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Supplement to
ALMOST PERIODIC FACTORIZATION OF CERTAIN BLOCK TRIANGULAR MATRIX FUNCTIONS

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5. Appendices

The following appendices provide explicit AP_W factorizations (1.2) for particular cases of the matrix (1.4), (1.5) mentioned in the body of the paper. These factorizations (as well as matrices $X_\pm$, G' in the proof of Theorem 4.1) were obtained using the symbolic computation program Maple; the code used to produce them is available from the authors upon request.

To reduce the number of cases involved, we make a repeated use of the factorizations (2.8) and

$$G_{f^T} = (-J_1 G_+^{-T})\Lambda^{-1}(G_-^{-T} J_1), \tag{5.1}$$

obtainable from (1.2) via taking transposes and inverses. Here $J_1 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, T is the transposition sign, and the abbreviation X^{-T} stands for $(X^{-1})^T$. We also observe that since

$$G_{RfS} = \operatorname{diag}[S^{-1}, R]\, G_f \, \operatorname{diag}[S, R^{-1}],$$

the factorization properties of G are not changed under the transformation $f \mapsto RfS$ where R, S are invertible constant $m \times m$ matrices. This includes, in particular, elementary row and column operations applied to $c_{\pm 1}$ and c_0 simultaneously.

Appendix A.

$$c_{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad c_0 = \begin{bmatrix} z & u & b \\ 0 & z & 0 \\ 0 & a & 0 \end{bmatrix}, \quad c_1 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and at least one of the entries a, b differs from zero.

Case 1: $ab \neq 0$. The AP_W factorization of G is given by

$$G_+ = \begin{bmatrix} -1 & 0 & e_\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{a} & e_{\lambda+\nu} & -e_\lambda \\ \frac{z}{b} & e_\lambda & -\frac{ze_\nu+1}{b} & -\frac{e_\alpha}{ab} & 0 & 0 \\ 0 & b & 0 & 0 & ue_\nu - e_\lambda & -u + e_\alpha \\ 0 & 0 & 0 & 0 & 1 + ze_\nu & -z \\ 0 & 0 & 0 & 0 & ae_\nu & -a \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -\frac{u}{a} \\ \frac{e_{-\nu}+z}{b} & 0 & 1 & \frac{e_{-\lambda}}{b} & 0 & \frac{e_{-\nu}-ue_{-\lambda}}{ab} \\ 1 & 0 & 0 & e_{-\alpha} & 0 & -\frac{ue_{-\alpha}}{a} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & e_{-\alpha} & -\frac{ze_{-\alpha}}{a} \\ 0 & 0 & 0 & 0 & 1 & -\frac{z+e_{-\nu}}{a} \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[e_\nu, 1, e_\alpha, 1, e_{-\nu}, e_{-\alpha}].$$

Case 2: Exactly one of the entries a, b differs from zero.

Using (5.1) and switching the first and the second row and column of f if necessary, we may suppose without loss of generality that $a = 0$, $b \neq 0$. This case splits into two subcases.

Case 2a: $z \neq 0$.

If, in addition, $\nu > \alpha$, then

$$G_+ = \begin{bmatrix} -1 & 0 & 0 & e_\nu & 0 & 0 \\ 0 & -\frac{1}{z} & \frac{b}{z} & 0 & \frac{ze_\lambda - e_\alpha}{z} & 0 \\ \frac{z}{b} & \frac{u-e_\alpha}{bz} & \frac{ze_\lambda - u + e_\alpha}{z} & -\frac{1+ze_\nu}{b} & \frac{ue_\alpha - uze_\lambda - e_{2\alpha}}{bz} & 0 \\ 0 & 0 & b & 0 & -e_\alpha & 0 \\ 0 & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ z+e_{-\nu} & u & b & e_{-\lambda} & 1 & 0 \\ \frac{z+e_{-\nu}}{b} & \frac{uz+e_{\alpha-\nu}}{bz} & 1 & \frac{e_{-\lambda}}{b} & \frac{e_{-\nu}}{bz} & 0 \\ 1 & 0 & 0 & e_{-\alpha} & 0 & 0 \\ 0 & \frac{z+e_{-\nu}}{z} & 0 & 0 & \frac{e_{-\lambda}}{z} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[e_\nu, 1, 1, e_\alpha, 1, e_{-\lambda}].$$

When $\nu < \alpha$, G can be represented as $G = X_+G'X_-$, where

$$X_+ = \begin{bmatrix} -1 & 0 & 0 & e_\nu & 0 & 0 \\ 0 & -\frac{1}{z} & -\frac{1}{z} & 0 & be_\lambda & 0 \\ \frac{z}{b} & \frac{u-e_\alpha}{bz} & \frac{u-ze_\lambda-e_\alpha}{bz} & -\frac{1+ze_\nu}{b} & e_{\lambda+\alpha} - ue_\lambda & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & bz & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is invertible in AP_W^+,

$$X_- = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ z+e_{-\nu} & u & b & e_{-\lambda} & 1 & 0 \\ -e_{-\lambda} - ze_{-\alpha} & 1 - ue_{-\alpha} & -be_{-\alpha} & -e_{-(\lambda+\alpha)} & 0 & 0 \\ 1 & 0 & 0 & e_{-\alpha} & 0 & 0 \\ \frac{(z+e_{-\nu})^2}{bz} & \frac{u(z+e_{-\nu})}{bz} & \frac{z+e_{-\nu}}{z} & \frac{e_{-(\lambda+\nu)}+ze_{-\lambda}}{bz} & \frac{e_{-\nu}}{bz} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is invertible in AP_W^-, and

$$G' = \begin{bmatrix} e_\nu & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & e_\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & e_\alpha & 0 & 0 \\ 0 & 0 & \frac{z+e_{-\nu}}{bz} & 0 & e_{-\alpha} & 0 \\ 0 & 0 & 0 & 0 & 0 & e_{-\lambda} \end{bmatrix}$$

splits into a direct sum of $\mathrm{diag}[e_\nu, 1, e_\alpha, e_{-\lambda}]$ with a 2×2 matrix $\begin{bmatrix} e_\alpha & 0 \\ \frac{z+e_{-\nu}}{bz} & e_{-\alpha} \end{bmatrix}$ that is AP_W factorable (with zero partial AP indices) due to Theorem 1.2. This means that G' (and therefore G itself) is also AP_W factorable.

Case 2b: $z = 0$. The AP_W factorization of G is given by

$$G_+ = \begin{bmatrix} -1 & 0 & 0 & e_\nu & 0 & 0 \\ 0 & -1 & 0 & 0 & e_{\lambda+\nu} & 0 \\ 0 & \frac{u-e_\alpha}{b} & e_\lambda & -\frac{1}{b} & 0 & 0 \\ 0 & 0 & b & 0 & ue_\nu - e_\lambda & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{e_{-\nu}}{b} & 0 & 1 & \frac{e_{-\lambda}}{b} & \frac{1-ue_{-\alpha}}{b} & 0 \\ 1 & 0 & 0 & e_{-\alpha} & 0 & 0 \\ 0 & 1 & 0 & 0 & e_{-\alpha} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[e_\nu, e_\nu, 1, e_\alpha, e_{-\nu}, e_{-\lambda}].$$

Appendix B.

$$c_0^{(1)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad c_{\pm 1}^{(1)} = \begin{bmatrix} x_\pm & y_\pm \\ 0 & z_\pm \end{bmatrix},$$

where $x_+z_+ = 0$, $x_-z_- = 0$, but at least one of the entries $x_\pm$, $z_\pm$ differs from zero.

Under these conditions, the number of non-zero entries among $x_\pm$, $z_\pm$ is either two or one. In the former situation, the possibilities are as follows:

(i) $x_+ = x_- = 0$, $z_\pm \neq 0$,

(ii) $z_+ = z_- = 0$, $x_\pm \neq 0$,

(iii) $x_+ = z_- = 0$, $x_-z_+ \neq 0$,

(iv) $z_+ = x_- = 0$, $x_+z_- \neq 0$.

With the use of (5.1), (ii) can be reduced to (i) and (iii) to (iv).

If exactly one of the coefficients $x_\pm$, $z_\pm$ differs from zero, the symmetries (2.8) and (5.1) allow us to suppose without loss of generality that $x_- \neq 0$, $x_+ = z_\pm = 0$. This leaves us with the following three cases.

Case 1: $x_+ = x_- = 0$, $z_\pm \neq 0$.

Subcase 1a: $A = y_-z_+ - y_+z_- \neq 0$. The AP_W factorization of G is given by

$$G_+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e_{\lambda+\nu} - \frac{z_-e_\nu}{z_+} & -e_\nu & -\frac{e_\alpha}{z_-} + \frac{z_+e_\lambda - z_-}{z_-A} \\ 0 & y_+e_\lambda + e_\nu + y_- - \frac{z_-y_+}{z_+} & -y_+ & -\frac{y_+e_{\alpha-\nu}}{z_-} + \frac{z_+ + z_+y_+e_\alpha}{z_-A} \\ 0 & z_+e_\lambda & -z_+ & -\frac{z_+e_{\alpha-\nu}}{z_-} + \frac{z_+^2 e_\alpha}{z_-A} \end{bmatrix},$$

$$G_- = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{z_+e_{-\alpha}}{A} & -\frac{z_+e_{\nu-\alpha}+y_+z_-e_{-\alpha}}{z_-A} \\ 0 & -\frac{z_-}{z_+} & 1 & -\frac{y_-z_+ + z_-e_{-\alpha}}{z_-z_+} \\ 0 & 0 & -z_-e_{-\nu} & 1 + y_-e_{-\nu} \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[e_\lambda, e_{-\nu}, e_{-\nu}, e_{\nu-\alpha}]$$

if $\nu < \alpha$, and by

$$G_+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (z_- - y_-)e_\nu + e_{2\nu} - \frac{z_- e_{\nu-\alpha}}{z_+} & \frac{z_- - z_+ e_\lambda}{A} & -\frac{e_\lambda}{z_-} \\ 0 & y_+ e_\nu - y_- y_+ + e_{\nu-\alpha} + \frac{z_- y_+{}^2}{z_+} & -\frac{z_+(1+y_+ e_\alpha)}{A} & -\frac{1+y_+ e_\alpha}{z_-} \\ 0 & z_+ e_\nu - A & -\frac{z_+^2 e_\alpha}{A} & -\frac{z_+ e_\alpha}{z_-} \end{bmatrix},$$

$$G_- = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \frac{z_- e_{-\alpha}}{A} & \frac{z_+ e_{-\alpha}}{A} & -\frac{y_+ e_{-\alpha} + e_{-2\alpha}}{A} \\ 0 & -\frac{z_-}{z_+} & 1 & -\frac{y_- z_+ + z_- e_{-\alpha}}{z_- z_+} \\ 0 & -\frac{A z_- e_{\alpha-\nu}}{z_+} & -z_- e_{-\nu} & 1 + \frac{z_- y_+ e_{-\nu}}{z_+} \end{bmatrix}$$

(with the same Λ) if $\nu > \alpha$.

Subcase 1b: $y_- z_+ - y_+ z_- = 0$. Using elementary row transformations, we may without loss of generality suppose that $y_+ = y_- = 0$. Then

$$G_+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e_\lambda - \frac{z_-}{z_+} & -\frac{e_\lambda}{z_-} & -e_\nu \\ 0 & 1 & -\frac{1}{z_-} & 0 \\ 0 & z_+ e_\alpha & -\frac{z_+ e_\alpha}{z_-} & -z_+ \end{bmatrix}, \quad G_- = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{e_{-\alpha}}{z_-} \\ 0 & 0 & -z_- e_{-\nu} & 1 \\ 0 & -\frac{z_-}{z_+} & 1 & -\frac{e_{-\alpha}}{z_+} \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[e_\lambda, 1, e_{-\alpha}, e_{-\nu}].$$

Case 2: $x_- = z_+ = 0$, $x_+ z_- \neq 0$.
The AP_W factorization of G is given by

$$G_+ = \begin{bmatrix} e_\nu & -\frac{y_+ e_{\lambda+\nu} + y_- e_\nu + e_{2\nu}}{x_+} & \frac{y_+}{x_+} & \frac{1+y_+ e_\alpha}{x_+ z_-} \\ 0 & e_{\lambda+\nu} & -1 & -\frac{e_\alpha}{z_-} \\ x_+ & 0 & 0 & 0 \\ 0 & z_- & 0 & 0 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 1 & \frac{y_- e_{-\lambda} + e_{-\alpha} + y_+}{x_+} & \frac{e_{-(\lambda+\alpha)}}{x_+} & 0 \\ 0 & 1 & 0 & \frac{e_{-\alpha}}{z_-} \\ 0 & 0 & 1 & -\frac{y_-}{z_-} \\ 0 & 0 & -z_- e_{-\nu} & 1 + y_- e_{-\nu} \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[e_\alpha, e_{-\nu}, 1, e_{\nu-\alpha}].$$

Case 3: $x_- \neq 0$, $x_+ = z_+ = z_- = 0$.
Then G can be represented as $G = X_+ G' X_-$ where

$$X_+ = \begin{bmatrix} e_{\lambda+\nu} & 0 & -\frac{1}{x_-} & -\frac{y_- + y_+ e_\lambda + e_\nu}{x_-} \\ 0 & 0 & 0 & 1 \\ x_- & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

is invertible in AP_W^+,

$$X_- = \begin{bmatrix} 1 & \frac{y_-}{x_-} & \frac{e_{-\alpha}}{x_-} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

is invertible in AP_W^-, and

$$G' = \begin{bmatrix} e_{-\nu} & 0 & 0 & \frac{y_+ e_\alpha + 1}{x_-} \\ 0 & e_{-\lambda} & 0 & 0 \\ 0 & 0 & e_\nu & 0 \\ 0 & 0 & 0 & e_\lambda \end{bmatrix}.$$

After a suitable permutation of rows and columns, the latter matrix splits into a direct sum of $\mathrm{diag}[e_{-\lambda}, e_\nu]$ and

$$G_1 = e_{\alpha/2} \begin{bmatrix} e_{\lambda-\alpha/2} & 0 \\ \frac{y_+ e_{\alpha/2} + e_{-\alpha/2}}{x_-} & e_{-\lambda+\alpha/2} \end{bmatrix}.$$

The matrix G_1 is AP_W factorable due to Theorem 1.2. This implies that G' and the original matrix G are both AP_W factorable.

Of course if $y_+ = 0$, then the matrix G itself satisfies the conditions of Theorem 1.2. Its explicit AP_W factorization in this case is given by

$$G_+ = \begin{bmatrix} y_- + e_\nu & 0 & -\frac{1}{x_-} & -\frac{y_- e_\lambda}{x_-} \\ -x_- & 0 & 0 & e_\lambda \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad G_- = \begin{bmatrix} 1 & \frac{y_-}{x_-} & \frac{e_{-\alpha}}{x_-} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ x_- e_{-\nu} & 1 + y_- e_{-\nu} & e_{-\lambda} & 0 \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[e_\alpha, e_{-\lambda}, e_\nu, 1].$$

Appendix C.

$$c_{\pm 1} = \begin{bmatrix} a_\pm & b_\pm & d_\pm \\ f_\pm & 0 & l_\pm \\ g_\pm & 0 & 0 \end{bmatrix}, \quad c_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where at least on of the entries $b_\pm$, $g_\pm$ differs from zero and (4.3), (4.5) hold.

We sort all the possible situations by the number of non-zero entries among $b_\pm$, $g_\pm$.

Situation 1: $b_\pm$, $g_\pm$ are all non-zero. Then, from (4.5), $l_\pm = 0$. Applying elementary row operations, we may suppose also that $a_- = d_- = f_- = 0$ (this, of course, changes a_+, f_+ and d_+, so that equalities (4.3) do not have to hold any more).

If the (new) values of d_+ and f_+ happen to be zero, the matrix G splits into a direct sum of $\mathrm{diag}[e_\lambda, e_{-\lambda}]$ and a matrix satisfying Corollary 2.5. It remains to consider the following cases.

Case 1: $d_+ f_+ \neq 0$. The AP_W factorization of G is then delivered by

$$G_+ = \begin{bmatrix} 0 & e_{\lambda+\nu} & 0 & -\frac{1}{f_+} & -\frac{e_\nu}{g_-} & 0 \\ e_\lambda & 0 & 0 & \frac{e_\nu}{b_- f_+} & \frac{e_{2\nu}}{b_- g_-} & -\frac{1}{b_-} \\ -\frac{b_+ e_\lambda + b_-}{d_+} & -\frac{a_+ e_{\lambda+\nu} + e_{2\nu}}{d_+} & e_\nu & \frac{a_+ b_- - b_+ e_\nu}{b_- d_+ f_+} & \frac{a_+ b_- e_\nu - b_+ e_{2\nu}}{b_- d_+ g_-} & \frac{b_+}{b_- d_+} \\ 0 & 0 & d_+ & 0 & 0 & 0 \\ 0 & f_+ e_\lambda & 0 & 0 & -\frac{f_+}{g_-} & 0 \\ 0 & g_+ e_\lambda + g_- & 0 & 0 & -\frac{g_+}{g_-} & 0 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 1 & 0 & \frac{e_{-\alpha}}{b_-} & \frac{g_+ e_{\nu-\alpha}}{b_- f_+ g_-} & -\frac{e_{\nu-\alpha}}{b_- g_-} \\ 1 & 0 & 0 & 0 & -\frac{g_+ e_{-\alpha}}{f_+ g_-} & \frac{e_{-\alpha}}{g_-} \\ \frac{e_{-\alpha} + a_+}{d_+} & \frac{b_+ + b_- e_{-\lambda}}{d_+} & 1 & \frac{e_{-(\lambda+\alpha)}}{d_+} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{g_+ + g_- e_{-\lambda}}{f_+} & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[1, e_{-\nu}, e_\alpha, e_{-\alpha}, 1, e_\nu]$$

if $\nu < \alpha$, and

$$G_+ = \begin{bmatrix} 0 & e_{\lambda+\nu} & 0 & -\frac{1}{f_+} & -\frac{e_\nu}{g_-} & 0 \\ e_\lambda & 0 & 0 & \frac{e_\nu}{b_- f_+} & 0 & -\frac{1}{b_-} \\ -\frac{b_- + b_+ e_\lambda}{d_+} & -\frac{a_+ e_{\lambda+\nu} + e_{2\nu}}{d_+} & e_\nu & \frac{a_+ b_- - b_+ e_\nu}{b_- d_+ f_+} & \frac{a_+ e_\nu + e_{\nu-\alpha}}{d_+ g_-} & \frac{b_+}{b_- d_+} \\ 0 & 0 & d_+ & 0 & 0 & 0 \\ 0 & f_+ e_\lambda & 0 & 0 & -\frac{f_+}{g_-} & 0 \\ 0 & g_+ e_\lambda + g_- & 0 & 0 & -\frac{g_+}{g_-} & 0 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 1 & 0 & \frac{e_{-\alpha}}{b_-} & -\frac{e_{-2\alpha}}{b_- f_+} & 0 \\ 1 & 0 & 0 & 0 & -\frac{g_+ e_{-\alpha}}{f_+ g_-} & \frac{e_{-\alpha}}{g_-} \\ \frac{e_{-\alpha} + a_+}{d_+} & \frac{b_+ + b_- e_{-\lambda}}{d_+} & 1 & \frac{e_{-(\lambda+\alpha)}}{d_+} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{g_+ + g_- e_{-\lambda}}{f_+} & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(with the same Λ) if $\nu > \alpha$.

Case 2: Exactly one of d_+, f_+ differs from zero. Using (5.1), we may without loss of generality suppose that $d_+ = 0$, $f_+ \neq 0$. The AP_W factorization of G is then delivered by

$$G_+ = \begin{bmatrix} e_{\lambda+\nu} & 0 & 0 & -\frac{e_\nu}{g_-} & 0 & -\frac{1}{f_+} \\ -\frac{e_{2\nu}}{b_+} & \frac{b_+ e_\lambda - b_-}{b_+} & 0 & \frac{e_{2\nu}}{b_- g_-} & -\frac{e_\nu}{b_-} & \frac{a_+}{b_+ f_+} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ a_+ e_\lambda & b_+ e_\alpha & 0 & \frac{b_+ e_\nu - a_+ b_-}{b_- g_-} & -\frac{b_+}{b_-} & 0 \\ f_+ e_\lambda & 0 & 0 & -\frac{f_+}{g_-} & 0 & 0 \\ g_+ e_\lambda + g_- & 0 & 0 & -\frac{g_+}{g_-} & 0 & 0 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{g_+ e_{-\alpha}}{f_+ g_-} & \frac{e_{-\alpha}}{g_-} \\ 0 & 1 & 0 & \frac{e_{-\alpha}}{b_-} & \frac{g_+ e_{\nu-\alpha}}{b_- f_+ g_-} & -\frac{e_{\nu-\alpha}}{b_- g_-} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{g_- e_{-\lambda} + g_+}{f_+} & 1 \\ -\frac{b_-}{b_+} & -\frac{b_-^2 e_{-\nu}}{b_+} & 0 & \frac{b_+ - b_- e_{-\lambda}}{b_+} & \frac{a_+ b_- e_{-\lambda} - b_+ e_{-\alpha}}{b_+ f_+} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[e_{-\nu}, 1, e_\lambda, 1, 1, e_{-\alpha}]$$

if $\nu < \alpha$, and

$$G_+ = \begin{bmatrix} e_{\lambda+\nu} & 0 & 0 & -\frac{e_\nu}{g_-} & 0 & -\frac{1}{f_+} \\ -\frac{e_{2\nu}}{b_+} & \frac{b_+ e_\lambda - b_-}{b_+} & 0 & \frac{e_{\nu-\alpha}}{b_+ g_-} & -\frac{e_\nu}{b_-} & \frac{a_+}{b_+ f_+} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ a_+ e_\lambda & b_+ e_\alpha & 0 & -\frac{a_+}{g_-} & -\frac{b_+}{b_-} & 0 \\ f_+ e_\lambda & 0 & 0 & -\frac{f_+}{g_-} & 0 & 0 \\ g_+ e_\lambda + g_- & 0 & 0 & -\frac{g_+}{g_-} & 0 & 0 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{g_+ e_{-\alpha}}{f_+ g_-} & \frac{e_{-\alpha}}{g_-} \\ 0 & 1 & 0 & \frac{e_{-\alpha}}{b_-} & -\frac{e_{-2\alpha}}{b_- f_+} & \frac{e_{-\alpha}}{g_-} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{g_- e_{-\lambda} + g_+}{f_+} & 1 \\ -\frac{b_-}{b_+} & -\frac{b_-^2 e_{-\nu}}{b_+} & 0 & \frac{b_+ - b_- e_{-\lambda}}{b_+} & \frac{a_+ b_- e_{-\lambda} - b_+ e_{-\alpha}}{b_+ f_+} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(with the same Λ) if $\nu > \alpha$.

Situation 2: Exactly one of the numbers $b_\pm$, $g_\pm$ is zero.

Using the symmetries (2.8) and (5.1), we may suppose that $b_- = 0$, $b_+, g_\pm \neq 0$. From (4.5), (4.3) it follows that $l_\pm = 0$. Applying elementary row and column operations, we may then ensure that $f_- = a_\pm = d_+ = 0$; of course, the latter transformations change the value of f_+. If the (new) f_+ and d_- both equal zero, then (as in Situation 1) G splits into a direct sum of a diagonal matrix with a matrix AP_W factorable due to Theorem 1.2. The remaining three possibilities are covered by the following cases.

Case 3: $d_- f_+ \neq 0$. The AP_W factorization of G is given by

$$G_+ = \begin{bmatrix} 0 & -\frac{e_\nu}{g_-} & 0 & -\frac{1}{f_+} & e_{\lambda+\nu} & 0 \\ 0 & 0 & e_\nu & 0 & -\frac{e_{2\nu}}{b_+} & -\frac{d_-}{b_+} \\ -\frac{1}{d_-} & \frac{e_{2\nu}}{d_- g_-} & 0 & \frac{e_\nu}{d_- f_+} & 0 & e_\lambda \\ 0 & 0 & b_+ & 0 & 0 & 0 \\ 0 & -\frac{f_+}{g_-} & 0 & 0 & f_+ e_\lambda & 0 \\ 0 & -\frac{g_+}{g_-} & 0 & 0 & g_+ e_\lambda + g_- & 0 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{g_+ + g_- e_{-\lambda}}{f_+} & 1 \\ \frac{e_{-\alpha}}{b_+} & 1 & \frac{d_- e_{-\lambda}}{b_+} & \frac{e_{-(\lambda+\alpha)}}{b_+} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -\frac{g_+ e_{-\alpha}}{f_+ g_-} & \frac{e_{-\alpha}}{g_-} \\ 0 & 0 & 1 & \frac{e_{-\alpha}}{d_-} & \frac{g_+ e_{\nu-\alpha}}{d_- f_+ g_-} & -\frac{e_{\nu-\alpha}}{d_- g_-} \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[e_\nu, 1, e_\alpha, e_{-\alpha}, e_{-\nu}, 1]$$

if $\nu < \alpha$, and

$$G_+ = \begin{bmatrix} 0 & -\frac{e_\nu}{g_-} & 0 & -\frac{1}{f_+} & e_{\lambda+\nu} & 0 \\ 0 & \frac{e_{\nu-\alpha}}{b_+ g_-} & e_\nu & 0 & -\frac{e_{2\nu}}{b_+} & -\frac{d_-}{b_+} \\ -\frac{1}{d_-} & 0 & 0 & \frac{e_\nu}{d_- f_+} & 0 & e_\lambda \\ 0 & 0 & b_+ & 0 & 0 & 0 \\ 0 & -\frac{f_+}{g_-} & 0 & 0 & f_+ e_\lambda & 0 \\ 0 & -\frac{g_+}{g_-} & 0 & 0 & g_+ e_\lambda + g_- & 0 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{g_+ + g_- e_{-\lambda}}{f_+} & 1 \\ \frac{e_{-\alpha}}{b_+} & 1 & \frac{d_- e_{-\lambda}}{b_+} & \frac{e_{-(\lambda+\alpha)}}{b_+} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -\frac{g_+ e_{-\alpha}}{f_+ g_-} & \frac{e_{-\alpha}}{g_-} \\ 0 & 0 & 1 & \frac{e_{-\alpha}}{d_-} & -\frac{e_{-2\alpha}}{d_- f_+} & 0 \end{bmatrix}$$

(with the same Λ) if $\nu > \alpha$.

Case 4: $d_- = 0$, $f_+ \neq 0$. The AP_W factorization of G is delivered by

$$G_+ = \begin{bmatrix} e_{\lambda+\nu} & 0 & 0 & -\frac{1}{f_+} & -e_\nu & -\frac{e_\alpha}{g_-} \\ 0 & e_\nu & 0 & 0 & 0 & \frac{1}{b_+ g_-} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ e_\nu & b_+ & 0 & 0 & 0 & 0 \\ f_+ e_\lambda & 0 & 0 & 0 & -f_+ & -\frac{f_+ e_{\alpha-\nu}}{g_-} \\ g_+ e_\lambda + g_- & 0 & 0 & 0 & -g_+ & -\frac{g_+ e_{\alpha-\nu}}{g_-} \end{bmatrix},$$

$$G_- = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{g_+e_{-\alpha}}{f_+g_-} & \frac{e_{-\alpha}}{g_-} \\ 0 & 1 & 0 & \frac{e_{-(\lambda+\alpha)}}{b_+} & \frac{g_+e_{-2\alpha}}{b_+f_+g_-} & -\frac{e_{-2\alpha}}{b_+g_-} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{e_{-\alpha}}{f_+} & 0 \\ 0 & 0 & 0 & -g_-e_{-\nu} & -\frac{g_+}{f_+} & 1 \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[e_{-\nu}, e_\alpha, e_\lambda, e_{-\alpha}, 1, e_{-\alpha}]$$

if $\nu < \alpha$, and

$$G_+ = \begin{bmatrix} e_{\lambda+\nu} & 0 & 0 & -\frac{1}{f_+} & -\frac{e_\nu}{g_-} & 0 \\ 0 & e_\nu & 0 & 0 & \frac{e_{\nu-\alpha}}{b_+g_-} & -\frac{1}{b_+} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ e_\nu & b_+ & 0 & 0 & 0 & 0 \\ f_+e_\lambda & 0 & 0 & 0 & -\frac{f_+}{g_-} & 0 \\ g_+e_\lambda + g_- & 0 & 0 & 0 & -\frac{g_+}{g_-} & 0 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{g_+e_{-\alpha}}{f_+g_-} & \frac{e_{-\alpha}}{g_-} \\ 0 & 1 & 0 & \frac{e_{-(\lambda+\alpha)}}{b_+} & \frac{g_+e_{-2\alpha}}{b_+f_+g_-} & -\frac{e_{-2\alpha}}{b_+g_-} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{g_++g_-e_{-\lambda}}{f_+} & 1 \\ 0 & 0 & 0 & 1 & -\frac{e_{-\alpha}}{f_+} & 0 \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[e_{-\nu}, e_\alpha, e_\lambda, e_{-\alpha}, 1, e_{-\alpha}]$$

if $\nu > \alpha$.

Case 5: $d_- \neq 0$, $f_+ = 0$. The AP_W factorization of G is given by

$$G_+ = \begin{bmatrix} 0 & 0 & 0 & e_\lambda - \frac{g_-}{g_+} & 0 & -\frac{e_\nu}{g_-} \\ 0 & -\frac{d_-}{b_+} & e_\nu & -\frac{e_\nu}{b_+} & 0 & 0 \\ -\frac{1}{d_-} & e_\lambda & 0 & \frac{g_-e_\nu}{d_-g_+} & 0 & \frac{e_{2\nu}}{d_-g_-} \\ 0 & 0 & b_+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g_+e_\alpha & 0 & -\frac{g_+}{g_-} \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{e_{-\alpha}}{d_-} & 0 & -\frac{e_{\nu-\alpha}}{d_-g_-} \\ \frac{e_{-\alpha}}{b_+} & 1 & \frac{d_-e_{-\lambda}}{b_+} & \frac{e_{-(\lambda+\alpha)}}{b_+} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \frac{e_{-\alpha}}{g_-} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{g_-^2e_{-\nu}}{g_+} & 0 & 0 & 0 & 0 & 1-\frac{g_-e_{-\lambda}}{g_+} \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[e_\nu, 1, e_\alpha, 1, e_{-\lambda}, 1]$$

if $\nu < \alpha$, and

$$G_+ = \begin{bmatrix} 0 & 0 & 0 & e_\lambda - \frac{g_-}{g_+} & 0 & -\frac{e_\nu}{g_-} \\ 0 & -\frac{d_-}{b_+} & e_\nu & -\frac{e_\nu}{b_+} & 0 & \frac{e_{\nu-\alpha}}{b_+g_-} \\ -\frac{1}{d_-} & e_\lambda & 0 & \frac{g_-e_\nu}{d_-g_+} & 0 & 0 \\ 0 & 0 & b_+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g_+e_\alpha & 0 & -\frac{g_+}{g_-} \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{g_-e_{-\alpha}}{d_-g_+} & 0 & 1 & \frac{e_{-\alpha}}{d_-} & 0 & -\frac{e_{-2\alpha}}{d_-g_+} \\ \frac{e_{-\alpha}}{b_+} & 1 & \frac{d_-e_{-\lambda}}{b_+} & \frac{e_{-(\lambda+\alpha)}}{b_+} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \frac{e_{-\alpha}}{g_-} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{g_-^2e_{-\nu}}{g_+} & 0 & 0 & 0 & 0 & 1-\frac{g_-e_{-\lambda}}{g_+} \end{bmatrix}$$

(with the same Λ) if $\nu > \alpha$.

We now pass to the situations in which exactly two of the coefficients $b_\pm$, $g_\pm$ differ from zero. There are six such situations overall, but due to the symmetries (2.8) and (5.1), half of them can be dropped. The remaining three possibilities are considered in Situations 3-5 below.

Situation 3: $b_+ = g_+ = 0$, $b_-g_- \neq 0$.

From (4.5), (4.3) it follows that $l_\pm = 0$. Using elementary row and column operations, we may without loss of generality suppose that $a_+ = f_+ = d_+ = 0$. This does not change the values of a_-, d_-, and the further classification of possible cases is based on the zero/non-zero pattern of d_-, f_-. If both of them equal zero, the matrix G splits into a direct sum of a diagonal matrix and a matrix satisfying Corollary 2.5 (compare with a similar case in Situation 1). The remaining options are covered by Cases 6 and 7.

Case 6: $d_-f_- \neq 0$. The AP_W factorization of G is delivered by

$$G_+ = \begin{bmatrix} 0 & -\frac{1}{g_+} & e_{\lambda+\nu} & -\frac{e_\nu}{f_-} & 0 & 0 \\ 0 & 0 & 0 & 0 & e_\nu & -\frac{d_-}{b_+} \\ -\frac{1}{d_-} & \frac{e_\nu+a_-}{d_-g_+} & -\frac{a_-e_{\lambda+\nu}}{d_-} & \frac{e_{2\nu}+a_-e_\nu}{d_-f_-} & 0 & e_\lambda \\ 0 & 0 & e_\nu & 0 & b_+ & 0 \\ 0 & 0 & f_- & 0 & 0 & 0 \\ 0 & 0 & g_+e_\lambda & -\frac{g_+}{f_-} & 0 & 0 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \frac{e_{-\alpha}}{f_-} & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{f_-e_{-\lambda}}{g_+} \\ \frac{a_-e_{-\lambda}}{b_+} & 1 & \frac{d_-e_{-\lambda}}{b_+} & \frac{e_{-(\lambda+\alpha)}}{b_+} & -\frac{e_{-2\alpha}}{b_+f_-} & 0 \\ \frac{a_-}{d_-} & 0 & 1 & \frac{e_{-\alpha}}{d_-} & -\frac{e_{\nu-\alpha}}{d_-f_-} & 0 \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[e_\nu, e_{-\alpha}, e_{-\nu}, 1, e_\alpha, 1]$$

if $\nu < \alpha$, and by

$$G_+ = \begin{bmatrix} 0 & -\frac{1}{g_+} & e_{\lambda+\nu} & -\frac{e_\nu}{f_-} & 0 & 0 \\ 0 & 0 & 0 & \frac{e_{\nu-\alpha}}{b_+ f_-} & e_\nu & -\frac{d_-}{b_+} \\ -\frac{1}{d_-} & \frac{e_\nu + a_-}{d_- g_+} & -\frac{a_- e_{\lambda+\nu}}{d_-} & \frac{a_- e_\nu}{d_- f_-} & 0 & e_\lambda \\ 0 & 0 & e_\nu & 0 & b_+ & 0 \\ 0 & 0 & f_- & 0 & 0 & 0 \\ 0 & 0 & g_+ e_\lambda & -\frac{g_+}{f_-} & 0 & 0 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \frac{e_{-\alpha}}{f_-} & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{f_- e_{-\lambda}}{g_+} \\ \frac{a_- e_{-\lambda}}{b_+} & 1 & \frac{d_- e_{-\lambda}}{b_+} & \frac{e_{-(\lambda+\alpha)}}{b_+} & -\frac{e_{-2\alpha}}{b_+ f_-} & 0 \\ \frac{a_-}{d_-} & 0 & 1 & \frac{e_{-\alpha}}{d_-} & 0 & -\frac{e_{-2\alpha}}{d_- g_+} \end{bmatrix}$$

(with the same Λ) if $\nu > \alpha$.

Case 7: Exactly one of the entries d_-, f_- differs from zero.

Using the symmetry (5.1), we may without loss of generality suppose that $d_- = 0$, $f_- \neq 0$. Then the APw factorization of G is delivered by

$$G_+ = \begin{bmatrix} e_{\lambda+\nu} & 0 & 0 & -e_\nu & -\frac{1}{g_+} & -\frac{e_\alpha}{f_-} \\ 0 & e_\nu & 0 & 0 & 0 & \frac{1}{b_+ f_-} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ e_\nu + a_- & b_+ & 0 & 0 & 0 & 0 \\ f_- & 0 & 0 & 0 & 0 & 0 \\ g_+ e_\lambda & 0 & 0 & -g_+ & 0 & -\frac{g_+ e_{\alpha-\nu}}{f_-} \end{bmatrix},$$

$$G_- = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{e_{-\alpha}}{f_-} & 0 \\ 0 & 1 & 0 & \frac{e_{-(\lambda+\alpha)}}{b_+} & -\frac{a_- e_{-(\lambda+\alpha)} + e_{-2\alpha}}{b_+ f_-} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{a_-}{f_-} & -\frac{e_{-\alpha}}{g_+} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -f_- e_{-\nu} & a_- e_{-\nu} + 1 & 0 \end{bmatrix},$$

$$\Lambda = \operatorname{diag}[e_{-\nu}, e_\alpha, e_\lambda, e_{-\nu}, e_{-\alpha}, e_{\nu-\alpha}]$$

if $\nu < \alpha$, and

$$G_+ = \begin{bmatrix} e_{\lambda+\nu} & 0 & 0 & 0 & -\frac{1}{g_+} & -\frac{e_\alpha}{f_-} \\ 0 & e_\nu & 0 & -\frac{1}{b_+} & 0 & \frac{e_{\nu-\alpha}}{b_+ f_-} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ e_\nu + a_- & b_+ & 0 & 0 & 0 & 0 \\ f_- & 0 & 0 & 0 & 0 & 0 \\ g_+ e_\lambda & 0 & 0 & 0 & 0 & -\frac{g_+}{f_-} \end{bmatrix},$$

$$G_- = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{e_{-\alpha}}{f_-} & 0 \\ 0 & 1 & 0 & -\frac{e_{-(\lambda+\alpha)}}{b_+} & -\frac{a_- e_{-(\lambda+\alpha)} + e_{-2\alpha}}{b_+ f_-} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{a_-}{f_-} & -\frac{e_{-\alpha}}{g_+} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -\frac{f_- e_{-\lambda}}{g_+} \end{bmatrix},$$

$$\Lambda = \operatorname{diag}[e_{-\nu}, e_\alpha, e_\lambda, e_{-\alpha}, e_{-\alpha}, 1]$$

if $\nu > \alpha$.

Situation 4: $b_\pm = 0$, $g_\pm \neq 0$.

Conditions (4.5) and the second equality of (4.3) are then satisfied automatically. The other two equalities of (4.3) take the form

$$l_+ g_- = l_- g_+, \quad d_+ g_- = d_- g_+. \tag{5.2}$$

Due to the first condition in (5.2), either $l_+ = l_- = 0$ or $l_\pm \neq 0$. We now consider these two cases separately.

Case 8: $l_\pm \neq 0$.

Using elementary row operations, we may without loss of generality suppose that $f_- = a_- = d_- = 0$; due to (5.2), we will simultaneously achieve that $d_+ = 0$. Applying elementary column operations, we may then assure that $f_+ = 0$. There are now two subcases, depending on the new value of a_+.

Subcase 8a: $a_+ \neq 0$. The desired APw factorization is given by

$$G_+ = \begin{bmatrix} 0 & 0 & e_{\lambda+\nu} & \frac{g_- - g_+ e_\lambda - a_+ g_- e_\alpha}{a_+ g_-^2} & 0 & \frac{g_+ e_{\lambda+\nu} - g_- e_\nu}{g_-} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ e_\lambda - \frac{l_-}{k_+} & 0 & 0 & 0 & -\frac{e_\nu}{l_-} & 0 \\ 0 & 0 & e_\nu + a_+ e_\lambda & -\frac{a_+ g_+ e_\alpha + g_+ + a_+^2 g_- e_{\alpha-\nu}}{a_+ g_-^2} & 0 & \frac{a_+ g_+ e_\lambda + g_+ e_\nu}{g_-} - a_+ \\ l_+ e_\alpha & 0 & 0 & 0 & -\frac{l_+}{l_-} & 0 \\ 0 & 0 & g_+ e_\lambda + g_- & -\frac{g_+ (a_+ g_- e_{\alpha-\nu} + g_+ e_\alpha)}{a_+ g_-^2}. & 0 & \frac{g_+^2 e_\lambda}{g_-} \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 0 & 1 & 0 & \frac{e_{-\alpha}}{l_-} & 0 \\ 0 & 1 & & 0 & 0 & 0 \\ 1 & 0 & 0 & -\frac{g_+}{g_-} & 0 & \frac{e_{-\alpha}}{g_-} \\ 0 & 0 & 0 & -g_- e_{-\nu} & 0 & 1 \\ 0 & 0 & -\frac{l_-^2 e_{-\nu}}{l_+} & 0 & 1 - \frac{l_- e_{-\lambda}}{l_+} & 0 \\ 0 & 0 & 0 & 1 - \frac{e_{-\alpha}}{a_+} & 0 & \frac{e_{\nu-\alpha}}{a_+ g_-} \end{bmatrix},$$

$$\Lambda = \operatorname{diag}[1, e_\lambda, e_{-\nu}, e_{\nu-\alpha}, 1, e_{-\nu}]$$

if $\nu < \alpha$, and by

$$G_+ = \begin{bmatrix} \frac{g_+ e_\lambda - g_-}{a_+ g_-} & 0 & e_{2\nu} + \frac{g_-(a_+ e_\nu - e_{\nu-\alpha})}{g_+} & 0 & 0 & -\frac{e_\lambda}{g_-} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_\lambda - \frac{l_-}{l_+} & -\frac{e_\nu}{l_-} & 0 \\ \frac{g_+(1 + a_+ e_\alpha)}{a_+ g_-} & 0 & a_+ e_\nu + e_{\nu-\alpha} + \frac{a_+^2 g_-}{g_+} & 0 & 0 & -\frac{1 + a_+ e_\alpha}{g_-} \\ 0 & 0 & 0 & l_+ e_\alpha & -\frac{l_+}{l_-} & 0 \\ \frac{g_+^2 e_\alpha}{a_+ g_-} & 0 & g_+ e_\nu + a_+ g_- & 0 & 0 & -\frac{g_+ e_\alpha}{g_-} \end{bmatrix},$$

$$G_- = \begin{bmatrix} -\frac{g_-}{g_+} & 0 & 0 & 1 & 0 & -\frac{e_{-\alpha}}{g_+} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 + \frac{e_{-\alpha}}{a_+} & 0 & 0 & -\frac{g_+ e_{-\alpha}}{a_+ g_-} & 0 & \frac{a_+ e_{-\alpha} + e_{-2\alpha}}{a_+ g_-} \\ 0 & 0 & 1 & 0 & \frac{e_{-\alpha}}{l_-} & 0 \\ 0 & 0 & -\frac{l_-^2 e_{-\nu}}{l_+} & 0 & 1 - \frac{l_- e_{-\lambda}}{l_+} & 0 \\ \frac{a_+ g_-^2 e_{\alpha-\nu}}{g_+} & 0 & 0 & -g_- e_{-\nu} & 0 & 1 + \frac{a_+ g_- e_{-\nu}}{g_+} \end{bmatrix},$$

$$\Lambda = \operatorname{diag}[e_{-\alpha}, e_\lambda, e_{\alpha-\nu}, 1, 1, e_{-\alpha}]$$

if $\nu > \alpha$.

Subcase 8b: $a_+ = 0$. The desired AP_W factorization is given by

$$G_+ = \begin{bmatrix} 0 & 0 & -\frac{e_\lambda}{g_-} & \frac{g_+ e_\lambda}{g_-} - 1 & 0 & e_{\lambda+\nu} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ e_\lambda - \frac{l_-}{l_+} & 0 & 0 & \frac{f_+ e_\lambda}{l_-} & -\frac{e_\nu}{l_-} & 0 \\ 0 & 0 & -\frac{1}{g_-} & \frac{g_+}{g_-} & 0 & e_\nu \\ l_+ e_\alpha & 0 & -\frac{f_+ e_\alpha}{g_-} & \frac{f_+ e_\alpha (g_+ l_- + g_- l_+)}{g_- l_-} & -\frac{l_+}{l_-} & f_+ e_\lambda \\ 0 & 0 & -\frac{g_+ e_\alpha}{g_-} & \frac{g_+^2 e_\alpha}{g_-} & 0 & g_+ e_\lambda + g_- \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 0 & 1 & -\frac{f_+}{l_-} & \frac{e_{-\alpha}}{l_-} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -g_- e_{-\nu} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{l_-^2 e_{-\nu}}{l_+} & \frac{f_+ l_- e_{-\nu}}{l_+} & 1 - \frac{l_- e_{-\lambda}}{l_+} & 0 \\ 1 & 0 & 0 & -\frac{g_+}{g_-} & 0 & \frac{e_{-\alpha}}{g_-} \end{bmatrix},$$

$$\Lambda = \operatorname{diag}[1, e_\lambda, e_{-\alpha}, 1, 1, e_{-\nu}].$$

Case 9: $l_+ = l_- = 0$.

Using elementary row operations, we can make $a_- = f_- = 0$. If in addition $f_+ = 0$, then G splits into a direct sum of a diagonal matrix $\operatorname{diag}[e_\lambda, e_{-\lambda}]$ and another matrix G_1 of the type (1.4), with $m = 2$ and

$$c_{\pm 1} = \begin{bmatrix} d_\pm & 0 \\ 0 & g_\pm \end{bmatrix}, \quad c_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Due to the second equation in (5.2), either $d_\pm \neq 0$ or $d_+ = d_- = 0$. In the former case, G_1 satisfies the conditions of Corollary 2.5. The latter case is covered by Case 1 of Appendix B. Either way, G_1 is AP_W factorable, which implies AP_W factorability of G.

It remains to consider the case $f_+ \neq 0$. Applying another elementary row operation, we may suppose that $a_+ = 0$. Due to (5.2), it is still true that either $d_+ = d_- = 0$ or $d_\pm \neq 0$. According to this, the following subcases are in order.

Subcase 9a: $d_+ = d_- = 0$. The desired AP_W factorization is given by

$$G_+ = \begin{bmatrix} 0 & 0 & -\frac{e_\lambda}{g_-} & \frac{g_+ e_\lambda - g_-}{f_+ g_-} & \frac{g_+ e_{\lambda+\nu}}{g_-} - e_\nu & e_{\lambda+\nu} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{g_-} & \frac{g_+}{f_+ g_-} & \frac{g_+ e_\nu}{g_-} & e_\nu \\ 0 & 0 & -\frac{f_+ e_\alpha}{g_-} & \frac{g_+ e_\alpha}{g_-} & \frac{f_+(g_+ e_\lambda - g_-)}{g_-} & f_+ e_\lambda \\ 0 & 0 & -\frac{g_+ e_\alpha}{g_-} & \frac{g_+^2 e_\alpha}{f_+ g_-} & \frac{g_+^2 e_\lambda}{g_-} & g_+ e_\lambda + g_- \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -g_- e_{-\nu} & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{e_{-\alpha}}{f_+} & 0 \\ 1 & 0 & 0 & -\frac{g_+}{g_-} & 0 & \frac{e_{-\alpha}}{g_-} \end{bmatrix},$$

$$\Lambda = \operatorname{diag}[e_\lambda, e_\lambda, e_{-\alpha}, e_{-\alpha}, e_{-\nu}, e_{-\nu}].$$

Subcase 9b: $d_\pm \neq 0$. The desired AP_W factorization of G is given by

$$G_+ = \begin{bmatrix} 0 & 0 & e_{\lambda+\nu} & \frac{g_+ e_\lambda - g_-}{f_+ g_-} & -\frac{e_\nu}{g_-} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{e_\nu}{d_-} & 0 & -\frac{e_{2\nu}}{d_+} & \frac{e_\nu (g_- d_+ - g_+ d_-)}{d_+ d_- f_+ g_-} & \frac{e_{2\nu}}{d_- g_-} & e_\lambda - \frac{d_-}{d_+} \\ -\frac{d_+}{d_-} & 0 & 0 & \frac{d_+}{d_- f_+} & \frac{d_+ e_\nu}{d_- g_-} & d_+ e_\alpha \\ 0 & 0 & f_+ e_\lambda & \frac{g_+ e_\alpha}{g_-} & -\frac{f_+}{g_-} & 0 \\ 0 & 0 & g_+ e_\lambda + g_- & \frac{g_+^2 e_\alpha}{f_+ e_-} & -\frac{g_+}{g_-} & 0 \end{bmatrix},$$

$$G_- = \begin{bmatrix} -\frac{d_-}{d_+} & 0 & -\frac{d_-^2 e_{-\nu}}{d_+} & 1 - \frac{d_- e_{-\lambda}}{d_+} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -\frac{g_+ e_{-\alpha}}{f_+ g_-} & \frac{e_{-\alpha}}{g_-} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{g_- e_{-\lambda}}{f_+} & 1 \\ 0 & 0 & 1 & \frac{e_{-\alpha}}{d_-} & 0 & -\frac{e_{\nu-\alpha}}{d_- g_-} \end{bmatrix},$$

$$\Lambda = \operatorname{diag}[1, e_\lambda, e_{-\nu}, e_{-\alpha}, 1, 1]$$

if $\nu < \alpha$, and by

$$G_+ = \begin{bmatrix} 0 & 0 & 0 & -\frac{e_\nu}{g_-} & e_{\lambda+\nu} - \frac{g_- e_\nu}{g_+} & \frac{g_+ e_\lambda - g_-}{f_+ g_-} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{e_\nu}{d_-} & 0 & e_\lambda - \frac{d_-}{d_+} & \frac{e_{\nu-\alpha}}{d_+ g_-} & \frac{e_{2\nu}(g_- d_+ - g_+ d_-)}{d_+ d_- g_+} & \frac{e_\nu (g_- d_+ - g_+ d_-)}{d_+ d_- f_+ g_-} \\ -\frac{d_+}{d_-} & 0 & d_+ e_\alpha & 0 & \frac{d_+ g_- e_\nu}{d_- g_+} & \frac{d_+}{d_- f_+} \\ 0 & 0 & 0 & -\frac{f_+}{g_-} & f_+ e_\lambda - \frac{f_+ g_-}{g_+} & \frac{g_+ e_\alpha}{g_-} \\ 0 & 0 & 0 & -\frac{g_+}{g_-} & g_+ e_\lambda & \frac{g_+^2 e_\alpha}{f_+ g_-} \end{bmatrix},$$

$$G_- = \begin{bmatrix} -\frac{d_-}{d_+} & 0 & -\frac{d_-^2 e_{-\nu}}{d_+} & 1 - \frac{d_- e_{-\lambda}}{d_+} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{g_- e_{-\alpha}}{d_- g_+} & 0 & 1 & \frac{e_{-\alpha}}{d_-} & 0 & -\frac{e_{-2\alpha}}{d_- g_+} \\ -\frac{g_-^2 e_{-\nu}}{g_+} & 0 & 0 & 0 & 0 & 1 - \frac{g_- e_{-\lambda}}{g_+} \\ 1 & 0 & 0 & 0 & -\frac{g_+ e_{-\alpha}}{f_+ g_-} & \frac{e_{-\alpha}}{g_-} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\Lambda = \operatorname{diag}[1, e_\lambda, 1, 1, e_{-\nu}, e_{-\alpha}]$$

if $\nu > \alpha$.

Situation 5: $b_- = g_+ = 0$, $b_+ g_- \neq 0$.

From the first two equations of (4.3) it follows that $l_\pm = 0$. Applying elementary row and column operations, we may suppose without loss of generality that $a_\pm = d_+ = f_- = 0$; of course, this leads to a change in the value of f_+. If the new f_+ and d_- equal zero, then the matrix G splits into a direct sum of a diagonal matrix $\operatorname{diag}[e_\lambda, e_{-\lambda}]$ and a matrix G_1 of the type (1.4), (1.5) with $m = 2$ and

$$c_{-1} = \begin{bmatrix} 0 & 0 \\ 0 & g_- \end{bmatrix}, \quad c_1 = \begin{bmatrix} b_+ & 0 \\ 0 & 0 \end{bmatrix}, \quad c_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The latter matrix is AP_W factorable due to Case 2 of Appendix B. Hence, the original matrix G is also AP_W factorable.

The remaining three possibilities (in terms of zero/non-zero pattern of the entries d_-, f_+) are covered by Cases 10–12.

Case 10: $d_- f_+ \neq 0$. The AP_W factorization of G is given by

$$G_+ = \begin{bmatrix} 0 & -\frac{e_\nu}{g_-} & 0 & -\frac{1}{f_+} & e_{\lambda+\nu} & 0 \\ 0 & 0 & e_\nu & 0 & -\frac{e_{2\nu}}{b_+} & -\frac{d_-}{b_+} \\ -\frac{1}{d_-} & \frac{e_{2\nu}}{d_- g_-} & 0 & \frac{e_\nu}{d_- f_+} & 0 & e_\lambda \\ 0 & 0 & b_+ & 0 & 0 & 0 \\ 0 & -\frac{f_+}{g_-} & 0 & 0 & f_+ e_\lambda & 0 \\ 0 & 0 & 0 & 0 & g_- & 0 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{g_- e_{-\lambda}}{f_+} & 1 \\ \frac{e_{-\alpha}}{b_+} & 1 & \frac{d_- e_{-\lambda}}{b_+} & \frac{e_{-(\lambda+\alpha)}}{b_+} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & \frac{e_{-\alpha}}{g_-} \\ 0 & 0 & 1 & \frac{e_{-\alpha}}{d_-} & 0 & -\frac{e_{\nu-\alpha}}{d_- g_-} \end{bmatrix},$$

$$\Lambda = \operatorname{diag}[e_\nu, 1, e_\alpha, e_{-\alpha}, e_{-\nu}, 1]$$

if $\nu < \alpha$, and by

$$G_+ = \begin{bmatrix} 0 & -\frac{e_\nu}{g_-} & 0 & -\frac{1}{f_+} & e_{\lambda+\nu} & 0 \\ 0 & \frac{e_{\nu-\alpha}}{b_+ g_-} & e_\nu & 0 & -\frac{e_{2\nu}}{b_+} & -\frac{d_-}{b_+} \\ -\frac{1}{d_-} & 0 & 0 & \frac{e_\nu}{d_- f_+} & 0 & e_\lambda \\ 0 & 0 & b_+ & 0 & 0 & 0 \\ 0 & -\frac{f_+}{g_-} & 0 & 0 & f_+ e_\lambda & 0 \\ 0 & 0 & 0 & 0 & g_- & 0 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{g_- e_{-\lambda}}{f_+} & 1 \\ \frac{e_{-\alpha}}{b_+} & 1 & \frac{d_- e_{-\lambda}}{b_+} & \frac{e_{-(\lambda+\alpha)}}{b_+} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & \frac{e_{-\alpha}}{g_-} \\ 0 & 0 & 1 & \frac{e_{-\alpha}}{d_-} & -\frac{e_{-2\alpha}}{d_- f_+} & 0 \end{bmatrix}$$

(with the same Λ) if $\nu > \alpha$.

Case 11: $d_- \neq 0$, $f_+ = 0$. The AP_W factorization of G is given by

$$G_+ = \begin{bmatrix} 0 & -\frac{1}{g_-} & 0 & e_{\lambda+\nu} & 0 & 0 \\ 0 & 0 & e_\nu & -\frac{e_{2\nu}}{b_+} & 0 & -\frac{d_-}{b_+} \\ -\frac{1}{d_-} & \frac{e_\nu}{d_- g_-} & 0 & 0 & 0 & e_\lambda \\ 0 & 0 & b_+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g_- & 0 & 0 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{e_{-\alpha}}{b_+} & 1 & \frac{d_- e_{-\lambda}}{b_+} & \frac{e_{-(\lambda+\alpha)}}{b_+} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \frac{e_{-\alpha}}{g_-} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{e_{-\alpha}}{d_-} & 0 & -\frac{e_{\nu-\alpha}}{d_- g_-} \end{bmatrix},$$

$$\Lambda = \operatorname{diag}[e_\nu, e_\nu, e_\alpha, e_{-\nu}, e_{-\lambda}, 1]$$

if $\nu < \alpha$, and

$$G_+ = \begin{bmatrix} 0 & -d_- & 0 & e_{\lambda+\nu} & 0 & -\frac{e_\alpha}{g_-} \\ 0 & 0 & e_\nu & -\frac{e_{2\nu}}{b_+} & 0 & \frac{1}{b_+ g_-} \\ -\frac{1}{d_-} & e_\nu & 0 & 0 & 0 & 0 \\ 0 & 0 & b_+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g_- & 0 & 0 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{e_{-\alpha}}{d_-} & 0 & 0 \\ \frac{e_{-\alpha}}{b_+} & 1 & \frac{d_- e_{-\lambda}}{b_+} & \frac{e_{-(\lambda+\alpha)}}{b_+} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \frac{e_{-\alpha}}{g_-} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -d_- g_- e_{\alpha-\nu} & -g_- e_{-\nu} & 0 & 1 \end{bmatrix},$$

$$\Lambda = \operatorname{diag}[e_\nu, e_\alpha, e_\alpha, e_{-\nu}, e_{-\lambda}, e_{\nu-\alpha}]$$

if $\nu > \alpha$.

Case 12: $d_- = 0$, $f_+ \neq 0$. The AP_W factorization of G is given by

$$G_+ = \begin{bmatrix} e_{\lambda+\nu} & 0 & 0 & -\frac{1}{f_+} & -e_\nu & -\frac{e_\alpha}{g_-} \\ 0 & e_\nu & 0 & 0 & 0 & \frac{1}{b_+ g_-} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ e_\nu & b_+ & 0 & 0 & 0 & 0 \\ f_+ e_\lambda & 0 & 0 & 0 & -f_+ & -\frac{f_+ e_{\alpha-\nu}}{g_-} \\ g_- & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{e_{-\alpha}}{g_-} \\ 0 & 1 & 0 & \frac{e_{-(\lambda+\alpha)}}{b_+} & 0 & -\frac{e_{-2\alpha}}{b_+ g_-} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{e_{-\alpha}}{f_+} & 0 \\ 0 & 0 & 0 & -g_- e_{-\nu} & 0 & 1 \end{bmatrix},$$

$$\Lambda = \operatorname{diag}[e_{-\nu}, e_\alpha, e_\lambda, e_{-\alpha}, e_{-\nu}, e_{\nu-\alpha}]$$

if $\nu < \alpha$, and

$$G_+ = \begin{bmatrix} e_{\lambda+\nu} & 0 & 0 & -\frac{1}{f_+} & -\frac{e_\nu}{g_-} & 0 \\ 0 & e_\nu & 0 & 0 & \frac{e_{\nu-\alpha}}{b_+ g_-} & -\frac{1}{b_+} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ e_\nu & b_+ & 0 & 0 & 0 & 0 \\ f_+ e_\lambda & 0 & 0 & 0 & -\frac{f_+}{g_-} & 0 \\ g_- & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{e_{-\alpha}}{g_-} \\ 0 & 1 & 0 & \frac{e_{-(\lambda+\alpha)}}{b_+} & 0 & -\frac{e_{-2\alpha}}{b_+ g_-} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{g_- e_{-\lambda}}{f_+} & 1 \\ 0 & 0 & 0 & 1 & -\frac{e_{-\alpha}}{f_+} & 0 \end{bmatrix},$$

$$\Lambda = \operatorname{diag}[e_{-\nu}, e_\alpha, e_\lambda, e_{-\alpha}, 1, e_{-\alpha}]$$

if $\nu > \alpha$.

Situation 6: Exactly one of the entries $b_\pm$, $g_\pm$ differs from zero.

Using symmetries (2.8), (5.1), we may without loss of generality suppose that $b_- \neq 0$, $b_+ = g_\pm = 0$. Condition (4.5) and the first equation of (4.3) are satisfied automatically. The other two equations of (4.3) imply that $l_+ = f_+ = 0$. Applying elementary column operations, we can make $a_- = d_- = 0$ without changing any other entries. The further classification depends on l_-.

Case 13: $l_- \neq 0$.

Using elementary column operations, we may assume that $f_- = 0$; these operations change the value of a_+. We now distinguish several subcases in terms of the new value of a_+ and d_+.

Subcase 13a: $a_+ d_+ \neq 0$. The AP_W factorization of G is given by

$$G_+ = \begin{bmatrix} \frac{d_+}{a_+ l_-} & 0 & -b_- e_\nu & e_\alpha & 0 & 0 \\ -\frac{d_+ e_\nu}{a_+ b_- l_-} & -\frac{1}{b_-} & e_{2\nu} & 0 & 0 & 0 \\ -\frac{1}{l_-} & 0 & 0 & -\frac{1}{d_+} & e_{\lambda+\nu} & 0 \\ 0 & 0 & -a_+ b_- & a_+ e_{\alpha-\nu} & d_+ e_\lambda & 0 \\ 0 & 0 & 0 & 0 & l_- & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$G_- = \begin{bmatrix} -\frac{l_-}{d_+} & -\frac{b_- l_- e_{-\nu}}{d_+} & 0 & -\frac{l_- e_{-\lambda}}{d_+} & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{e_{\nu-\alpha}}{a_+ b_-} & 1 - \frac{e_{-\alpha}}{a_+} & 0 & \frac{a_+ e_{-\alpha} - e_{-2\alpha}}{a_+ b_-} & \frac{d_+ e_{\nu-\alpha}}{a_+ b_- l_-} & 0 \\ 1 & b_- e_{-\nu} & 0 & e_{-\lambda} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{e_{-\alpha}}{l_-} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Lambda = \operatorname{diag}[e_\nu, e_\nu, e_{\alpha-\nu}, e_\nu, e_{-\nu}, e_{-\lambda}]$$

if $\nu < \alpha$, and by

$$G_+ = \begin{bmatrix} -b_- & 0 & e_\nu & \frac{d_+ e_\alpha}{a_+ l_-} & 0 & 0 \\ e_\nu & -\frac{1}{b_-} & 0 & 0 & 0 & 0 \\ \frac{a_+ b_-}{d_+} & 0 & -\frac{e_{\nu-\alpha}}{d_+} & -\frac{1+a_+ e_\alpha}{a_+ l_-} & e_{\lambda+\nu} & 0 \\ 0 & 0 & a_+ & 0 & d_+ e_\lambda & 0 \\ 0 & 0 & 0 & 0 & l_- & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 1 & 0 & \frac{e_{-\alpha}}{b_-} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 + \frac{e_{-\alpha}}{a_+} & \frac{b_- e_{-\lambda}}{a_+} & 0 & \frac{e_{-(\lambda+\alpha)}}{a_+} & -\frac{d_+ e_{-\alpha}}{a_+ l_-} & 0 \\ -\frac{l_-}{d_+} & \frac{b_- l_- (a_+ e_{\alpha-\nu} - e_{-\nu})}{d_+} & 0 & \frac{l_- (a_+ e_{-\nu} - e_{-\lambda})}{d_+} & 1 & 0 \\ 0 & 0 & 1 & 0 & \frac{e_{-\alpha}}{l_-} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Lambda = \operatorname{diag}[e_\alpha, e_\nu, e_\alpha, e_{\nu-\alpha}, e_{-\nu}, e_{-\lambda}]$$

if $\nu > \alpha$.

Subcase 13b: $a_+ = 0$. The AP_W factorization of G is given by

$$G_+ = \begin{bmatrix} e_\lambda & 0 & \frac{d_+ e_\alpha}{l_-} & -b_- & 0 & 0 \\ 0 & -\frac{1}{b_-} & 0 & e_\nu & 0 & 0 \\ 0 & 0 & -\frac{1}{l_-} & 0 & e_{\lambda+\nu} & 0 \\ 1 & 0 & 0 & 0 & d_+ e_\lambda & 0 \\ 0 & 0 & 0 & 0 & l_- & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 1 & b_- e_{-\nu} & 0 & e_{-\lambda} & -\frac{d_+}{l_-} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{e_{-\alpha}}{b_-} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{e_{-\alpha}}{l_-} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Lambda = \operatorname{diag}[1, e_\nu, e_\nu, e_\alpha, e_{-\nu}, e_{-\lambda}].$$

Subcase 13c: $a_+ \neq 0$, $d_+ = 0$. Then G can be represented as $G = X_+ G' X_-$, where

$$X_+ = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{a_+ e_\lambda + e_\nu}{b_-} & -\frac{1}{b_-} & 0 & e_{\lambda+\nu} & 0 & 0 \\ 0 & 0 & -\frac{1}{l_-} & 0 & e_{\lambda+\nu} & 0 \\ 0 & 0 & 0 & b_- & 0 & 0 \\ 0 & 0 & 0 & 0 & l_- & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is invertible in AP_W^+,

$$X_- = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{e_{-\alpha}}{b_-} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{e_{-\alpha}}{l_-} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is invertible in AP_W^-, and

$$G' = \begin{bmatrix} e_\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & e_\nu & 0 & 0 & 0 & 0 \\ 0 & 0 & e_\nu & 0 & 0 & 0 \\ \frac{1+a_+ e_\alpha}{b_-} & 0 & 0 & e_{-\nu} & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{-\nu} & 0 \\ 0 & 0 & 0 & 0 & 0 & e_{-\lambda} \end{bmatrix}.$$

Obviously, G' can be represented as a direct sum of a diagonal matrix $\operatorname{diag}[e_\nu, e_\nu, e_{-\nu}, e_{-\lambda}]$ with a 2×2 matrix

$$e_{\alpha/2} \begin{bmatrix} e_{\nu+\alpha/2} & 0 \\ \frac{e_{-\alpha/2} + a_+ e_{\alpha/2}}{b_-} & e_{-\nu-\alpha/2} \end{bmatrix}.$$

The latter matrix is AP_W factorable due to Theorem 1.2. This in turn guarantees the AP_W factorability of G.

Case 14: $l_- = 0$.

If in addition $d_+ = 0$, then G splits into a direct sum of a diagonal matrix $\mathrm{diag}[e_\lambda, e_{-\lambda}]$ with another matrix G_1 of the type (1.4), for which $m = 2$ and

$$c_{-1} = \begin{bmatrix} b_- & 0 \\ 0 & f_- \end{bmatrix}, \quad c_1 = \begin{bmatrix} 0 & a_+ \\ 0 & 0 \end{bmatrix}, \quad c_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The matrix G_1 satisfies Corollary 2.5 if $f_- \neq 0$, and falls into Case 3 of Appendix B otherwise. Either way, G_1 is AP_W factorable. The AP_W factorability of G follows immediately.

If $d_+ \neq 0$, we can use an elementary column transformation to make $a_+ = 0$. It remains to consider two subcases, depending on the value of f_-.

Subcase 14a: $f_- \neq 0$. The AP_W factorization of G is given by

$$G_+ = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{f_-} & e_{\lambda+\nu} & 0 \\ 0 & -\frac{1}{b_-} & e_\lambda & \frac{e_\nu}{b_- f_-} & 0 & 0 \\ e_\nu & 0 & -\frac{b_-}{d_+} & 0 & 0 & 0 \\ d_+ & 0 & 0 & 0 & e_\nu & 0 \\ 0 & 0 & 0 & 0 & f_- & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & \frac{b_- e_{-\lambda}}{d_+} & 1 & \frac{e_{-(\lambda+\alpha)}}{d_+} & -\frac{e_{-2\alpha}}{d_+ f_-} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{e_{-\alpha}}{b_-} & -\frac{e_{\nu-\alpha}}{b_- f_-} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & \frac{e_{-\alpha}}{f_-} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[e_\alpha, e_\nu, 1, e_\nu, e_{-\nu}, e_{-\lambda}]$$

if $\nu < \alpha$, and by

$$G_+ = \begin{bmatrix} 0 & 0 & -\frac{e_\alpha}{f_-} & -b_- & e_{\lambda+\nu} & 0 \\ 0 & -\frac{1}{b_-} & 0 & e_\nu & 0 & 0 \\ e_\nu & 0 & \frac{1}{d_+ f_-} & 0 & 0 & 0 \\ d_+ & 0 & 0 & 0 & e_\nu & 0 \\ 0 & 0 & 0 & 0 & f_- & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & \frac{b_- e_{-\lambda}}{d_+} & 1 & \frac{e_{-(\lambda+\alpha)}}{d_+} & -\frac{e_{-2\alpha}}{d_+ f_-} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -b_- f_- e_{\alpha-\nu} & 0 & -f_- e_{-\nu} & 1 & 0 \\ 0 & 1 & 0 & \frac{e_{-\alpha}}{b_-} & 0 & 0 \\ 1 & 0 & 0 & 0 & \frac{e_{-\alpha}}{f_-} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[e_\alpha, e_\nu, e_{\nu-\alpha}, e_\alpha, e_{-\nu}, e_{-\lambda}]$$

if $\nu > \alpha$.

Subcase 14b: $f_- = 0$. The AP_W factorization of G is given by

$$G_+ = \begin{bmatrix} 0 & 0 & e_\alpha & -b_- & 0 & 0 \\ 0 & -\frac{1}{b_-} & 0 & e_\nu & 0 & 0 \\ e_\nu & 0 & -\frac{1}{d_+} & 0 & 0 & 0 \\ d_+ & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$G_- = \begin{bmatrix} \frac{e_{-\alpha}}{d_+} & \frac{b_- e_{-\lambda}}{d_+} & 1 & \frac{e_{-(\lambda+\alpha)}}{d_+} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & b_- e_{-\nu} & 0 & e_{-\lambda} & 0 & 0 \\ 0 & 1 & 0 & \frac{e_{-\alpha}}{b_-} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[e_\alpha, e_\nu, e_\nu, e_\alpha, e_{-\lambda}, e_{-\lambda}].$$

Appendix D.

$$c_{-1} = \begin{bmatrix} a_- & 0 \\ 0 & l_- \end{bmatrix}, \quad c_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad c_1 = \begin{bmatrix} a_+ & d_+ \\ f_+ & l_+ \end{bmatrix},$$

where $l_- \neq 0$ and either $l_+ = 0$ and exactly one of d_+, f_+ differs from zero (Situation 1), or $l_+ \neq 0$, $a_- \det c_1 = 0$ (Situation 2).

In Situation 1, we can use symmetry (5.1) to suppose without loss of generality that $d_+ = 0$, $f_+ \neq 0$. We now consider the four separate cases, depending on the zero/non-zero pattern of $a_\pm$.

Case 1: $a_\pm \neq 0$.

For $\nu < \alpha$, G can be represented as $G = X_+ G' X_-$, where

$$X_+ = \begin{bmatrix} e_\lambda - \frac{a_-}{a_+} & \frac{a_+ l_- e_\nu}{f_+} & -\frac{l_-}{f_+} & 0 \\ \frac{a_- f_+ e_\lambda}{a_+ l_-} & 0 & e_\lambda & -\frac{1}{l_-} \\ a_+ e_\alpha + 1 & \frac{a_+^2 l_-}{f_+} & 0 & 0 \\ f_+ e_\alpha & a_+ l_- & 0 & 0 \end{bmatrix}$$

is invertible in AP_W^+,

$$X_- = \begin{bmatrix} 1 + a_- e_{-\nu} & -\frac{a_+ l_- e_{-\nu}}{f_+} & e_{-\lambda} & -\frac{a_+ e_{-\lambda}}{f_+} \\ -\frac{a_- f_+}{a_+ l_-} & 1 + \frac{e_{-\alpha}}{a_+} & -\frac{f_+ e_{-\alpha}}{a_+ l_-} & \frac{e_{-2\alpha} + a_+ e_{-\alpha}}{a_+ l_-} \\ 0 & -\frac{1 + a_- e_{-\nu}}{a_+} & \frac{f_+}{a_+ l_-} & -\frac{a_- e_{-\lambda} + e_{-\alpha}}{a_+ l_-} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is invertible in AP_W^-, and

$$G' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e_{\alpha-\nu} & 0 & 0 \\ 0 & a_- e_{-\nu} + 1 & e_{-\alpha} & 0 \\ 0 & 0 & 0 & e_\nu \end{bmatrix}$$

splits into a direct sum of a diagonal matrix $\mathrm{diag}[1, e_\nu]$ with the matrix

$$e_{-\nu/2} \begin{bmatrix} e_{\alpha-\nu/2} & 0 \\ a_- e_{-\nu/2} + e_{\nu/2} & e_{-\alpha+\nu/2} \end{bmatrix}.$$

The latter matrix is AP_W factorable due to Theorem 1.2. This implies the AP_W factorability of G' and ultimately of G itself.

When $\nu > \alpha$, the AP_W factorization of G is delivered by

$$G_+ = \begin{bmatrix} -\frac{e_\lambda}{a_-} & \frac{l_-(a_+ e_\nu - e_{\nu-\alpha})}{f_+} & e_\lambda - \frac{a_-}{a_+} & 0 \\ 0 & e_{2\nu} & \frac{a_- f_+ e_\lambda}{a_+ l_-} & -\frac{1}{l_-} \\ -\frac{1 + a_+ e_\alpha}{a_-} & \frac{a_+^2 l_-}{f_+} & 1 + a_+ e_\alpha & 0 \\ -\frac{f_+ e_\alpha}{a_-} & a_+ l_- & f_+ e_\alpha & 0 \end{bmatrix},$$

$$G_- = \begin{bmatrix} -a_-^2 e_{\alpha-\nu} & \frac{l_-(a_+a_-e_{\alpha-\nu}-1)}{f_+} & 1 - a_-e_{-\nu} & \frac{a_+a_-e_{-\nu}-e_{-\alpha}}{f_+} \\ -\frac{a_-f_+}{a_+l_-} & 1 + \frac{e_{-\alpha}}{a_+} & -\frac{f_+e_{-\alpha}}{a_+l_-} & \frac{e_{-2\alpha}+a_+e_{-\alpha}}{a_+l_-} \\ 1 & -\frac{l_-e_{-\alpha}}{a_-f_+} & \frac{e_{-\alpha}}{a_-} & -\frac{e_{-2\alpha}}{a_-f_+} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[e_{-\alpha}, e_{\alpha-\nu}, 1, e_\nu].$$

Case 2: $a_+ \neq 0$, $a_- = 0$.
The AP_W factorization of G is delivered by

$$G_+ = \begin{bmatrix} e_\lambda & \frac{l_-e_\nu}{f_+} & 1 - a_+e_\alpha & 0 \\ 0 & 0 & -\frac{f_+e_\lambda}{l_-} & -\frac{1}{l_-} \\ 1 + a_+e_\alpha & \frac{a_+l_-}{f_+} & -a_+^2 e_{\alpha-\nu} & 0 \\ f_+e_\alpha & l_- & -a_+f_+e_{\alpha-\nu} & 0 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 1 & -\frac{a_+l_-e_{-\nu}}{f_+} & e_{-\lambda} & -\frac{a_+e_{-\lambda}}{f_+} \\ 0 & 1 & -\frac{f_+}{l_-} & \frac{e_{-\alpha}}{l_-} \\ 0 & -\frac{l_-}{f_+} & 0 & -\frac{e_{-\alpha}}{f_+} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[1, e_{-\nu}, 1, e_\nu]$$

when $\nu < \alpha$, and by

$$G_+ = \begin{bmatrix} 1 & \frac{l_-(a_+e_\nu - e_{\nu-\alpha})}{a_+f_+} & e_\lambda & 0 \\ -\frac{f_+e_\lambda}{l_-} & \frac{e_{2\nu}}{a_+} & 0 & -\frac{1}{l_-} \\ 0 & \frac{a_+l_-}{f_+} & 1 + a_+e_\alpha & 0 \\ 0 & l_- & f_+e_\alpha & 0 \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & \frac{l_-}{a_+f_+} & -\frac{1}{a_+} & \frac{e_{-\alpha}}{a_+f_+} \\ 0 & e_{-\alpha} + a_+ & -\frac{f_+e_{-\alpha}}{l_-} & \frac{e_{-2\alpha}+a_+e_{-\alpha}}{l_-} \\ 1 & -\frac{a_+l_-e_{-\nu}}{f_+} & e_{-\lambda} & -\frac{a_+e_{-\lambda}}{f_+} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[e_{-\alpha}, e_{\alpha-\nu}, 1, e_\nu]$$

if $\nu > \alpha$.
Case 3: $a_+ = 0$, $a_- \neq 0$.
If $\nu < \alpha$, G can be represented as $G = X_+G'X_-$, where

$$X_+ = \begin{bmatrix} -\frac{l_-}{f_+} & 0 & -a_-e_\nu & e_\lambda \\ e_\lambda & -\frac{1}{l_-} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -a_-f_+ & f_+e_\alpha \end{bmatrix}$$

is invertible in AP_W^+,

$$X_- = \begin{bmatrix} 0 & 1 & 0 & \frac{e_{-\alpha}}{l_-} \\ 0 & 0 & 0 & 1 \\ 1 & -\frac{l_-e_{-\alpha}}{a_-f_+} & \frac{e_{-\alpha}}{a_-} & -\frac{e_{-2\alpha}}{a_-f_+} \\ 0 & \frac{l_-(1+a_-e_{-\nu})}{a_-f_+} & -\frac{1}{a_-} & \frac{a_-e_{-\lambda}+e_{-\alpha}}{a_-f_+} \end{bmatrix}$$

is invertible in AP_W^-, and

$$G' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e_\nu & 0 & 0 \\ 0 & 0 & e_{\alpha-\nu} & 0 \\ 0 & 0 & 1 + a_-e_{-\nu} & e_{-\alpha} \end{bmatrix}.$$

As in Case 1, G' splits into a direct sum of a diagonal matrix and another matrix of the type (1.4), AP_W factorable due to Theorem 1.2. This implies AP_W factorability of G.
If $\nu > \alpha$, an explicit AP_W factorization of G is delivered by

$$G_+ = \begin{bmatrix} e_{2\nu} - a_-e_\nu & 0 & -\frac{l_-}{f_+} & e_\lambda \\ 0 & -\frac{1}{l_-} & e_\lambda & 0 \\ e_{\nu-\alpha} & 0 & 0 & 1 \\ f_+(e_\nu - a_-) & 0 & 0 & f_+e_\alpha \end{bmatrix},$$

$$G_- = \begin{bmatrix} 1 & -\frac{l_-e_{-\alpha}}{a_-f_+} & \frac{e_{-\alpha}}{a_-} & -\frac{e_{-2\alpha}}{a_-f_+} \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{e_{-\alpha}}{l_-} \\ a_-e_{\alpha-\nu} & \frac{l_-}{a_-f_+} & e_{-\nu} - \frac{1}{a_-} & \frac{e_{-\alpha}}{a_-f_+} \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[e_{\alpha-\nu}, e_\nu, 1, e_{-\alpha}].$$

Case 4: $a_+ = a_- = 0$.
The AP_W factorization of G is given by

$$G_+ = \begin{bmatrix} -\frac{l_-}{f_+} & 0 & e_\lambda & -e_\nu \\ e_\lambda & -\frac{1}{l_-} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & f_+e_\alpha & -f_+ \end{bmatrix}, \quad G_- = \begin{bmatrix} 0 & 1 & 0 & \frac{e_{-\alpha}}{l_-} \\ 0 & 0 & 0 & 1 \\ 1 & 0 & e_{-\lambda} & 0 \\ 0 & -\frac{l_-}{f_+} & 1 & -\frac{e_{-\alpha}}{f_+} \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[1, e_\nu, 1, e_{-\nu}].$$

In Situation 2, we distinguish between the following cases, depending on $D = \det c_1$.
Case 5: $D \neq 0$.
Then $a_- = 0$, and G can be represented as $G = X_+G'X_-$, where

$$X_+ = \begin{bmatrix} 1 & \frac{d_+l_-e_\alpha}{l_+} & 0 & 0 \\ -\frac{f_+}{l_+} & e_\lambda - \frac{l_-}{l_+} & -\frac{e_\nu}{l_-} & \frac{f_+e_\nu}{l_+D} \\ 0 & \frac{d_+(l_+e_\alpha + a_+l_-e_{\alpha-\nu})}{l_+} & -\frac{d_+}{l_-} & \frac{a_+}{D} \\ 0 & l_+e_\alpha + \frac{d_+f_+l_-e_{\alpha-\nu}}{l_+} & -\frac{l_+}{l_-} & \frac{f_+}{D} \end{bmatrix},$$

$$X_- = \begin{bmatrix} 1 & -\frac{d_+l_-e_{-\nu}}{l_+} & 0 & -\frac{d_+e_{-\lambda}}{l_+} \\ 0 & 1 & 0 & \frac{e_{-\alpha}}{l_-} \\ \frac{f_+l_-}{D} & -\frac{a_+l_-^2e_{-\nu}}{D} & \frac{f_+l_-e_{-\lambda}}{D} & 1 - \frac{a_+l_-e_{-\lambda}}{D} \\ 0 & 0 & l_+ & 0 \end{bmatrix}, \quad G' = \begin{bmatrix} e_\lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ l_+ + De_\alpha & 0 & 0 & e_{-\lambda} \end{bmatrix}$$

if $\nu < \alpha$, and

$$X_+ = \begin{bmatrix} \frac{l_+^2}{D^2} & \frac{d_+l_-}{D} & 0 & \frac{e_\nu}{D} - \frac{l_+e_{\nu-\alpha}}{D^2} \\ -\frac{f_+l_+}{D^2} & e_\lambda - \frac{a_+l_-}{D} & -\frac{e_\nu}{l_-} & \frac{f_+e_{\nu-\alpha}}{D^2} \\ 0 & d_+e_\alpha & -\frac{d_+}{l_-} & \frac{a_+}{D} \\ 0 & l_+e_\alpha & -\frac{l_+}{l_-} & \frac{f_+}{D} \end{bmatrix}, \quad G' = \begin{bmatrix} e_{\nu-\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ l_+ + De_\alpha & 0 & 0 & e_{\alpha-\nu} \end{bmatrix},$$

$$X_- = \begin{bmatrix} 1 & -\frac{d_+l_-e_{-\nu}}{l_+} & e_{-\lambda} - a_+e_{-\nu} + \frac{d_+f_+e_{-\nu}}{l_+} & -\frac{d_+e_{-\lambda}}{l_+} \\ 0 & 1 & 0 & \frac{e_{-\alpha}}{l_-} \\ \frac{f_+l_-}{D} & -\frac{a_+l_-^2e_{-\nu}}{D} & \frac{f_+l_-e_{-\lambda}}{D} & 1 - \frac{a_+l_-e_{-\lambda}}{D} \\ 0 & \frac{d_+l_-D}{l_+} & \frac{D^2}{l_+} & \frac{d_+De_{-\alpha}}{l_+} \end{bmatrix}$$

if $\nu > \alpha$.

In both cases, X_+ is invertible in AP_W^+, X_- is invertible in AP_W^-, and G' splits into a direct sum of I_2 with yet another 2×2 matrix G_2 of type (1.4), AP_W factorable due to Theorem 1.2. From AP_W factorability of G_2 we conclude that G' is AP_W factorable as well. This in turn implies the AP_W factorability of G.

Case 6: D=0.

Applying an elementary row operation, we may without loss of generality suppose that $a_+ = d_+ = 0$. However, this changes the upper right entry of c_{-1} so that d_- is now not necessarily zero. In other words, we need to consider matrices (1.4), (1.5) with

$$c_{-1} = \begin{bmatrix} a_- & d_- \\ 0 & l_- \end{bmatrix},\quad c_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\quad c_1 = \begin{bmatrix} 0 & 0 \\ f_+ & l_+ \end{bmatrix}$$

and $l_\pm \neq 0$.

We partition this case into three subcases.

Subcase 6a: $a_-d_- \neq 0$.

If $\nu < \alpha$, G can be represented as $G = X_+G'X_-$, where

$$X_+ = \begin{bmatrix} a_-d_- & 0 & -d_-e_\alpha & \frac{e_\lambda}{l_+} \\ -\frac{d_-f_+(a_-+e_\nu)}{l_+} & -\frac{e_\nu}{l_-} & e_\lambda + \frac{d_-f_+e_\alpha - l_-}{l_+} & -\frac{a_-(l_+e_\lambda - l_-)}{d_-l_+^2} \\ 0 & 0 & 0 & \frac{1}{l_+} \\ -d_-f_+ & -\frac{l_+}{l_-} & l_+e_\alpha & \frac{(d_-f_+ - a_-l_+)e_\alpha}{d_-l_+} \end{bmatrix}$$

is invertible in AP_W^+,

$$X_- = \begin{bmatrix} \frac{a_-}{d_-} & 1 & \frac{a_-e_{-\alpha} - e_{\nu-\alpha}}{a_-d_-} & \frac{e_{\nu-\alpha}}{a_-l_-} \\ 0 & -\frac{l_-^2e_{-\nu}}{l_+} & \frac{f_+l_-}{a_-l_+} & 1 - \frac{a_-l_-e_{-\lambda} + d_-f_+}{a_-l_+} \\ 0 & 0 & \frac{1}{a_-d_-} & -\frac{1}{a_-l_-} \\ l_+(a_-e_{-\nu}+1) & d_-l_+e_{-\nu} & l_+e_{-\lambda} & 0 \end{bmatrix}$$

is invertible in AP_W^-, and

$$G' = \begin{bmatrix} e_{\alpha-\nu} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_-e_{-\nu}+1 & 0 & e_{\nu-\alpha} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

splits into a direct sum of I_2 and a 2×2 matrix $\begin{bmatrix} e_{\alpha-\nu} & 0 \\ a_-e_{-\nu}+1 & e_{\nu-\alpha} \end{bmatrix}$, AP_W factorable due to Theorem 1.2.

When $\nu > \alpha$ the AP_W factorization of G is given by

$$G_+ = \begin{bmatrix} -d_-e_\alpha & 0 & -a_-d_- & \frac{e_\lambda}{l_+} \\ e_\lambda + \frac{d_-f_+e_\alpha - l_-}{l_+} & -\frac{e_\nu}{l_-} & \frac{a_-d_-f_+}{l_+} & -\frac{a_-(l_+e_\lambda - l_-)}{d_-l_+^2} \\ 0 & 0 & 0 & \frac{1}{l_+} \\ l_+e_\alpha & -\frac{l_+}{l_-} & 0 & \frac{(d_-f_+ - a_-l_+)e_\alpha}{d_-l_+} \end{bmatrix},$$

$$G_- = \begin{bmatrix} \frac{a_-(a_-e_{-\nu}+1)}{d_-} & a_-e_{-\nu}+1 & \frac{a_-e_{-\lambda}}{d_-} & \frac{e_{-\alpha}}{l_-} \\ \frac{a_-f_+l_-e_{\alpha-\nu}}{l_+} & \frac{l_-(d_-f_+e_{\alpha-\nu} - l_-e_{-\nu})}{l_+} & \frac{l_-f_+e_{-\nu}}{l_+} & 1 - \frac{l_-e_{-\lambda}}{l_+} \\ -\frac{a_-e_{\alpha-\nu}}{d_-} & -e_{\alpha-\nu} & \frac{1-a_-e_{-\nu}}{a_-d_-} & -\frac{1}{a_-l_-} \\ l_+(1+a_-e_{-\nu}) & d_-l_+e_{-\nu} & l_+e_{-\lambda} & 0 \end{bmatrix}$$

and its partial AP indices equal zero, that is, $\Lambda = I$.

Subcase 6b: $a_- = 0$.

The AP_W factorization of G is delivered by

$$G_+ = \begin{bmatrix} -d_-e_\alpha & 0 & -1 & \frac{e_\lambda}{l_+} \\ e_\lambda - \frac{l_-}{l_+} & -\frac{e_\nu}{l_-} & \frac{f_+}{l_+} & 0 \\ 0 & 0 & 0 & \frac{1}{l_+} \\ l_+e_\alpha - d_-f_+e_{\alpha-\nu} & -\frac{l_+}{l_-} & 0 & \frac{f_+e_\alpha}{l_+} \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 1 & 0 & \frac{e_{-\alpha}}{l_-} \\ 0 & -\frac{l_-^2e_{-\nu}}{l_+} & \frac{f_+l_-e_{-\nu}}{l_+} & 1 - \frac{l_-e_{-\lambda} + d_-f_+e_{-\nu}}{l_+} \\ 0 & 0 & 1 & -\frac{d_-}{l_-} \\ l_+ & d_-l_+e_{-\nu} & l_+e_{-\lambda} & 0 \end{bmatrix}$$

if $\nu < \alpha$, and

$$G_+ = \begin{bmatrix} -d_-e_\alpha & 0 & -1 & \frac{e_\lambda}{l_+} \\ e_\lambda - \frac{d_-f_+e_\alpha - l_-}{l_+} & -\frac{e_\nu}{l_-} & \frac{f_+}{l_+} & 0 \\ 0 & 0 & 0 & \frac{1}{l_+} \\ l_+e_\alpha & -\frac{l_+}{l_-} & 0 & \frac{f_+e_\alpha}{l_+} \end{bmatrix},$$

$$G_- = \begin{bmatrix} 0 & 1 & 0 & \frac{e_{-\alpha}}{l_-} \\ 0 & \frac{l_-(d_-f_+e_{\alpha-\nu} - l_-e_{-\nu})}{l_+} & \frac{f_+l_-e_{-\nu}}{l_+} & 1 - \frac{l_-e_{-\lambda}}{l_+} \\ 0 & 0 & 1 & -\frac{d_-}{l_-} \\ l_+ & d_-l_+e_{-\nu} & l_+e_{-\lambda} & 0 \end{bmatrix}$$

if $\nu > \alpha$; in both cases all partial AP indices of G equal zero.

Subcase 6c: $a_- \neq 0$, $d_- = 0$.

If $\nu < \alpha$, G can be represented as $G = X_+G'X_-$, where

$$X_+ = \begin{bmatrix} -a_- & 0 & e_\lambda & 0 \\ \frac{f_+(a_-+e_\nu)}{l_+} & e_\lambda - \frac{l_-}{l_+} & -\frac{f_+e_\lambda}{l_+} & -\frac{e_\nu}{l_-} \\ 0 & 0 & 1 & 0 \\ f_+ & l_+e_\alpha & 0 & -\frac{l_+}{l_-} \end{bmatrix}$$

is invertible in AP_W^+,

$$X_- = \begin{bmatrix} 1 & 0 & \frac{e_{-\alpha}}{a_-} & 0 \\ 0 & 1 & 0 & \frac{e_{-\alpha}}{l_-} \\ 0 & 0 & -\frac{1}{a_-} & 0 \\ 0 & -\frac{l_-^2e_{-\nu}}{l_+} & \frac{f_+l_-}{a_-l_+} & 1 - \frac{l_-e_{-\lambda}}{l_+} \end{bmatrix}$$

is invertible in AP_W^-, and

$$G' = \begin{bmatrix} e_\alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 + a_-e_{-\nu} & 0 & e_{-\alpha} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

splits into a direct sum of I_2 and the matrix $\begin{bmatrix} e_\alpha & 0 \\ 1+a_-e_{-\nu} & e_{-\alpha} \end{bmatrix}$, AP_W factorable due to Theorem 1.2.

When $\nu > \alpha$, G is AP_W factorable with zero partial AP indices and

$$G_+ = \begin{bmatrix} a_- & 0 & e_\lambda - a_-e_\alpha & 0 \\ -\frac{f_+(a_-+e_\nu)}{l_+} & e_\lambda - \frac{l_-}{l_+} & \frac{a_-f_+e_\alpha}{l_+} & -\frac{e_\lambda}{l_-} \\ 0 & 0 & 1 & 0 \\ -f_+ & l_+e_\alpha & f_+e_\alpha & -\frac{l_+}{l_-} \end{bmatrix},$$

$$G_- = \begin{bmatrix} a_-e_{\alpha-\nu} & 0 & e_{-\nu} - \frac{1}{a_-} & 0 \\ 0 & 1 & 0 & \frac{e_{-\alpha}}{l_-} \\ 1+a_-e_{-\nu} & 0 & e_{-\lambda} & 0 \\ 0 & -\frac{l_-^2e_{-\nu}}{l_+} & \frac{f_+l_-}{a_-l_+} & 1-\frac{l_-e_{-\lambda}}{l_+} \end{bmatrix}$$

Appendix E.

$$c_{\pm 1} = \begin{bmatrix} d_\pm & f_\pm & b_\pm \\ 0 & d_\pm & 0 \\ 0 & a_\pm & 0 \end{bmatrix}, \quad c_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where

$$a_+a_-b_+b_- \neq 0 \quad \text{and} \quad a_+b_- = a_-b_+.$$

In this appendix we shall use the following abbreviations:

$$A = a_-d_+ - a_+d_- \qquad B = a_-f_+ - a_+f_-$$
$$C = b_-d_+ - b_+d_- \qquad D = b_-f_+ - b_+f_-.$$

Case 1: $C \neq 0$.

The AP_W factorization of G is given by

$$G_+ = \begin{bmatrix} -b_-e_\nu & 0 & e_\alpha & \frac{b_+}{C} & 0 & 0 \\ 0 & e_{\lambda+\nu} - \frac{a_-}{a_+}e_\nu & 0 & 0 & -\frac{e_\alpha}{a_-} - \frac{a_+e_\lambda - a}{a_-A} & -e_\nu \\ e_{2\nu} + d_-e_\nu & \frac{a_-f_+e_\nu}{a_+b_+} - \frac{f_-e_{\lambda+\nu}}{b_-} & -\frac{d_-e_\alpha}{b_-} - \frac{1}{b_+} & -\frac{b_+e_\nu + d_+b_-}{b_-C} & \frac{f_+e_\alpha}{a_-b_+} - \frac{a_-b_-f_+ - a_+b_+f_-e_\lambda}{a_-b_-b_+A} & \frac{f_-e_\nu}{b_+} \\ b_+e_\nu + b_+d_- - d_+b_- & \frac{De_\lambda}{b_-} & \frac{Be_{\alpha-\nu}}{b_-} & -\frac{b_+^2}{b_-C} & -\frac{a_+De_\alpha}{a_-b_-A} & 0 \\ 0 & e_\nu + d_- + d_+e_\lambda - \frac{a_-d_+}{a_+} & 0 & 0 & \frac{d_+e_{\alpha-\nu}}{a_-} - \frac{a_+ + a_+d_+e_\alpha}{a_-A} & -d_+ \\ 0 & a_+e_\lambda & 0 & 0 & -\frac{a_+e_{\alpha-\nu}}{a_-} - \frac{a_+^2e_\alpha}{a_-A} & -a_+ \end{bmatrix},$$

$$G_- = \begin{bmatrix} \frac{d_-}{b_-} - \frac{d_-e_{-\alpha} + e_{\nu-\alpha}}{C} & \frac{f_-}{b_-} - \frac{f_-e_{-\alpha}}{C} & 1 - \frac{b_-e_{-\alpha}}{C} & \frac{e_{-\alpha}}{b_-} - \frac{b_-e_{-2\alpha} - b_+e_{\nu-\alpha}}{C} & 0 & \frac{De_{\nu-\alpha}}{a_-b_-C} \\ 0 & 1 & 0 & 0 & -\frac{a_+e_{-\alpha}}{A} & \frac{a_+e_{\nu-\alpha} + a_-d_+e_{-\alpha}}{a_-A} \\ 1+d_-e_{-\nu} & f_-e_{-\nu} & b_-e_{-\nu} & e_{-\lambda} & 0 & 0 \\ -\frac{b_-(1+d_-e_{-\nu})}{b_+} & -\frac{b_-f_-e_{-\nu}}{b_+} & -\frac{b_-^2e_{-\nu}}{b_+} & 1 - \frac{b_-e_{-\lambda}}{b_+} & 0 & \frac{D}{a_-b_+} \\ 0 & 0 & 0 & 0 & -a_-e_{-\nu} & 1+d_-e_{-\nu} \\ 0 & -\frac{a_-}{a_+} & 0 & 0 & 1 & -\frac{a_+d_- + a_-e_{-\alpha}}{a_+a_-} \end{bmatrix},$$

$$\Lambda = \text{diag}[e_{\alpha-\nu}, e_{-\nu}, e_{-\nu}, e_\nu, e_\nu, e_\nu, e_{\nu-\alpha}, e_{-\nu}]$$

if $\nu < \alpha$, and by

$$G_+ = \begin{bmatrix} -b_- & 0 & \frac{b_+e_\alpha}{C} & e_\nu & 0 & 0 \\ 0 & \frac{a_+e_\lambda - a_-}{A} & 0 & 0 & -\frac{e_\lambda}{a_-} & e_{2\nu} - d_-e_\nu + \frac{a_-d_+e_\nu - a_-e_{\nu-\alpha}}{a_+} \\ e_\nu + \frac{b_-d_+}{b_+} & \frac{a_-b_-f_+ - a_+f_-b_+e_\lambda}{b_-b_+A} & -\frac{1+d_+e_\alpha}{C} & -\frac{e_{\nu-\alpha}}{b_+} - \frac{d_-e_\nu}{b_-} & \frac{f_+e_\lambda}{a_-b_+} & \frac{a_-b_-f_+e_{\nu-\alpha} - b_-f_+Ae_\nu - a_+b_+f_-e_{2\nu}}{a_+b_-b_+} \\ b_+ & \frac{Da_+e_\alpha}{Ab_-} & 0 & \frac{C}{b_-} & 0 & \frac{De_\nu}{b_-} \\ 0 & \frac{a_+(1+d_+e_\alpha)}{A} & 0 & 0 & -\frac{1+d_+e_\alpha}{a_-} & d_+e_\nu - d_-d_+ + e_{\nu-\alpha} + \frac{a_-d_+^2}{a_+} \\ 0 & \frac{a_+^2e_\alpha}{A} & 0 & 0 & -\frac{a_+e_\alpha}{a_-} & A + a_+e_\nu \end{bmatrix}$$

$$G_- = \begin{bmatrix} \frac{d_-}{b_-} & \frac{f_-}{b_-} & 1 & \frac{e_{-\alpha}}{b_-} & 0 & 0 \\ 0 & -\frac{a_-}{a_+} & 0 & 0 & 1 & -\frac{d_-}{a_-} - \frac{e_{-\alpha}}{a_+} \\ \frac{d_-Ce_{\alpha-\nu} - b_- - b_-d_-e_{-\nu}}{b_+} & \frac{f_-(Ce_{\alpha-\nu} - b_-e_{-\nu})}{b_+} & \frac{b_-(Ce_{\alpha-\nu} - b_-e_{-\nu})}{b_+} & 1 - d_-e_{-\nu} + \frac{b_-d_+e_{-\nu} - b_-e_{-\lambda}}{b_+} & 0 & \frac{D}{a_-b_+} \\ 1 \div b_-\frac{e_{-\alpha} + d_-e_{-\lambda}}{C} & \frac{b_-f_-e_{-\lambda}}{C} & \frac{b_-^2e_{-\lambda}}{C} & \frac{b_-e_{-(\lambda+\alpha)} - b_+e_{-\alpha}}{C} & 0 & -\frac{De_{-\alpha}}{a_-C} \\ 0 & \frac{a_-Ae_{\alpha-\nu}}{a_+} & 0 & 0 & -a_-e_{-\nu} & 1 + \frac{a_-d_+e_{-\nu}}{a_+} \\ 0 & 1 + \frac{a_-e_{-\alpha}}{A} & 0 & 0 & -\frac{a_+e_{-\alpha}}{A} & \frac{d_+e_{-\alpha} + e_{-2\alpha}}{A} \end{bmatrix}$$

$$\Lambda = \mathrm{diag}[e_\alpha, e_{-\alpha}, e_{\nu-\alpha}, e_\alpha, e_{-\alpha}, e_{\alpha-\nu}]$$

if $\nu > \alpha$.

Observe that the condition $a_+b_- = a_-b_+$ was not used in the above factorizations.

Case 2: $C = 0$.

From this and the equality $a_+b_- = a_-b_+$ it follows that $d_-b_+ = d_+b_- = d_-a_+ = d_+a_-$, which allows us (by using elementary row and column operations) to suppose without loss of generality that $d_+ = d_- = 0$. Under these additional conditions, the AP_W factorization of G is given by

$$G_+ = \begin{bmatrix} 0 & 0 & e_\alpha & -b_- & 0 & 0 \\ 0 & e_\lambda - \frac{a_-}{a_+} & 0 & 0 & -\frac{e_\lambda}{a_-} & -e_\nu \\ -\frac{e_\nu}{b_-} & \frac{a_-b_-f_+ - a_+b_+f_-e_\lambda}{a_+b_-b_+} & -\frac{1}{b_+} & e_\nu & \frac{f_+e_\lambda}{a_-b_+} & \frac{f_+e_\nu}{b_+} \\ -\frac{b_+}{b_-} & \frac{De_\alpha}{b_-} & 0 & b_+ & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{a_-} & 0 \\ 0 & a_+e_\alpha & 0 & 0 & -\frac{a_+e_\alpha}{a_-} & -a_+ \end{bmatrix},$$

$$G_- = \begin{bmatrix} -\frac{b_-}{b_+} & -\frac{b_-f_-e_{-\nu}}{b_+} & -\frac{b_-^2e_{-\nu}}{b_+} & 1 - \frac{b_-e_{-\lambda}}{b_+} & 0 & \frac{D}{a_-b_+} \\ 0 & 1 & 0 & 0 & 0 & \frac{e_{-\alpha}}{a_-} \\ 1 & f_-e_{-\nu} & b_-e_{-\nu} & e_{-\lambda} & 0 & 0 \\ 0 & \frac{f_-}{b_-} & 1 & \frac{e_{-\alpha}}{b_-} & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_-e_{-\nu} & 1 \\ 0 & -\frac{a_-}{a_+} & 0 & 0 & 1 & -\frac{e_{-\alpha}}{a_+} \end{bmatrix},$$

$$\Lambda = \mathrm{diag}[1, 1, e_\nu, e_\alpha, e_{-\alpha}, e_{-\nu}].$$