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EXPLICIT UPPER BOUNDS FOR EXPONENTIAL SUMS OVER PRIMES

HEDI DABOUSSI AND JOËL RIVAT

Dedicated to the memory of Chen Jing Run

ABSTRACT. We give explicit upper bounds for linear trigonometric sums over primes.

1. INTRODUCTION

In 1937 I.M. Vinogradov [12] proved that every sufficiently large odd number is the sum of three prime numbers. Later Chen and Wang [2] gave a lower bound for the result of Vinogradov, which is very large, around 10^{43000}. The method used is the Hardy-Littlewood circle method, and the following sums play an important role in the proof:

$$\sum_{p\le x} \mathrm{e}(\alpha p), \qquad S(x,\alpha) = \sum_{n\le x} \Lambda(n)\ \mathrm{e}(n\alpha),$$

where Λ is the function of Von Mangoldt and $\mathrm{e}(\alpha) = e^{2i\pi\alpha}$.

In [1] Chen proved that if $\alpha = \frac{a}{q} + \frac{\beta}{q^2}$, $|\beta| \le 1$, $q \le x$, then

$$\left|\sum_{p\le x} \mathrm{e}(\alpha p)\right| \le 1.2\ x\,(\log x)^{3/4} \log\log x \left(\sqrt{\frac{5}{q} + \frac{q\log q}{x}} + \sqrt{\log q}\ \exp -\frac{1}{2}\sqrt{\log x}\right).$$

More recently in [3] Chen and Wang proved that

$$|S(x,\alpha)| \le 0.177\ \frac{x}{\sqrt{q}}\,(\log x)^3 + 3.8\ x^{4/5}(\log x)^{2.2} + 0.08\ \sqrt{xq}\,(\log x)^{3.5}.$$

Our purpose is to improve on these two estimates. By a classical elementary transformation it suffices to consider $S(x,\alpha)$.

In order to estimate this sum, a useful identity has been proved by R.C. Vaughan [11]. Recently Daboussi [4] gave another identity, which has the advantage of involving nice coefficients. This permits us to give a new explicit upper bound for $|S(x,\alpha)|$.

In this paper we will need sharp versions of some classical inequalities which have their own independent interest. We will prove

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TABLE 1. Upper bounds for $x^{-1}|S(x,\alpha)|$ when $q = (\log x)^3$

x	$x^{-1}\|S(x,\alpha)\|$
10^{200}	0.385
10^{300}	0.293
10^{400}	0.241
10^{500}	0.207
10^{1000}	0.129
10^{2000}	0.080
10^{5000}	0.042
10^{10000}	0.026
10^{20000}	0.016
10^{43000}	0.010

Theorem 1. *For $\alpha = \frac{a}{q} + \frac{\beta}{q^2}$, $|\beta| \le 1$, $q \le x$, we have*

$$\begin{aligned}|S(x,\alpha)| \quad &\le \quad 14.86\sqrt{\log\log x + 0.5}\; x\,(\log x)^{3/4}\sqrt{\frac{1}{q} + \frac{q\log 4q}{\pi x}} \\ &\quad + 6.45\sqrt{\log\log x + 0.5}\; x\,(\log x)^{5/4}\exp\left(-\frac{1}{2}\sqrt{\log x}\right).\end{aligned}$$

From this theorem we can compute numerical upper bounds for $x^{-1}|S(x,\alpha)|$ (cf. Table 1) with the choice $q = (\log x)^3$ for which the result of Chen and Wang is not even as good as the trivial upper bound.

Definitions and notations. For x real we will denote by $\lfloor x \rfloor$ the greatest integer $\le x$, $\{x\}$ the fractional part of x, $\lceil x \rceil$ the smallest integer $\ge x$, $\|x\|$ the distance from x to the nearest integer, $\lfloor x \rceil$ the smallest integer n such that $|x-n| \le 1/2$ (n is unique if $\{x\} \ne 1/2$). The letter p denotes always a prime number, $\pi(x)$ denotes the number of primes $\le x$. We denote by μ and φ the Möbius and Euler functions, respectively. The functions $\Omega(n)$ and $\omega(n)$ count the number of prime factors of n, respectively, with and without multiplicity. We define the functions u_z and v_z by $u_z(m) = 1$ if $(\forall p,\ p\,|\,m \Rightarrow p > z)$ and $u_z(m) = 0$ otherwise, and $v_z(m) = 1$ if $(\forall p,\ p\,|\,m \Rightarrow p \le z)$ and $v_z(m) = 0$ otherwise.

2. VINOGRADOV TYPE LEMMAS

In [1] Chen improved Vinogradov Lemmas 8a and 8b [13]. In this section, we further improve the results of Chen.

Lemma 1. *Let $x \in \mathbb{R}$, $\alpha = \frac{a}{q} + \frac{\beta}{q^2}$, $|\beta| \le 1$, $(a,q) = 1$, $U > 0$. We then have*

$$\sum_{x<n\le x+q} \min\left(U, \frac{1}{|\sin(\pi\alpha n)|}\right) \le 2\,U + \frac{2}{\pi}\,q\log 4q.$$

Remark 1. This is the analog of Vinogradov Lemma 8a. Chen obtained $5\,U + q\log q$. The factor 2 instead of 5 is obtained by using $t = \lfloor t \rceil + \delta$ with $|\delta| \le 1/2$ which is more precise than the classical $t = \lfloor t \rfloor + \{t\}$. The factor $2/\pi$ has been obtained by dealing directly with $(\sin t)^{-1}$ without using the classical inequality $\sin t \ge 2t/\pi$ for

$0 \le t \le \pi/2$. Indeed we simply used the fact that $(\log\tan(t/2))' = (\sin t)^{-1}$. We acknowledge the referee's improvement of this lemma (see below).

Proof. The result is trivial for $q \le 2$. We therefore suppose $q \ge 3$.

Let $m_0 = \lfloor x \rfloor + \lfloor (q+1)/2 \rfloor$. We have

$$\sum_{x<n\le x+q} \min\left(U, \frac{1}{|\sin(\pi\alpha n)|}\right) = \sum_{-q/2<m\le q/2} \min\left(U, \frac{1}{|\sin(\pi\alpha(m_0+m))|}\right).$$

Now writing

$$b = \left\lfloor a m_0 + \frac{\beta m_0}{q} \right\rceil \quad \text{and} \quad b+\delta = a m_0 + \frac{\beta m_0}{q}$$

(hence $|\delta| \le 1/2$), we obtain

$$\alpha(m_0+m) = \frac{1}{q}\left(am + am_0 + \frac{\beta m_0}{q} + \frac{\beta m}{q}\right) = \frac{1}{q}(am+b) + \frac{1}{q}\left(\delta + \frac{\beta m}{q}\right).$$

When m runs through the integers in the interval $-q/2 < m \le q/2$, $am+b$ runs through a complete set of residue classes modulo q. We introduce r such that $am + b \equiv r \bmod q$ and $-q/2 < r \le q/2$. Using $\left|\delta + \frac{\beta m}{q}\right| \le 1$, we get for $|r| \ge 2$

$$\min\left(U, \frac{1}{|\sin(\pi\alpha(m_0+m))|}\right) \le \sin\left(\frac{\pi}{q}(|r|-1)\right).$$

For $r = \pm 1$, the referee observed that

$$\|\alpha(m_0+m)\| = \left\|\frac{f(r)}{q}\right\|,$$

where $f(r) = r + \delta + \theta(r)$, with $|\theta(r)| \le \frac{1}{2}$. It follows that $f(1) - f(-1) \ge 1$ so that

$$\max\left\{\left\|\frac{f(1)}{q}\right\|, \left\|\frac{f(-1)}{q}\right\|\right\} \ge \frac{1}{2q}.$$

Thus one of the two terms for $r = \pm 1$ can be bounded by $|\sin(\pi/2q)|^{-1}$. Hence we obtain

$$\sum_{x<n\le x+q} \min\left(U, \frac{1}{|\sin(\pi\alpha n)|}\right) \le 2\,U + \frac{1}{\sin\left(\frac{\pi}{2q}\right)} + 2\sum_{2\le r\le q/2} \frac{1}{\sin\left(\frac{\pi}{q}(r-1)\right)}$$

(the sum on the right hand side is empty for $q = 3$).

Using the convexity of the function $t \longmapsto 1/\sin(\pi t/q)$ for $0 < t \le q/2$, we obtain

$$\sum_{1\le r\le \frac{q}{2}-1} \frac{1}{\sin\frac{\pi}{q}r} \le \int_{\frac{1}{2}}^{\frac{q-1}{2}} \frac{dt}{\sin\frac{\pi t}{q}} \le \frac{q}{\pi}\log\cot\frac{\pi}{4q} \le \frac{q}{\pi}\log\frac{4q}{\pi},$$

and we have for $q \ge 3$

$$\frac{1}{\sin\left(\frac{\pi}{2q}\right)} + \frac{2q}{\pi}\log\frac{4q}{\pi} \le \frac{2q}{\pi}\log 4q,$$

which completes the proof of Lemma 1. □

Lemma 2. *Let $N \ge 1$, $\alpha = \frac{a}{q} + \frac{\beta}{q^2}$, $|\beta| \le 1$, $(a,q) = 1$, $U > 0$. We then have*

$$\sum_{1\le n\le N} \min\left(U, \frac{1}{|\sin(\pi\alpha n)|}\right) \le \left\lceil\frac{N}{q}\right\rceil\left(2\,U + \frac{2}{\pi}\,q\log 4q\right).$$

Proof. We divide the interval $1 \leq n \leq N$ into subintervals $kq+1 \leq n \leq (k+1)q$, for which we apply Lemma 1. There are at most $\left\lceil \frac{N}{q} \right\rceil$ such subintervals. □

Lemma 3. *Let* $N \geq 1$, $\alpha = \frac{a}{q} + \frac{\beta}{q^2}$, $|\beta| \leq 1$, $(a,q)=1$, $x > 0$. *We then have*

$$\sum_{1 \leq n \leq N} \min\left(\frac{x}{n}, \frac{1}{|\sin(\pi\alpha n)|}\right) \leq 2\,\frac{x}{q}\log\left(\frac{8N}{q}+4\right) + \frac{2}{\pi}\,N\log 4q + \frac{3}{\pi}\,q\log 5q.$$

Remark 2. This is the analog of Vinogradov Lemma 8b.

Proof. We can assume without loss of generality that N is an integer.

Using the convexity of $t \longmapsto \frac{1}{t}$ for $t > 0$, we obtain for $N \geq 1$

$$\sum_{1 \leq n \leq N} \frac{1}{n} \leq \int_{\frac{1}{2}}^{N+\frac{1}{2}} \frac{dt}{t} = \log(2N+1).$$

This proves the result for $q \leq 2$. We can now suppose $q \geq 3$.

Writing $K = \left\lceil \frac{N}{q} - \frac{1}{2} \right\rceil$, we have $K \geq \frac{N}{q} - \frac{1}{2}$ and $Kq + \frac{q}{2} \geq N$. Hence

$$\sum_{1 \leq n \leq N} \min\left(\frac{x}{n}, \frac{1}{|\sin(\pi\alpha n)|}\right) \leq \sum_{k=0}^{K} S_k,$$

where

$$S_0 = \sum_{1 \leq n \leq q/2} \min\left(\frac{x}{n}, \frac{1}{|\sin(\pi\alpha n)|}\right),$$

and for $k \geq 1$

$$S_k = \sum_{kq-q/2 < n \leq kq+q/2} \min\left(\frac{x}{n}, \frac{1}{|\sin(\pi\alpha n)|}\right).$$

For $1 \leq n \leq q/2$, we have $an \not\equiv 0 \bmod q$ and $\alpha n = \frac{an}{q} + \frac{1}{q}\frac{\beta n}{q}$ with $\left|\frac{\beta n}{q}\right| \leq \frac{1}{2}$. Hence for $an \equiv r \bmod q$ with $-q/2 \leq r \leq q/2$ and $r \neq 0$, we have

$$|\sin(\pi\alpha n)| \geq \left|\sin\left(\frac{\pi}{q}\left(|r| - \frac{1}{2}\right)\right)\right|$$

and

$$\begin{aligned} S_0 &\leq 2\sum_{1 \leq r \leq q/2} \left|\sin\left(\frac{\pi}{q}\left(r - \frac{1}{2}\right)\right)\right|^{-1} \\ &\leq 2\left|\sin\left(\frac{\pi}{2q}\right)\right|^{-1} + 2\int_{\frac{3}{2}}^{\frac{q+1}{2}} \frac{dt}{\sin\left(\frac{\pi}{q}\left(t - \frac{1}{2}\right)\right)} \\ &\leq \frac{2}{\sin\left(\frac{\pi}{2q}\right)} + \frac{2q}{\pi}\log\cot\left(\frac{\pi}{2q}\right) \\ &\leq \frac{2q}{\pi}\left(2\frac{\frac{\pi}{2q}}{\sin\left(\frac{\pi}{2q}\right)} + \log\left(\frac{2q}{\pi}\right)\right) \\ &\leq \frac{2q}{\pi}\log 5q \quad \text{for } q \geq 4. \end{aligned}$$

For $q = 3$ we also have $S_0 \le 2|\sin \frac{\pi}{2q}|^{-1} = 4 \le \frac{2q}{\pi} \log 5q$. This proves the result for $K < 1$, so from now we assume that $K \ge 1$.

By Lemma 1 we have

$$\begin{aligned}
\sum_{1\le k\le K} S_k &\le \sum_{1\le k\le K} \sum_{kq-q/2<n\le kq+q/2} \min\left(\frac{x}{q(k-\frac{1}{2})}, \frac{1}{|\sin(\pi\alpha n)|}\right) \\
&\le \sum_{1\le k\le K} \left(2\,\frac{x}{q(k-\frac{1}{2})} + \frac{2}{\pi} q \log 4q\right) \\
&\le \frac{2}{\pi} K q \log 4q + \frac{2x}{q} \sum_{1\le k\le K} \frac{1}{(k-\frac{1}{2})}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{1\le k\le K} S_k &\le \frac{2}{\pi} K q \log 4q + \frac{2x}{q}\left(2 + \int_{\frac{3}{2}}^{K+\frac{1}{2}} \frac{dt}{(t-\frac{1}{2})}\right) \\
&\le \frac{2}{\pi} K q \log 4q + \frac{2x}{q}(2 + \log K) \\
&\le \frac{2}{\pi}\left(\frac{N}{q} + \frac{1}{2}\right) q \log 4q + \frac{2x}{q} \log\left(e^2 K\right) \\
&\le \frac{2}{\pi} N \log 4q + \frac{q}{\pi} \log 4q + \frac{2x}{q} \log\left(e^2 K\right).
\end{aligned}$$

Finally,

$$\begin{aligned}
&\sum_{1\le n\le N} \min\left(\frac{x}{n}, \frac{1}{|\sin(\pi\alpha n)|}\right) \\
&\le \frac{2}{\pi} N \log 4q + \frac{2q}{\pi}\left(\log q + \log 5 + \frac{\log q}{2} + \frac{\log 4}{2}\right) + \frac{2x}{q} \log\left(e^2 K\right) \\
&\le \frac{2}{\pi} N \log 4q + \frac{3q}{\pi}\left(\log q + \frac{2\log 5}{3} + \frac{\log 4}{3}\right) + \frac{2x}{q} \log\left(e^2 K\right) \\
&\le \frac{2}{\pi} N \log 4q + \frac{3q}{\pi} \log 5q + \frac{2x}{q} \log\left(e^2 K\right) \\
&\le \frac{2}{\pi} N \log 4q + \frac{3q}{\pi} \log 5q + \frac{2x}{q} \log\left(\frac{8N}{q} + 4\right),
\end{aligned}$$

which completes the proof. □

3. Rankin's method

Elliott [6, pages 81-83] has given an effective version of Rankin's method. In this section we generalize and improve his results numerically.

Lemma 4. *Let $z \ge 2$, f a multiplicative function with $f \ge 0$, and*

$$S = \sum_{p\le z} \frac{f(p)}{1+f(p)} \log p.$$

We assume $S > 0$ and write $K(t) = \log t - 1 + \frac{1}{t}$ for $t \ge 1$.

For any y with $\log y \geq S$ we have

$$\begin{aligned}\sum_{n>y} v_z(n)\mu^2(n)f(n) &\leq \left(\prod_{p\leq z}(1+f(p))\right)\exp\left(-\frac{\log y}{\log z}K\left(\frac{\log y}{S}\right)\right),\\ \sum_{n\leq y} v_z(n)\mu^2(n)f(n) &\geq \left(\prod_{p\leq z}(1+f(p))\right)\left(1-\exp\left(-\frac{\log y}{\log z}K\left(\frac{\log y}{S}\right)\right)\right).\end{aligned}$$

In particular for any y with $\log y \geq 7S$ we have

$$\begin{aligned}\sum_{n>y} v_z(n)\mu^2(n)f(n) &\leq \left(\prod_{p\leq z}(1+f(p))\right)\exp\left(-\frac{\log y}{\log z}\right),\\ \sum_{n\leq y} v_z(n)\mu^2(n)f(n) &\geq \left(\prod_{p\leq z}(1+f(p))\right)\left(1-\exp\left(-\frac{\log y}{\log z}\right)\right).\end{aligned}$$

Proof. The special case for $\log y \geq 7S$ is a direct consequence of the general case $\log y \geq S$, as for all $t \geq 7$ we have $K(t) \geq K(7) \geq 1$.

We note that

$$\sum_{n\leq y} v_z(n)\mu^2(n)f(n) + \sum_{n>y} v_z(n)\mu^2(n)f(n) = \prod_{p\leq z}(1+f(p)),$$

which shows that the required lower bound for the first sum will follow from the required upper bound for the second sum.

For all $\eta \geq 0$ we have

$$\begin{aligned}\sum_{n>y} v_z(n)\mu^2(n)f(n) &\leq \sum_{n=1}^{\infty} v_z(n)\mu^2(n)f(n)\left(\frac{n}{y}\right)^{\eta}\\ &\leq y^{-\eta}\prod_{p\leq z}\left(1+f(p)\,p^{\eta}\right).\end{aligned}$$

Now

$$\prod_{p\leq z}\left(1+f(p)\,p^{\eta}\right) = \left(\prod_{p\leq z}(1+f(p))\right)\left(\prod_{p\leq z}\left(1+\frac{f(p)}{1+f(p)}\,(p^{\eta}-1)\right)\right).$$

Using $\log(1+u) \leq u$ for $u \geq 0$ we get

$$\prod_{p\leq z}\left(1+\frac{f(p)}{1+f(p)}\,(p^{\eta}-1)\right) \leq \exp\left(\sum_{p\leq z}\frac{f(p)}{1+f(p)}\,(p^{\eta}-1)\right),$$

$$\begin{aligned}
\sum_{p\le z}\frac{f(p)}{1+f(p)}\left(p^{\eta}-1\right) &\le \sum_{p\le z}\frac{f(p)}{1+f(p)}\left(\exp(\eta\log p)-1\right)\\
&\le \sum_{p\le z}\frac{f(p)}{1+f(p)}\sum_{k=1}^{\infty}\frac{(\eta\log p)^k}{k!}\\
&\le \sum_{k=1}^{\infty}\frac{\eta^k(\log z)^{k-1}}{k!}\sum_{p\le z}\frac{f(p)}{1+f(p)}\log p\\
&\le \frac{S}{\log z}\sum_{k=1}^{\infty}\frac{\eta^k(\log z)^k}{k!}\\
&\le \frac{S}{\log z}\left(\exp(\eta\log z)-1\right).
\end{aligned}$$

Writing $\nu=\eta\log z$ we get

$$y^{-\eta}\prod_{p\le z}\left(1+\frac{f(p)}{1+f(p)}\left(p^{\eta}-1\right)\right)\le\exp\left(\frac{S}{\log z}\left(\exp(\nu)-1-\nu\frac{\log y}{S}\right)\right).$$

The last inequality is valid for any $\nu\ge 0$, in particular for $\nu=\log\left(\frac{\log y}{S}\right)$. Hence

$$\begin{aligned}
&\sum_{n>y} v_z(n)\mu^2(n)f(n)\\
&\le \left(\prod_{p\le z}(1+f(p))\right)\exp\left(\frac{S}{\log z}\left(\frac{\log y}{S}-1-\frac{\log y}{S}\log\left(\frac{\log y}{S}\right)\right)\right)\\
&\le \left(\prod_{p\le z}(1+f(p))\right)\exp\left(-\frac{\log y}{\log z}K\left(\frac{\log y}{S}\right)\right).
\end{aligned}$$

□

4. Effective inequalities

Lemma 5. *For all $x>1$ we have*

$$\pi(2x)-\pi(x)<\frac{x}{\log x}.$$

Proof. P. Dusart [5] improved some results of [9] and proved for $x\ge 60184$ that

$$\frac{x}{\log(x)-1}<\pi(x)<\frac{x}{\log(x)-1.1}.$$

This implies for $x\ge 60184$ that

$$\pi(2x)-\pi(x)\le\frac{2x}{\log x-0.41}-\frac{x}{\log x-1}\le\frac{x}{\log x}\left(1-\frac{0.016}{\log x}\right)<\frac{x}{\log x}$$

using the inequalities $1+u<\frac{1}{1-u}<1+\frac{6}{5}u$ (valid for $0<u<1/6$). The result can be easily extented for all $x>1$ by computer evidence. □

Remark 3. We note that the result of this lemma is sharp for $x=113/2$ for which

$$\pi(113)-\pi(113/2)=14<\frac{113/2}{\log(113/2)}=14.0051\ldots.$$

Lemma 6. *For $z \geq 2$ we have*

$$\sum_{q \leq z} \left(1 + \frac{q}{z}\right)^{-1} \frac{\mu^2(q)}{\varphi(q)} \geq \log z.$$

Proof. By Lemma 8 of Montgomery-Vaughan [7] we have for $z \geq 100$

$$\sum_{q \leq z} \left(1 + \frac{q}{z}\right)^{-1} \frac{\mu^2(q)}{\varphi(q)} \geq \log z + 0.361$$

and the result follows by a direct computation for $2 \leq z < 100$. □

Lemma 7. *For $2 \leq z \leq x$ we have*

$$\sum_{x < m \leq 2x} u_z(m) \leq \frac{x}{\log z}.$$

Proof. Suppose first that $\sqrt{2x} \leq z \leq x$. For $x < m \leq 2x$, we have $u_z(m) = 1$ if and only if m is prime. Using Lemma 5 we obtain

$$\sum_{x < m \leq 2x} u_z(m) = \pi(2x) - \pi(x) \leq \frac{x}{\log x} \leq \frac{x}{\log z}.$$

Hence we can suppose $z < \sqrt{2x}$.

By Corollary 1 of Montgomery-Vaughan [7] we have for any positive number z,

$$\sum_{x < m \leq 2x} u_z(m) \leq x \left(\sum_{q \leq z} \left(1 + \frac{3qz}{2x}\right)^{-1} \frac{\mu^2(q)}{\varphi(q)} \right)^{-1},$$

and using Lemma 6 we obtain for $z \leq \sqrt{\frac{2}{3}x}$

$$\sum_{x < m \leq 2x} u_z(m) \leq x \left(\sum_{q \leq z} \left(1 + \frac{q}{z}\right)^{-1} \frac{\mu^2(q)}{\varphi(q)} \right)^{-1} \leq \frac{x}{\log z}.$$

Thus we can suppose $\sqrt{\frac{2}{3}x} < z < \sqrt{2x}$.

For $x > 15$ we have $(2x)^{1/3} < \sqrt{\frac{2}{3}x} < z < \sqrt{2x}$ and

$$\sum_{x < m \leq 2x} u_z(m) = \pi(2x) - \pi(x) + \pi_2(2x, z) - \pi_2(x, z),$$

where

$$\pi_2(x, z) = \#\{n \leq x,\ \Omega(n) = 2,\ p \mid n \Longrightarrow p > z\}.$$

We have using Lemma 5

$$\pi_2(2x, z) - \pi_2(x, z) \leq \sum_{z < p \leq \sqrt{2x}} \left(\pi\left(\frac{2x}{p}\right) - \pi\left(\frac{x}{p}\right) \right) \leq \sum_{z < p \leq \sqrt{2x}} \frac{x/p}{\log(x/p)}.$$

For $x > 15$ we have $x/z > e$ and the function $t \longmapsto t/\log t$ is increasing for $t > e$. Hence

$$\pi_2(2x, z) - \pi_2(x, z) \leq \frac{x/z}{\log(x/z)} \left(\pi(\sqrt{2x}) - \pi(z) \right) \leq \frac{x/z}{\log(x/z)} \left(\pi(2z) - \pi(z) \right),$$

and using Lemma 5 we obtain

$$\pi_2(2x,z)-\pi_2(x,z)\le \frac{x}{\log(x/z)\log z}.$$

Therefore we have for $x>15$

$$\sum_{x<m\le 2x} u_z(m)\le \frac{x}{\log x}+\frac{x}{\log(x/z)\log z},$$

and using the inequality $z<\sqrt{2x}$ we obtain for $x>200$

$$\sum_{x<m\le 2x} u_z(m)\le \frac{x}{\log z}\left(\frac{\log\sqrt{2x}}{\log x}+\frac{1}{\log\sqrt{x/2}}\right)\le \frac{x}{\log z}$$

and it suffices to show the result for $x\le 200$ and $z<\sqrt{2x}$, which can be verified easily by computer. This completes the proof of Lemma 7. □

Corollary 1. *For $2\le z\le x$ we have*

$$\sum_{x\le m<2x} u_z(m)\le \frac{x}{\log z}.$$

Proof. If x is not an integer or if x is an integer and $z=x$ (in this case $u_z(x)=0$), we have

$$\sum_{x\le d<2x} u_z(d)\le \sum_{x<d\le 2x} u_z(d)\le \frac{x}{\log z}.$$

If x is an integer and $z<x$, we have

$$\sum_{x\le d<2x} u_z(d)=\sum_{x^-<d\le 2x^-} u_z(d)\le \frac{x}{\log z}.$$

□

Lemma 8. *For $x\ge 2$ and $1\le h\le x$ we have*

$$\sum_{n\le x}\Lambda(n)\Lambda(n+h)\le 15\ x\ (\log\log x+0.5).$$

Proof. If h is odd, we have $\Lambda(n)\Lambda(n+h)=0$ if n is not a power of 2. Hence, when h is odd,

$$\sum_{n\le x}\Lambda(n)\Lambda(n+h)\le \sum_{r\le\frac{\log x}{\log 2}}\log 2\ \log 2x\le \log x\ \log 2x\le 2x.$$

We can suppose that h is even, and $\Lambda(n)\Lambda(n+h)\neq 0$ implies that n is odd; therefore $n\ge 3$ and $x\ge 3$.

The contribution of the terms for which n and $n+h$ are not both primes is at most

$$2\log 2x\sum_{\substack{p^r\le 2x\\ r\ge 2}}\log p\le 2\ \pi(\sqrt{2x})\ \log^2 2x.$$

By inequality 3.6 of Rosser and Schoenfeld [8] we have

$$\forall x>1,\quad \pi(x)<1.25506\ \frac{x}{\log x};$$

therefore the contribution of the terms for which n and $n+h$ are not both primes is at most

$$7.1\ \sqrt{x}\ \log 2x.$$

By the theorem of Siebert [10] the number of primes $p \leq x$ such that $p+h$ is prime is at most

$$16 \prod_{p \geq 3} \left(1 - \frac{1}{(p-1)^2}\right) \frac{x}{\log^2 x} \prod_{\substack{p \mid h \\ p \geq 3}} \frac{p-1}{p-2}.$$

We remark that

$$\frac{p-1}{p-2} = \left(1 - \frac{1}{p}\right)^{-1} \left(1 - \frac{1}{(p-1)^2}\right)^{-1},$$

and when h is even

$$\prod_{\substack{p \mid h \\ p \geq 3}} \left(1 - \frac{1}{p}\right) = 2\ \frac{\varphi(h)}{h}$$

so that Siebert's expression can be written as

$$8 \prod_{\substack{p \\ (p,h)=1}} \left(1 - \frac{1}{(p-1)^2}\right) \frac{x}{\log^2 x} \cdot \frac{h}{\varphi(h)} \leq \frac{8\,x}{\log^2 x} \cdot \frac{h}{\varphi(h)}.$$

By inequality 3.41 and 3.42 of Rosser and Schoenfeld [8] we have for $h \geq 3$

$$\frac{h}{\varphi(h)} \leq e^{\gamma} \log\log h + \frac{2.50637}{\log\log h};$$

hence for $x \geq 3$ we have

$$\sum_{n \leq x} \Lambda(n)\Lambda(n+h) \leq 8\ \frac{\log 2x}{\log x} x \left(e^{\gamma} \log\log x + \frac{2.50637}{\log\log x}\right) + 7.1\ \sqrt{x}\ \log 2x,$$

and for $x \geq 10^8$ we obtain

$$\sum_{n \leq x} \Lambda(n)\Lambda(n+h) \leq 15\ x\ (\log\log x + 0.5).$$

For $x < 10^8$ we have

$$\sum_{n \leq x} \Lambda(n)\Lambda(n+h) \leq \log 2x \sum_{n \leq x} \Lambda(n),$$

and we use the inequality 3.35 of Rosser and Schoenfeld [8]

$$\sum_{n \leq x} \Lambda(n) < 1.03883\ x \quad \text{for all } x > 0,$$

which gives for $10 \leq x < 10^8$

$$\sum_{n \leq x} \Lambda(n)\Lambda(n+h) \leq 1.03883\ \log(2.10^8)\ x \leq 20\ x \leq 15\ x\ (\log\log x + 0.5).$$

For $x < 10$ the inequality is verified by direct computation. □

5. Sums of type I and II

For $z \geq 3$ depending on x only we can split $S(x,\alpha)$ as follows:

$$S(x,\alpha) = \sum_{n\leq x} \Lambda(n)\ \mathrm{e}(n\alpha) = S_1(x,\alpha) + S_2(x,\alpha),$$

where

$$\begin{aligned} S_1(x,\alpha) &= \sum_{n\leq x} v_z(n)\Lambda(n)\ \mathrm{e}(n\alpha), \\ S_2(x,\alpha) &= \sum_{n\leq x} u_z(n)\Lambda(n)\ \mathrm{e}(n\alpha). \end{aligned}$$

We can estimate $S_1(x,\alpha)$ trivially:

$$|S_1(x,\alpha)| \leq \sum_{\substack{p^r\leq x\\ p\leq z}} \log p = \sum_{p\leq z} \log p \left\lfloor \frac{\log x}{\log p} \right\rfloor \leq \pi(z)\log x \leq z \log x.$$

Now we split $S_2(x,\alpha)$ into $B_1(x,\alpha) - B_2(x,\alpha)$ (see [4] for details) where

$$\begin{aligned} B_1(x,\alpha) &= \sum_{n\leq x} u_z(n)\ \log(n)\ \mathrm{e}(n\alpha), \\ B_2(x,\alpha) &= \sum_{z\leq d\leq x/z} u_z(d) \sum_{z\leq m\leq x/d} u_z(m)\ \Lambda(m)\ \mathrm{e}(md\alpha). \end{aligned}$$

6. Sums of type I

Lemma 9. *For $\alpha = \frac{a}{q} + \frac{\beta}{q^2}$, $|\beta| \leq 1$, $(a,q)=1$, $3^7 \leq z^7 \leq y \leq x$, we have*

$$\begin{aligned} |B_1(x,\alpha)| \leq\ & \frac{2}{3}\ e^{\gamma} x \log x \log 3z\ \exp\left(-\frac{\log y}{\log z}\right) \\ & + 2\ \frac{x}{q} \log x \log 3y + \frac{2}{\pi}\ y \log x \log 4q + \frac{3}{\pi}\ q \log x \log 5q. \end{aligned}$$

Proof. We write

$$B_1(x,\alpha) = \sum_{n\leq x} u_z(n)\ \mathrm{e}(n\alpha) \int_1^n \frac{dt}{t} = \int_1^x \frac{dt}{t} \sum_{t\leq n\leq x} u_z(n)\ \mathrm{e}(n\alpha).$$

Introducing $T_1(t,x,\alpha) = \sum_{t\leq n\leq x} u_z(n)\ \mathrm{e}(n\alpha)$, we see that

$$|B_1(x,\alpha)| \leq \log x \sup_{1\leq t\leq x} |T_1(t,x,\alpha)|.$$

By the Möbius inversion formula

$$T_1(t,x,\alpha) = \sum_{\substack{n,d\\ t\leq nd\leq x}} v_z(n)\ \mu(n)\ \mathrm{e}(nd\alpha).$$

Let y such that $z^7 \leq y \leq x$. We have

$$\begin{aligned} T_{1,1}(t,x,\alpha) &= \sum_{n\leq y} \sum_{t\leq nd\leq x} v_z(n)\ \mu(n)\ \mathrm{e}(nd\alpha), \\ T_{1,2}(t,x,\alpha) &= \sum_{y<n\leq x} \sum_{t\leq nd\leq x} v_z(n)\ \mu(n)\ \mathrm{e}(nd\alpha). \end{aligned}$$

Clearly

$$|T_{1,1}(t,x,\alpha)| \le \sum_{n\le y} \min\left(\frac{x}{n}, \frac{1}{|\sin(\pi n\alpha)|}\right).$$

Hence by Lemma 3

$$|T_{1,1}(t,x,\alpha)| \le 2\ \frac{x}{q}\log 3y + \frac{2}{\pi}\ y\log 4q + \frac{3}{\pi}\ q\log 5q.$$

We have

$$|T_{1,2}(t,x,\alpha)| \le \sum_{n>y} v_z(n)\ |\mu(n)| \sum_{d\le x/n} 1 \le x\sum_{n>y} \frac{v_z(n)}{n}\mu^2(n).$$

In [8], Rosser and Schoenfeld proved (inequality 3.24) that

$$\sum_{p\le x} \frac{\log p}{p} < \log x \quad \text{for all } x > 1.$$

Using this inequality we get

$$0 < S = \sum_{p\le z} \frac{\frac{1}{p}}{1+\frac{1}{p}}\log p \le \sum_{p\le z}\frac{\log p}{p} \le \log z \le \frac{\log y}{7}.$$

Hence by Rankin's method (Lemma 4) we get

$$|T_{1,2}(t,x,\alpha)| \le x\left(\prod_{p\le z}\left(1+\frac{1}{p}\right)\right)\exp\left(-\frac{\log y}{\log z}\right).$$

Now for $z \ge 3$ we have

$$\prod_{p\le z}\left(1+\frac{1}{p}\right)\prod_{p\le z}\left(1-\frac{1}{p}\right) = \prod_{p\le z}\left(1-\frac{1}{p^2}\right) \le \frac{3}{4}\cdot\frac{8}{9} = \frac{2}{3}.$$

In [8], Rosser and Schoenfeld proved (inequality 3.31) that

$$\prod_{p\le x}\frac{p}{p-1} < e^\gamma \sum_{1\le n\le x}\frac{1}{n} \quad \text{for all } x \ge 1.$$

Using these inequalities we obtain

$$|T_{1,2}(t,x,\alpha)| \le \frac{2}{3}\ e^\gamma x\left(\sum_{1\le n\le z}\frac{1}{n}\right)\exp\left(-\frac{\log y}{\log z}\right),$$

and finally

$$|T_{1,2}(t,x,\alpha)| \le \frac{2}{3}\ e^\gamma x\log 3z\exp\left(-\frac{\log y}{\log z}\right),$$

which completes the proof. □

7. Sums of type II

Let J satisfy $2^J \lceil z \rceil \leq x/z < 2^{J+1} \lceil z \rceil$. We have

$$|B_2(x,\alpha)| \leq \sum_{0\leq j\leq J} \sum_{2^j\lceil z\rceil \leq d < 2^{j+1}\lceil z\rceil} u_z(d) \left| \sum_{z\leq m\leq x/d} u_z(m)\ \Lambda(m)\ \mathrm{e}(md\alpha)\right|.$$

We observe that $J\log 2 \leq \log x - 2\log z \leq \log x - \log 2$, and we define

$$T_2(M) = \sum_{M\leq d<2M} u_z(d) \left| \sum_{z\leq m\leq x/d} u_z(m)\ \Lambda(m)\ \mathrm{e}(md\alpha)\right|.$$

We get

$$|B_2(x,\alpha)| \leq \frac{\log x}{\log 2} \sup_{\substack{z\leq M\leq x/z\\ M\in\mathbb{N}}} |T_2(M)|.$$

By the Cauchy-Schwarz inequality

$$|T_2(M)|^2 \leq \left(\sum_{M\leq d<2M} u_z^2(d)\right) \sum_{M\leq d<2M} \left| \sum_{z\leq m\leq x/d} u_z(m)\ \Lambda(m)\ \mathrm{e}(md\alpha)\right|^2.$$

By Corollary 1, we have

$$\sum_{M\leq d<2M} u_z^2(d) \leq \frac{M}{\log z}.$$

Expanding the square and summing first over the d's we obtain

$$|T_2(M)|^2 \leq \frac{M}{\log z} \sum_{z\leq m\leq x/M} \Lambda(m) \sum_{z\leq m'\leq x/M} \Lambda(m') \left| \sum_{d\in I(m,m')} \mathrm{e}((m-m')d\alpha)\right|,$$

where $I(m,m')$ is the interval of d's such that $M \leq d \leq \min(2M-1, \frac{x}{m}, \frac{x}{m'})$.

We distinguish $m = m'$ and $m \neq m'$ and obtain

$$|T_2(M)|^2 \leq |T_{2,1}(M)|^2 + |T_{2,2}(M)|^2,$$

where

$$|T_{2,1}(M)|^2 = \frac{M^2}{\log z} \sum_{z\leq m\leq x/M} \Lambda^2(m)$$

and

$$|T_{2,2}(M)|^2 = 2\,\frac{M}{\log z} \sum_{1\leq h\leq x/M}\ \sum_{z\leq m\leq x/M} \Lambda(m)\ \Lambda(m+h) \left| \sum_{d\in I(m,m+h)} \mathrm{e}(hd\alpha)\right|.$$

We have

$$\left| \sum_{d\in I(m,m+h)} \mathrm{e}(hd\alpha)\right| \leq \min\left(M, \frac{1}{|\sin(\pi h\alpha)|}\right).$$

By Lemma 8

$$\sum_{z\le m\le x/M} \Lambda(m)\ \Lambda(m+h) \le 15\ (\log\log x + 0.5)\ \frac{x}{M}.$$

So

$$|T_{2,2}(M)|^2 \le 30\ (\log\log x + 0.5)\frac{x}{\log z} \sum_{1\le h\le x/M} \min\left(M, \frac{1}{|\sin(\pi h\alpha)|}\right).$$

Using Lemma 2 and $z \le M \le x/z$ we get

$$\begin{aligned}
\sum_{1\le h\le x/M} \min\left(M, \frac{1}{|\sin(\pi h\alpha)|}\right) &\le \left(\frac{x}{Mq}+1\right)\left(2M + \frac{2}{\pi} q\log 4q\right) \\
&\le \frac{2x}{q} + 2M + \frac{2x\log 4q}{\pi M} + \frac{2}{\pi} q\log 4q \\
&\le \frac{2x}{q} + \frac{2x}{z} + \frac{2x\log 4q}{\pi z} + \frac{2}{\pi} q\log 4q \\
&\le 2\,x\left(\frac{1}{q} + \frac{\pi+\log 4q}{\pi z} + \frac{q\log 4q}{\pi x}\right).
\end{aligned}$$

We obtain

$$|T_{2,2}(M)|^2 \le 60\ (\log\log x + 0.5)\ \frac{x^2}{\log z}\left(\frac{1}{q} + \frac{\log 93q}{\pi z} + \frac{q\log 4q}{\pi x}\right),$$

$$\sum_{n\le x} \Lambda(n) < 1.03883\ x \quad \text{for all } x > 0.$$

For $z \ge 3$, the function $M \longmapsto M\log(x/M)$ is increasing on $[z, x/z]$. Hence by Rosser and Schoenfeld [8] inequality 3.35 we have

$$|T_{2,1}(M)|^2 \le 1.03883\ xM\frac{\log(x/M)}{\log z} \le 1.03883\ \frac{x^2}{z}$$

and

$$\begin{aligned}
|B_2(x,\alpha)| \le\ & \frac{\sqrt{1.03883}}{\log 2}\frac{x}{\sqrt{z}}\log x \\
&+ \frac{1}{\log 2}\sqrt{60\ (\log\log x + 0.5)}\frac{x\log x}{\sqrt{\log z}}\left(\sqrt{\frac{1}{q} + \frac{q\log 4q}{\pi x}} + \sqrt{\frac{\log 93q}{\pi z}}\right).
\end{aligned}$$

Finally

$$\begin{aligned}
|B_2(x,\alpha)| \le\ & 1.48\,\frac{x}{\sqrt{z}}\log x \\
&+ 11.18\sqrt{\log\log x + 0.5}\frac{x\log x}{\sqrt{\log z}}\left(\sqrt{\frac{1}{q} + \frac{q\log 4q}{\pi x}} + \sqrt{\frac{\log 93q}{\pi z}}\right).
\end{aligned}$$

8. Proof of Theorem 1

We can suppose $x \geq 10^{184}$; otherwise

$$\sqrt{\log\log x + 0.5}\; x\,(\log x)^{5/4} \exp\left(-\frac{1}{2}\sqrt{\log x}\right) > 0.166$$

and the result is trivial using Rosser and Schoenfeld [8] inequality 3.35. Furthermore we can suppose

$$(\log x)^{3/2}\log\log x \leq q \leq \frac{x}{(\log x)^{5/2}\log\log x};$$

otherwise the result is trivial.

We choose $\log z = \sqrt{\log x}$ and we obtain

$$\begin{aligned}
|B_2(x,\alpha)| &\leq 11.18\sqrt{\log\log x + 0.5}\; x\,(\log x)^{3/4}\sqrt{\frac{1}{q} + \frac{q\log 4q}{\pi x}}\\
&\quad + 1.48\, x\,\log x\,\exp\left(-\frac{1}{2}\sqrt{\log x}\right)\\
&\quad + 6.31\sqrt{\log\log x + 0.5}\; x\,(\log x)^{5/4}\exp\left(-\frac{1}{2}\sqrt{\log x}\right)\\
&\leq 11.18\sqrt{\log\log x + 0.5}\; x\,(\log x)^{3/4}\sqrt{\frac{1}{q} + \frac{q\log 4q}{\pi x}}\\
&\quad + 6.44\sqrt{\log\log x + 0.5}\; x\,(\log x)^{5/4}\exp\left(-\frac{1}{2}\sqrt{\log x}\right).
\end{aligned}$$

Let us first suppose that

$$(\log x)^{3/2}\log\log x \leq q \leq (\log x)^3.$$

Let $\log y = \sqrt{\log x}\log q$. We then have for $x \geq 10^{184}$

$$\begin{aligned}
|B_1(x,\alpha)| &\leq 1.19\,\frac{x}{q}\,\log x(\log z + \log 3) + 2\,\frac{x}{q}\,\log x(\log y + \log 3)\\
&\quad + 0.64\,\exp(\sqrt{\log x}\log q)\log x\log 4q + 0.96\; q\log x\log 5q\\
&\leq 3.68\sqrt{\log\log x + 0.5}\; x\,(\log x)^{3/4}\sqrt{\frac{1}{q}}.
\end{aligned}$$

Now let us suppose that

$$(\log x)^3 < q \leq \frac{x}{(\log x)^{5/2}\log\log x}.$$

We choose

$$y = x\,(\log x)^{-1/2}\exp(-\sqrt{\log x}),$$

$$\begin{aligned}
|B_1(x,\alpha)| &\leq 3.95\, x\,(\log x)^{3/2} \exp(-\sqrt{\log x}) + 2\,\frac{x}{q}\,(\log x)^2 \\
&\quad + \frac{2}{\pi}\, x\,(\log x)^{3/2} \exp(-\sqrt{\log x}) + \frac{3}{\pi}\, q\, \log x \log 5q \\
&\leq 5.59\, x\,(\log x)^{3/2} \exp(-\sqrt{\log x}) + 0.45\, x\,(\log x)^{3/4} \sqrt{\frac{1}{q}} \\
&\quad + 0.11\, x\,(\log x)^{3/4} \sqrt{\frac{q\log 4q}{\pi x}} \\
&\leq 0.56\, x\,(\log x)^{3/4} \sqrt{\frac{1}{q} + \frac{2}{3\pi}\cdot\frac{q\log 4q}{x}} \\
&\quad + 4.59\, x\,(\log x)^{3/2} \exp(-\sqrt{\log x}).
\end{aligned}$$

Hence for all q we have

$$\begin{aligned}
|B_1(x,\alpha)| &\leq 3.68\sqrt{\log\log x + 0.5}\; x\,(\log x)^{3/4} \sqrt{\frac{1}{q} + \frac{2}{3\pi}\cdot\frac{q\log 4q}{x}} \\
&\quad + 0.01\sqrt{\log\log x + 0.5}\; x\,(\log x)^{5/4} \exp\left(-\frac{1}{2}\sqrt{\log x}\right).
\end{aligned}$$

Finally we have

$$\begin{aligned}
|S(x,\alpha)| &\leq |S_1(x,\alpha)| + |B_1(x,\alpha)| + |B_2(x,\alpha)| \\
&\leq 14.86\sqrt{\log\log x + 0.5}\; x\,(\log x)^{3/4} \sqrt{\frac{1}{q} + \frac{2}{3\pi}\cdot\frac{q\log 4q}{x}} \\
&\quad + 6.45\sqrt{\log\log x + 0.5}\; x\,(\log x)^{5/4} \exp\left(-\frac{1}{2}\sqrt{\log x}\right).
\end{aligned}$$

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Faculté de Mathématiques et d'Informatique, 33 rue Saint-Leu, 80039 Amiens, France
Current address: UMR CNRS 8752, Mathématiques, Université Paris-Sud, 91405 Orsay Cedex, France
E-mail address: `daboussi@math.u-psud.fr`

Institut Girard Desargues, CNRS UPRES-A 5028, Université Lyon I, 43, boulevard du 11 novembre 1918, 69622 Villeurbanne cedex, France
Current address: Institut Elie Cartan, Université Henri Poincaré, B.P. 239, 54506 Vandoeuvre cedex, France
E-mail address: `rivat@iecn.u-nancy.fr`