of 4 terms of this sequence and noted the five apparent largest numbers which are not represented, denoted by $n_4 < n_3 < n_2 < n_1 < n_0$. From the last observed exception $n_0$, we test all the interval $[n_0\,,\, 50 \times n_0]$ in order to morally convince ourselves that it is the likely last exception. We get the results shown in Table A-1.

We first notice that densities are not symmetric. The histograms and distribution functions seems to show that we have for $n_i$, $i = 0, \ldots 4$, $\Gamma$-distributions with two parameters; the experimental agreement is rather good but we have no theoretical reason supporting such a $\Gamma$-distribution. The figures below show in the first column the empirical distribution functions for $n_0$ and $n_1$, and in the second column the histograms compared with the density of a $\Gamma$-distribution.

TABLE A-1. Five last exceptions $n_i, (i = 0 \ldots 4)$ for sums of 4 pseudo-cubes.

| Statistics | $n_0$ | $n_1$ | $n_2$ | $n_3$ | $n_4$ |
|---|---|---|---|---|---|
| minimum | 56 | 50 | 48 | 44 | 42 |
| lower quartile | 7011 | 4926 | 4006 | 3451 | 3118 |
| median | 17222 | 12994 | 10951.5 | 9852.5 | 8979 |
| mean | 26861.9 | 20478.34 | 17587.65 | 15906.79 | 14656.75 |
| upper quartile | 35741.5 | 27328 | 23634.5 | 21159 | 19457 |
| maximum | 190752 | 181169 | 171348 | 138612 | 134196 |
| variance | 840460536 | 510788421 | 390318881 | 330451800 | 287465045 |
| standard deviation | 28990.7 | 22600.63 | 19756.49 | 18178.33 | 16954.79 |

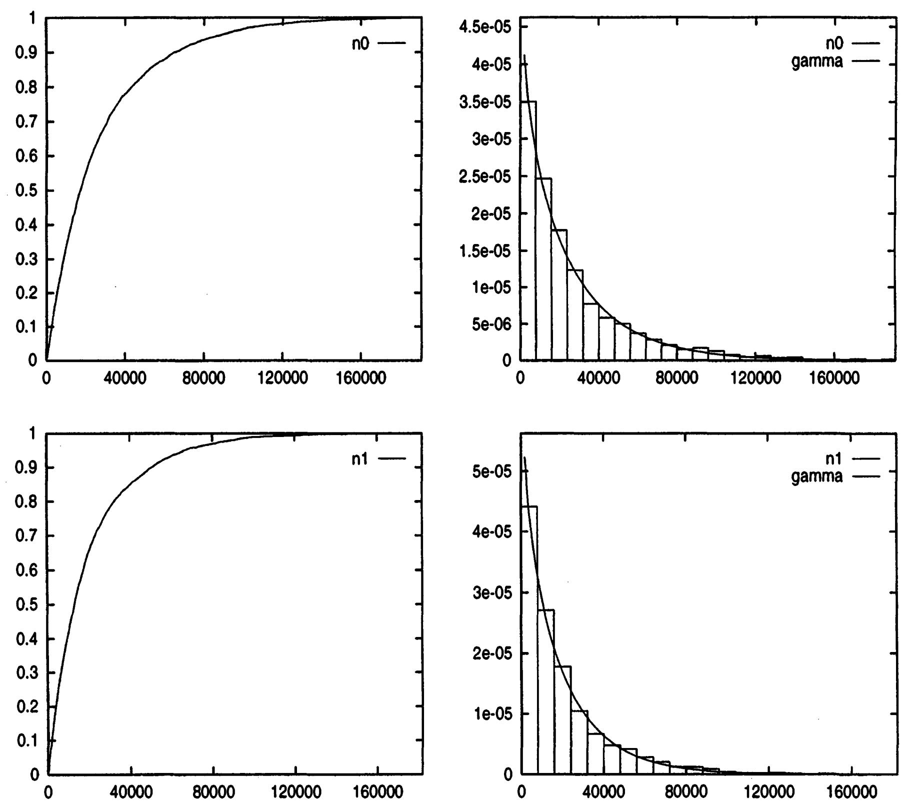


FIGURE A-1. Empirical distribution functions and histograms of $n_0$ and $n_1$ for sums of 4 pseudo-cubes.

The density of the $\Gamma$-distribution with parameters $p$ and $\theta$ is given by

$$f(x;p,\theta) = \frac{1}{\Gamma(p)\,\theta^p} x^{p-1} \exp\{-x/\theta\} \mathbb{1}_{\{x \geq 0\}}, \qquad \theta > 0. \tag{1}$$

Let $\mathbb{X} = (X_1, X_2, \ldots, X_n)^T$ be a random vector in $\mathbb{R}^n$, where $X_i (i = 1, \ldots, n)$ are independent and identically distributed with distribution function $F$ and density function $f$. We denote by $x_i$ the realizations of $X_i$ $(i = 1, \ldots, n)$. Estimates of $p$

TABLE A-2. Parameters of the gamma distribution $(p,\ \theta)$ for the five last exceptions for sums of four pseudo-cubes.

| Parameter | $n_0$ | $n_1$ | $n_2$ | $n_3$ | $n_4$ |
|---|---|---|---|---|---|
| $p$ | 0.88021 | 0.86390 | 0.82823 | 0.77149 | 0.79989 |
| $\theta$ | 28892.120 | 22500.829 | 20123.119 | 19994.318 | 18432.074 |

and $\theta$ can be obtained by finding the maximum of the likelihood function

$$L(\mathbb{X};p,\theta)=\prod_{i=1}^{n} f(x_i;p,\theta).$$

This gives the following numerical estimates.

## A.3. DISTRIBUTION OF $n_0/n_1$

Figure A-2 shows the ratio $n_i/n_0$ $(i = 1,2)$ as a plotted function of $n_0$. We clearly observe that when $n_0$ increases, the minimum of $(n_0/n_1)$ increases also and is bigger than 0.1 with extremely few exceptions, which means that the last exception is almost always smaller than ten times the previous one.

We also observe that the distribution of $Z = (n_0/n_1) - 1$ looks like a $\Gamma$-distribution with two parameters. By (1), the density of $X = n_0/n_1$ is then

$$f(x;p,\theta)=\frac{1}{\Gamma(p)\,\theta^p}(x-1)^{p-1}\exp\{-(x-1)/\theta\},\quad x\geq 1,\ \ \theta>0.$$

In Figure A-3 we have plotted the histograms compared with the density of a $\Gamma$-distribution.

From this, we have computed in Table A-3 under the hypothesis of a $\Gamma$-distribution for $Z$ with estimated parameters $p$ and $\theta$, the probability that $X = n_0/n_1 > k$ for $k = 5, \ldots, 10$.

More simulations have been performed around this subject and can be seen in [2]. From this, it appears that a large multiplicative gap between the two last exceptions for sums of four cubes seems to be very unlikely. Therefore this appendix supports,

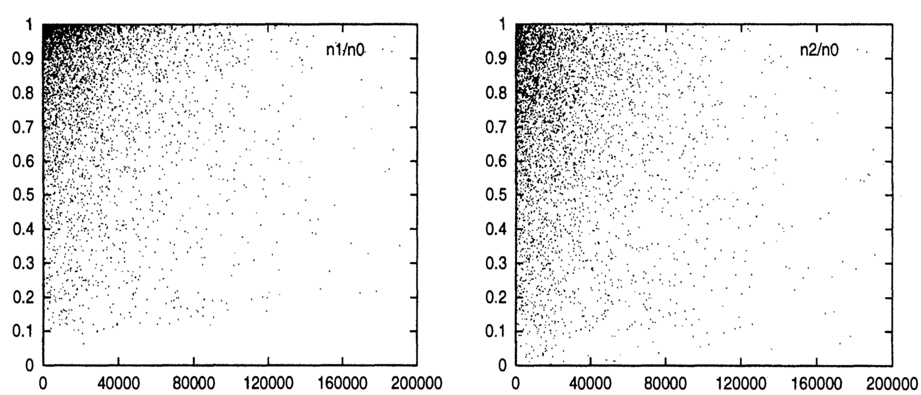


FIGURE A-2. Ratio $n_i/n_0$ $(i = 1,2)$ as a function of $n_0$ for sums of four pseudo-cubes.

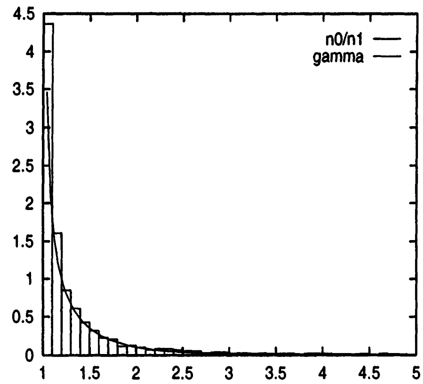


FIGURE A-3. Density of $n_0/n_1$ observed and estimated for sums of four pseudo-cubes.

TABLE A-3. Probability $\left\{X = \frac{n_0}{n_1} \geq k\right\}$ for $k = 5, \dots, 10$.

| $X = \frac{n_0}{n_1}$ | gamma $(p, \theta)$ $p = 0.37510$ $\theta = 0.65032$ |
|---|---|
| $\widehat{\mathrm{P}}(X \geq 5)$ | $2.6531 \cdot 10^{-4}$ |
| $\widehat{\mathrm{P}}(X \geq 6)$ | $5.0347 \cdot 10^{-5}$ |
| $\widehat{\mathrm{P}}(X \geq 7)$ | $9.7559 \cdot 10^{-6}$ |
| $\widehat{\mathrm{P}}(X \geq 8)$ | $1.9187 \cdot 10^{-6}$ |
| $\widehat{\mathrm{P}}(X \geq 9)$ | $3.8155 \cdot 10^{-7}$ |
| $\widehat{\mathrm{P}}(X \geq 10)$ | $7.6530 \cdot 10^{-8}$ |

to some extent, the choice of the factor 10 in the method used in the body of this paper for determining the likely largest number which is not a sum of four cubes.

## REFERENCES TO APPENDIX

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