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REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the American Mathematical Society classification scheme. The 2000 Mathematics Subject Classification can be found in print starting with the 1999 annual index of *Mathematical Reviews.* The classifications are also accessible from www.ams.org/msc/.

1[35L60, 65M06, 73V15, 76M20, 78A05, 80A22]—*Level set methods and fast marching methods: evolving interfaces in computational geometry, fluid mechanics, computer vision, and materials science*, by J. A. Sethian, Cambridge University Press, New York, NY 1999, xx+378 pp.; 23 cm, hardcover \$74.95, softcover \$29.95

The level set method was devised in [OSe] as a versatile and useful tool for analyzing the motion of fronts. It has proven to be phenomenally successful as both a theoretical and computational device. A level set calculus has been developed, there have been many extensions, and several related techniques have been constructed. One of these related techniques is the fast marching method devised in [T1], [T2].

The idea behind the level set method, as developed in [OSe], is merely to define a smooth (at least Lipschitz continuous) function $\varphi(\vec{x},t)$ that represents the interface Γ as the set where $\varphi(\vec{x},t)=0$. Here $\vec{x}=(x_1,\dots,x_n)\in R^n$ and Γ bounds (a possibly multiply connected) region Ω. This level set function $\varphi(\vec{x},t)$ has the following properties: $\varphi(\vec{x},t)>0$ for $\vec{x}\in\bar{\Omega}^c$, $\varphi(x,t)<0$ for $x\in\Omega$, $\varphi(\vec{x},t)=0$ for $\vec{x}=\partial\Omega=\Gamma(t)$.

The interface is to be captured for all time by merely locating the set $\Gamma(t)$ where φ vanishes. This deceptively trivial statement is of great significance because topological changes, such as merging and breaking, are well defined and easily performed.

The motion is to be analyzed by convecting the φ values (levels) with the velocity field $\vec{v}$. This elementary equation is

$$\frac{\partial\varphi}{\partial t}+\vec{v}\cdot\nabla\varphi=0. \tag{1}$$

Here, $\vec{v}$ is the desired velocity on the interfaces, i.e., where $\varphi=0$ and is arbitrary elsewhere.

Actually, only the normal component of v is needed: $v_n=\vec{v}\cdot\frac{\nabla\varphi}{|\nabla\varphi|}$, so (1) becomes

$$\frac{\partial\varphi}{\partial t}+v_n|\nabla\varphi|=0. \tag{2}$$

This is all easy to implement in the presence of boundary singularities, topological changes, and in 2 and 3 dimensions. Moreover, when the normal velocity v_n is a function only of the direction of the unit normal, then equation (2) becomes the first order Hamilton-Jacobi equation

$$\frac{\partial\varphi}{\partial t}+|\nabla\varphi|\gamma\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right)=0. \tag{3}$$

High order accurate, essentially nonoscillatory approximations to general Hamilton-Jacobi equations, including (3), were obtained in [OSe],[OSh]. Theoretical justification of these methods for geometrically based motion came through the theory of viscosity solutions, e.g., [CGG], [ES], [ESS].

The fast marching method, which gets equal billing in this book, is an optimal numerical algorithm for obtaining convergent approximations to the viscosity solution of the eikonal equation, e.g., in R^2

$$\sqrt{\varphi_x^2 + \varphi_y^2} = a(x, y) > 0$$

with φ given to be zero on a closed set. The function φ is the positive distance to this set using a weighted metric. Very slight generalizations are admitted. This too is a useful algorithm, as the author demonstrates. However, its scope (at this point) is far more limited than that of the level set method.

The author, who is the coinventor of the level set method, writes well and this book could indeed serve as a beginner's "user's guide" to this subject. However, the author is less than generous in his attribution of who was first responsible for many of the key ideas he describes.

Here are a couple of examples:

(1) The fast marching method was invented by John N. Tsitsiklis in [T1] (see also [T2]). Everything is in these papers—eikonal equation, viscosity solution, Hamilton-Jacobi equations, Dijkstra's algorithm, heap sort, precise complexity estimates (including a parallelized version), a rigorous proof that it works, and the application to optimal path planning (which is the subject of section 20.1 of this book).

The method was rediscovered independently and simultaneously by John Helmsen, see [HPCD] and the author [Se]. Remarkably, there is no mention of [T1], [T2] or [HPCD] in the book being reviewed here, whose title includes the words "Fast Marching Methods".

(2) On page 219 the author states that "the strategy of using level set methods for shape recovery was invented by Malladi". However, the earliest work in this area is the very substantial paper [CCCD].

The topics covered for the most part involve motion which is not coupled to external physical systems, for which v_n in equation (2) of this review depends only on the geometry and location of the shape. Topics involving external physics are briefly touched upon in chapter 18 and elsewhere. The versatility and utility of the level set method is exemplified in the book by applications such as computer vision, etching and deposition and image processing. Of course the author's understandable bias toward his work comes through, for example, in his choice for min/max flow for image restoration, which is probably not the most frequently used or useful approach.

The exposition of numerical methods is a bit confused and is lacking in detail. In sections (6.4), (6.5), (6.6) he calls "convex" the schemes coming from [OSe] for Hamiltonians closely related to the eikonal equation. These schemes would certainly not work for general convex Hamiltonians such as $p\varphi_x^2 + q\varphi_y^2 - 2r\varphi_x\varphi_y$ with $p, q > 0$ and $pq > r^2$. He calls "nonconvex" everything else, including convex Hamiltonians, for which he advocates using the schemes in [OSh].

Also the book (inevitably) fails to cover interesting, relevant topics such as Wulff shapes, island dynamics models in epitaxial growth, motion of higher codimensional objects, e.g., curves in R^3, the ghost fluid method for multiphase flow, etc. However it is a useful introduction to a remarkably dynamic area.

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