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WAVELET BASES IN H(div) AND H(curl)

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ABSTRACT. Some years ago, compactly supported divergence-free wavelets were constructed which also gave rise to a stable (biorthogonal) wavelet splitting of $\mathbf{H}(\mathrm{div};\Omega)$. These bases have successfully been used both in the analysis and numerical treatment of the Stokes and Navier–Stokes equations.

In this paper, we construct stable wavelet bases for the stream function spaces $\mathbf{H}(\mathbf{curl};\Omega)$. Moreover, **curl**-free vector wavelets are constructed and analysed. The relationship between $\mathbf{H}(\mathrm{div};\Omega)$ and $\mathbf{H}(\mathbf{curl};\Omega)$ are expressed in terms of these wavelets. We obtain discrete (orthogonal) Hodge decompositions.

Our construction works independently of the space dimension, but in terms of general assumptions on the underlying wavelet systems in $L^2(\Omega)$ that are used as building blocks. We give concrete examples of such bases for tensor product and certain more general domains $\Omega \subset \mathbb{R}^n$. As an application, we obtain wavelet multilevel preconditioners in $\mathbf{H}(\mathrm{div};\Omega)$ and $\mathbf{H}(\mathbf{curl};\Omega)$.

1. INTRODUCTION

The spaces $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$ arise naturally in the variational formulation of a whole variety of partial differential equations. Two prominent examples are the *Navier–Stokes equations* that describe the flow of a viscous, incompressible fluid and *Maxwell's equations* in electromagnetism. For the Navier–Stokes equations, $\boldsymbol{H}(\mathrm{div};\Omega)$ plays an important role for modelling the velocity-field of the flow. The space $\boldsymbol{H}(\mathbf{curl};\Omega)$ has to be considered, when one is interested in a formulation in nonprimitive variables such as stream function, vorticity and vector potential, [25].

Certain electromagnetic phenomena are known to be modelled by Maxwell's equations. Here, the space $\boldsymbol{H}(\mathbf{curl};\Omega)$ appears when linking the quantities electric and magnetic field, magnetic induction, and flux density, see for example [4, 6, 24] and the references therein. For the numerical treatment of these equations, it is very helpful to have at hand bases for the kernel of the **curl** operator and its orthogonal complement.

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Wavelets have become a powerful tool in both pure and applied mathematics during the past years. For example, they allow us to extend classical results of Fourier analysis to a much wider class of function spaces [29]. On the other hand, wavelet and multilevel systems are by now very widely used in many fields of science and technology, such as signal analysis, data compression and image processing [22, 31, 34]. More recently, starting from [5], they have shown promising features for the construction of efficient numerical schemes for solving operator equations, see e.g., [11, 16]. Moreover, it has been shown that the flexibility and freedom in the construction of biorthogonal wavelets can be used to adapt these bases to special requirements imposed by the problem at hand. For instance, wavelets have been adapted to the Stokes and Navier–Stokes equations [17, 26, 32, 33].

In this paper, we adapt wavelet bases to the spaces $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$. We construct a biorthogonal wavelet basis for $\boldsymbol{H}(\mathbf{curl};\Omega)$ and introduce **curl**-free vector wavelets. It is well known that the spaces $\boldsymbol{H}^0(\boldsymbol{H}\mathrm{div};\Omega)$ and $\boldsymbol{H}^0(\mathbf{curl};\Omega)$ of divergence-free, resp. **curl**-free, vector fiels in $\boldsymbol{L}^2(\Omega)$ (the space of square-integrable vector fields) are linked by certain interrelations. These play an important role when one considers the splitting of $\boldsymbol{L}^2(\Omega)$ into $\boldsymbol{H}^0(\boldsymbol{H}\mathrm{div};\Omega)$, $\boldsymbol{H}^0(\mathbf{curl};\Omega)$, respectively, and some complement. Using divergence-free wavelets [26, 32], it was shown that there exists a stable (biorthogonal) Helmholtz decomposition of $\boldsymbol{H}(\mathrm{div};\Omega)$ [33], but this is no orthogonal decomposition. Now, with the construction of wavelet bases in $\boldsymbol{H}(\mathbf{curl};\Omega)$ at hand, we can give an explicit *orthogonal* decomposition of $\boldsymbol{L}^2(\Omega)$. Using this decomposition, we construct wavelet multilevel preconditioners in $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$ that give rise to uniformly bounded condition numbers.

Our construction starts with some wavelet systems on Ω that have to fulfill certain conditions. These conditions are formulated in a rather general assumption that allow us to proceed with the construction in $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$. It turns out that this assumption is mostly a condition on the domain Ω. After proving the results concerning the new wavelet bases, we show concrete examples of domains and corresponding wavelet bases that realize our general conditions. These examples are at least twofold: Firstly, all tensor product domains are covered by taking wavelets that are built by tensor products of univariate systems. However, our construction is *not* restricted to this case. In particular, we give at least two extensions, namely when Ω is the image of a reference cube $\hat{\Omega} = (0,1)^n$ under certain parametric mappings.

The paper is organized as follows. In Section 2, we review the basic facts both on wavelets and on the spaces of vector fields under consideration. Section 3 is devoted to our construction of wavelets in $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$. Also, the general assumption mentioned above is stated there. In Section 4, we show examples of domains and wavelet bases fulfilling the general hypothesis and compute **curl**-free vector wavelets explicitly starting from biorthogonal spline wavelets. We finish with one application, namely multilevel preconditioning in $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$ in Section 5.

2. Notation and basic facts

In this section, we set up our notation and review the basic facts both on wavelet bases and the spaces of vector fields under consideration that are needed here. We begin with the spaces of vector fields and follow the description in [25].

2.1. **Stream function spaces.** Let $\Omega \subset \mathbb{R}^n$ be some open bounded domain with Lipschitz-continuous boundary. Then, for $\phi \in L^2(\Omega)$ and a two-dimensional vector field $\boldsymbol{v} = (v_1, v_2)^T$, we define

$$\mathbf{curl}\,\phi := (\partial_2\phi, -\partial_1\phi)^T, \qquad \mathrm{curl}\,\boldsymbol{v} := \partial_1 v_2 - \partial_2 v_1. \tag{2.1}$$

Here and in the sequel, we use the short-hand notation $\partial_i := \frac{\partial}{\partial x_i}$ for $1 \le i \le n$ and we assume the partial derivatives in (2.1) to exist in the distributional sense. We will always denote vector fields and any related quantities such as function spaces, by boldface characters. If needed, the spaces of vector fields have an additional superscript indicating the number of components of the vector fields. For three-dimensional vector fields $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3)^T$, we set

$$\mathbf{curl}\,\boldsymbol{\zeta} := (\partial_2\zeta_3 - \partial_3\zeta_2, \partial_3\zeta_1 - \partial_1\zeta_3, \partial_1\zeta_2 - \partial_2\zeta_1)^T, \tag{2.2}$$

where again the partial derivatives should exist in the distributional sense.

With these definitions at hand, we define the following stream function spaces (for $n = 3$):

$$\boldsymbol{H}(\mathbf{curl};\Omega) := \{\boldsymbol{\zeta} \in \boldsymbol{L}^2(\Omega) : \mathbf{curl}\,\boldsymbol{\zeta} \in \boldsymbol{L}^2(\Omega)\}, \tag{2.3}$$

$$\boldsymbol{H}^0(\mathbf{curl};\Omega) := \{\boldsymbol{\zeta} \in \boldsymbol{H}(\mathbf{curl};\Omega) : \mathbf{curl}\,\boldsymbol{\zeta} = \mathbf{0}\}. \tag{2.4}$$

We will be mainly concerned with the 3D case here. When nothing else is said, $n = 3$ is assumed. We will treat the curl-spaces for $n = 2$ separately. All these spaces are Hilbert spaces with the corresponding norm

$$\|\boldsymbol{\zeta}\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega)} := \|\boldsymbol{\zeta}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{\zeta}\|^2_{\boldsymbol{L}^2(\Omega)}. \tag{2.5}$$

Finally, we define

$$\boldsymbol{H}_0(\mathbf{curl};\Omega) := \mathrm{clos}_{\boldsymbol{H}(\mathbf{curl};\Omega)}\, C_0^\infty(\Omega) \tag{2.6}$$

and

$$\boldsymbol{H}^0_0(\mathbf{curl};\Omega) := \boldsymbol{H}_0(\mathbf{curl};\Omega) \cap \boldsymbol{H}^0(\mathbf{curl};\Omega). \tag{2.7}$$

Since these spaces model the stream function in the Navier–Stokes equations for nonprimitive variables, we also call them *stream function spaces.*

2.2. **Flux spaces.** It is known that the stream function spaces are heavily linked to a second class of spaces induced by the divergence operator. Since these spaces often model the flux of some physical quantity, we follow [24] and call these spaces *flux spaces.* As usual, the divergence operator is defined for any vector field $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n)^T$ by

$$\mathrm{div}\,\boldsymbol{\zeta} := \sum_{i=1}^{n} \partial_i \zeta_i, \tag{2.8}$$

where it is again assumed that the partial derivatives are well defined in the distributional sense. Then, we define

$$\begin{aligned} \boldsymbol{H}(\mathrm{div};\Omega) &:= \{\boldsymbol{\zeta} \in \boldsymbol{L}^2(\Omega) : \mathrm{div}\,\boldsymbol{\zeta} \in L^2(\Omega)\}, \\ \boldsymbol{H}^0(\boldsymbol{H}\mathrm{div};\Omega) &:= \{\boldsymbol{\zeta} \in \boldsymbol{H}(\mathrm{div};\Omega) : \mathrm{div}\,\boldsymbol{\zeta} = 0\}. \end{aligned}$$

Also these spaces are Hilbert spaces under the norm

$$\|\boldsymbol{\zeta}\|^2_{\boldsymbol{H}(\mathrm{div};\Omega)} := \|\boldsymbol{\zeta}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\,\boldsymbol{\zeta}\|^2_{\boldsymbol{L}^2(\Omega)}, \tag{2.9}$$

and we define

$$\boldsymbol{H}_0(\mathrm{div};\Omega) := \mathrm{clos}_{\boldsymbol{H}(\mathrm{div};\Omega)}\, C_0^\infty(\Omega), \tag{2.10}$$

as well as

$$\boldsymbol{H}_0^0(\mathrm{div};\Omega) := \boldsymbol{H}_0(\mathrm{div};\Omega) \cap \boldsymbol{H}^0(\boldsymbol{H}\mathrm{div};\Omega). \tag{2.11}$$

2.3. **Hodge decompositions.** There are some well-known relationships between stream function and flux spaces, which we will review now. Especially for the analysis and numerical treatment of partial differential equations involving div-operators and **curl**-operators, one is interested in the following orthogonal decompositions:

$$\boldsymbol{L}^2(\Omega) = \boldsymbol{H}_0^0(\mathrm{div};\Omega) \oplus^\perp \boldsymbol{H}_0^0(\mathrm{div};\Omega)^\perp \tag{2.12}$$

$$= \boldsymbol{H}_0^0(\mathbf{curl};\Omega) \oplus^\perp \boldsymbol{H}_0^0(\mathbf{curl};\Omega)^\perp, \tag{2.13}$$

which are often referred to as *Hodge decompositions*. Both decompositions are also of great interest for replacing $\boldsymbol{L}^2(\Omega)$ by $\boldsymbol{H}_0(\mathrm{div};\Omega)$, $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, respectively.

For (2.12), it is known that

$$\boldsymbol{H}_0^0(\mathrm{div};\Omega)^\perp = \{\mathbf{grad}\, q : q \in H^1(\Omega)\} = \boldsymbol{H}^0(\mathbf{curl};\Omega) \tag{2.14}$$

for simply connected bounded Lipschitz domains Ω (see [25]). The second decomposition (2.13) for $\boldsymbol{L}^2(\Omega)$ replaced by $\boldsymbol{H}(\mathrm{curl};\Omega)$ is somewhat more involved, one has (see for example [25], p. 50) that

$$\boldsymbol{H}_0^0(\mathbf{curl};\Omega)^\perp = \{\boldsymbol{\zeta} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) : (\boldsymbol{\zeta}, \mathbf{grad}\,\varphi)_{L^2(\Omega)} = 0 \;\forall\, \varphi \in H_0^1(\Omega)\}. \tag{2.15}$$

Moreover, the fact that $\boldsymbol{H}_0^0(\mathbf{curl};\Omega)^\perp$ is isomorphic to $\boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}^0(\boldsymbol{H}\mathrm{div};\Omega) \cap \boldsymbol{H}_0(\mathbf{curl};\Omega)$ can also be found in [25].

2.4. **Multiscale methods and wavelets.** Let us briefly review the main properties of multiscale methods and wavelets. We will focus on only those facts that we will need in this paper and refer to [11, 16] for extensive surveys.

We call a system of $L^2(\Omega)$-functions $\Psi := \{\psi_\lambda : \lambda \in \nabla\}$ a (primal) *wavelet basis* if they form a *Riesz basis* for $L^2(\Omega)$, i.e., each $L^2(\Omega)$-function can be expressed as a linear combination of the functions in Ψ, and

$$\Big\| \sum_{\lambda\in\nabla} d_\lambda \psi_\lambda \Big\|_{L^2(\Omega)} \sim \|\boldsymbol{d}\|_{\ell^2(\nabla)} = \left(\sum_{\lambda\in\nabla} |d_\lambda|^2 \right)^{1/2}, \qquad \boldsymbol{d} := \{d_\lambda\}_{\lambda\in\nabla}. \tag{2.16}$$

Here, ∇ denotes some appropriate (infinite) set of indices and by $A \sim B$ we abbreviate $cA \le B \le CA$ with absolute constants $0 < c \le C$. By Riesz's representation theorem, (2.16) implies the existence of a *biorthogonal* wavelet basis $\tilde\Psi$ of $L^2(\Omega)$, i.e., (2.16) holds for $\tilde\Psi$ and one has

$$(\psi_\lambda, \tilde\psi_{\lambda'})_{L^2(\Omega)} = \delta_{\lambda,\lambda'}, \qquad \lambda, \lambda' \in \nabla.$$

With the aid of this system, the expansion coefficients d_λ, $\tilde d_\lambda$ of a function

$$L^2(\Omega) \ni f = \sum_{\lambda\in\nabla} d_\lambda \psi_\lambda = \sum_{\lambda\in\nabla} \tilde d_\lambda \tilde\psi_\lambda$$

with respect to the wavelet basis can be written as

$$d_\lambda = (f, \tilde\psi_\lambda)_{L^2(\Omega)}, \qquad \tilde d_\lambda = (f, \psi_\lambda)_{L^2(\Omega)}, \qquad \lambda \in \nabla.$$

Hence, (2.16) (for the primal as well as the biorthogonal wavelets) may be rewritten as

$$\|f\|_{L^2(\Omega)} \sim \left(\sum_{\lambda \in \nabla} |(f, \tilde{\psi}_\lambda)_{L^2(\Omega)}|^2 \right)^{1/2} \sim \left(\sum_{\lambda \in \nabla} |(f, \psi_\lambda)_{L^2(\Omega)}|^2 \right)^{1/2}. \tag{2.17}$$

The couple Ψ, $\tilde{\Psi}$ is often referred to as *biorthogonal system*. Note that here Ψ needs not to be an orthonormal basis. However, orthogonal wavelets like Daubechies' wavelets [22] of course fulfull (2.16) with "=" instead of "$\sim$". In this case, one has $\Psi = \tilde{\Psi}$.

One may think of the indices $\lambda \in \nabla$ as a couple $\lambda = (j, k)$, where $j \in \mathbb{Z}$ denotes the *scale* or *level* of a function, whereas k refers to the location in space and the type of wavelet. We will often use

$$|\lambda| := j, \ \lambda = (j, k),$$

to abbreviate the level and

$$\nabla_j := \{\mu \in \nabla : \ |\mu| = j\}$$

to denote all indices on a certain level.

Moreover, we will frequently use the biorthogonal projectors induced by Ψ, $\tilde{\Psi}$. To define them, we set for any subset $\Lambda \subset \nabla$

$$\Psi_\Lambda := \{\psi_\lambda : \ \lambda \in \Lambda\}, \qquad \tilde{\Psi}_\Lambda := \{\tilde{\psi}_\lambda : \ \lambda \in \Lambda\} \tag{2.18}$$

as well as the generated wavelet spaces

$$S_\Lambda := \operatorname{clos}_{L^2(\Omega)} \operatorname{span} \Psi_\Lambda, \qquad \tilde{S}_\Lambda := \operatorname{clos}_{L^2(\Omega)} \operatorname{span} \tilde{\Psi}_\Lambda.$$

Then, the projectors $Q_\Lambda : L^2(\Omega) \to S_\Lambda$, $\tilde{Q}_\Lambda : L^2(\Omega) \to \tilde{S}_\Lambda$ are defined as

$$Q_\Lambda f := \sum_{\lambda \in \Lambda} (f, \tilde{\psi}_\lambda)_{L^2(\Omega)} \, \psi_\lambda, \qquad \tilde{Q}_\Lambda f := \sum_{\lambda \in \Lambda} (f, \psi_\lambda)_{L^2(\Omega)} \, \tilde{\psi}_\lambda, \tag{2.19}$$

and it can easily be seen that they are adjoint operators. Finally, we will use $Q := Q_\nabla$ as well as $Q_j := Q_{\nabla_j}$.

Remark 2.1. Of course, one can formulate all that is said about wavelets so far also for Hilbert spaces H instead of $L^2(\Omega)$ [16]. Note that in this case, the biorthogonal wavelet bases $\tilde{\Psi}$ usually characterize the *dual* space H'. This will be of some importance in the sequel. □

Remark 2.2. In many cases, the estimate (2.16) ((2.17) resp.) can be extended from $L^2(\Omega)$ to a whole range of Sobolev (and even Besov) spaces including $L^2(\Omega)$. In fact, under appropriate assumptions on Ψ and $\tilde{\Psi}$ there exist constants $\gamma, \tilde{\gamma} > 0$ (depending on the approximation properties and the regularity of Ψ and $\tilde{\Psi}$) such that

$$\left\| \sum_{\lambda \in \nabla} d_\lambda \psi_\lambda \right\|_{H^\sigma(\Omega)} \sim \left(\sum_{\lambda \in \nabla} 2^{2\sigma|\lambda|} |d_\lambda|^2 \right)^{1/2}, \tag{2.20}$$

holds for all $\sigma \in (-\tilde{\gamma}, \gamma)$ [16]. This means that the Sobolev norm of a function can be estimated by a weighted ℓ^2-norm of its wavelet coefficients. □

Vector fields For the space $\boldsymbol{L}^2(\Omega)$ of square integrable vector fields, we denote wavelet systems by boldface characters, i.e., $\boldsymbol{\Psi}$, and in (2.16) one uses the standard norm

$$\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} := \sum_{i=1}^{n} \|f_i\|^2_{L^2(\Omega)}$$

for $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)^n$. Moreover, in many cases, we have to equip the index $\lambda \in \nabla$ labeling the scalar wavelets with some additional index indicating the component of the vector field. For example, let $\Psi^{[\nu]} := \{\psi_\lambda^{[\nu]} : \lambda \in \nabla^{[\nu]}\}$, $\tilde{\Psi}^{[\nu]} := \{\tilde{\psi}_\lambda^{[\nu]} : \lambda \in \nabla^{[\nu]}\}$, $1 \le \nu \le n$, be (possibly different) biorthogonal systems in $L^2(\Omega)$. Then, the vector fields

$$\boldsymbol{\psi}_{(i,\lambda)} := \psi_\lambda^{[i]}\, \boldsymbol{\delta}_i, \qquad \tilde{\boldsymbol{\psi}}_{(i,\lambda)} := \tilde{\psi}_\lambda^{[i]}\, \boldsymbol{\delta}_i, \qquad \lambda \in \nabla^{[i]},\ 1 \le i \le n,$$

obviously form a biorthogonal wavelet basis for $\boldsymbol{L}^2(\Omega)$. Here $\boldsymbol{\delta}_i := (\delta_{1,i}, \dots, \delta_{n,i})^T$, $1 \le i \le n$ denotes the canonical unit vector in $\mathbb{R}^n$. Denoting by

$$\boldsymbol{\nabla} := \bigcup_{i=1}^{n} \bigcup_{\lambda \in \nabla^{[i]}} (i,\lambda), \qquad \boldsymbol{\lambda} := (i,\lambda),$$

the corresponding set of indices, we obtain the biorthogonal vector wavelet system

$$\boldsymbol{\Psi} = \{\boldsymbol{\psi}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}\}, \qquad \tilde{\boldsymbol{\Psi}} = \{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}\}.$$

3. Wavelet bases for stream function and flux spaces

In this section, we construct wavelet bases on the various stream function spaces defined in Section 2. Moreover, we recall the known construction of wavelets in the flux spaces and show relationships. We end up with Hodge decompositions of $\boldsymbol{L}^2(\Omega)$.

3.1. General assumptions on the wavelet systems.

We will formulate our results in terms of quite general assumptions with respect to wavelet systems in $L^2(\Omega)$. Since the construction of wavelets on general domains is still a field of intensive research (see, e.g., [9, 10, 15, 19, 20]), we want to clearly identify those properties of wavelet systems on Ω that are needed to proceed with the subsequent construction of wavelet bases for stream function and flux spaces. Thus, the task of constructing appropriate wavelets on $L^2(\Omega)$ is separated. We will show two concrete examples of bases fulfilling the latter assumptions in Section 4.

The assumption we will formulate now, will, roughly speaking, allow us to "play" with partial derivatives and wavelet systems. The motivation for this is the following result of P.G. Lemarié-Rieusset:

Theorem 3.1. [26] *Let $\psi^{(1)}$, $\tilde{\psi}^{(1)} \in L^2(\mathbb{R})$ be biorthogonal wavelets on $\mathbb{R}$ (i.e., $\psi^{(1)}_{j,k}$, $\tilde{\psi}^{(1)}_{j,k}$, $j,k \in \mathbb{Z}$, form a biorthogonal system, where $f_{j,k}(x) := 2^{j/2} f(2^j x - k)$, $x \in \mathbb{R}$), such that $\psi^{(1)} \in H^1_0(\mathbb{R})$. Then, there exists a second pair $\psi^{(0)}$, $\tilde{\psi}^{(0)}$ of biorthogonal wavelets fulfilling*

$$\frac{d}{dx}\psi^{(1)}(x) = 4\,\psi^{(0)}(x), \qquad \frac{d}{dx}\tilde{\psi}^{(0)}(x) = (-4)\,\tilde{\psi}^{(1)}(x). \qquad \square$$

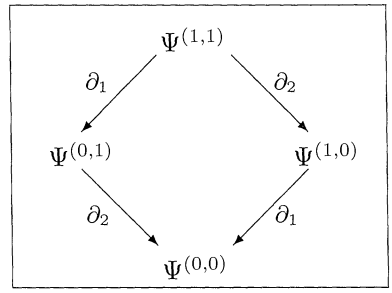

FIGURE 1. Relationships between wavelet systems for $n = 2$ according to Assumption 1.

What we need is a suitable generalization to the n-dimensional case, partial derivatives and domains $\Omega \subset \mathbb{R}^n$.

Assumption 1. For all $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)^T \in E^n := \{0,1\}^n$, and all $i \in \{1, \dots, n\}$ such that $\boldsymbol{\gamma} - \boldsymbol{\delta}_i \in E^n$, there exist biorthogonal systems $\Psi^{(\boldsymbol{\gamma})}$, $\tilde{\Psi}^{(\boldsymbol{\gamma})}$ on $L^2(\Omega)$ with the following properties:

(a) $\Psi^{(\boldsymbol{\gamma})}$, $\tilde{\Psi}^{(\boldsymbol{\gamma})}$ correspond to the *same* set of indices $\nabla = \{(j,k) : j \geq j_0,\ k \in \nabla_j\}$ for all $\boldsymbol{\gamma} \in E^n$, where $j_0 \in \mathbb{Z}$ denotes some coarse level.

(b) The functions $\psi_\lambda^{(\boldsymbol{\gamma})}$, $\lambda \in \nabla$ are compactly supported.

(c) One has

$$\partial_i \Psi^{(\boldsymbol{\gamma})} = D^{(i)} \Psi^{(\boldsymbol{\gamma} - \boldsymbol{\delta}_i)}, \tag{3.1}$$

where here $\Psi^{(\boldsymbol{\gamma})}$ is viewed as a column vector of functions and $D^{(i)}$ are *sparse* transformations. By "sparse" we mean here that for each $\lambda \in \nabla$, there is only a small and fixed number of nonzero entries $D^{(i)}_{\lambda,\mu}$ of $D^{(i)}$ ($\mu \in \nabla$).

(d) For all $\lambda \in \nabla$ with $|\lambda| > j_0$ (i.e., ψ_λ is *not* a coarse level function), there exists an index $i_\lambda \in \{1, \dots, n\}$ such that

$$D^{(i_\lambda)}_{i_\lambda} := (D^{(i_\lambda)}_{\mu,\mu'})_{\mu,\mu' \in \nabla_{|\lambda|}; i_\mu = i_{\mu'} = i_\lambda} \tag{3.2}$$

is invertible and

$$\|(D^{(i_\lambda)}_{i_\lambda})^{-1} D^{(i_\mu)}_{i_\mu}\| \lesssim 1$$

holds for all $\mu \in \nabla_{|\lambda|}$ independently of $|\lambda|$, where $\|\cdot\|$ denotes some matrix norm.

(e) We assume that $D^{(i)}_{\lambda,\mu} \neq 0$ only if $|\mu| = |\lambda|$ and $i_\lambda = i_\mu$. □

Let us illustrate this assumption for $n = 2$ by Figure 1. Assumption 1 states here, roughly speaking, that we can apply partial derivatives in the indicated way. Moreover, it states that the partial derivative for each wavelet can be expressed by a finite linear combination of few wavelets of a "lower" system. The conditions (d) and (e) are of a more technical nature. However, they are very important for the subsequent construction, since they allow us to express each wavelet in terms of linear combinations of certain partial derivatives of a "higher" system. The easiest example to think of occurs for $\Omega = \mathbb{R}^2$ and simply taking tensor products of the functions arising in Theorem 3.1.

Remark that condition (a) is by far not automatically fulfilled. It is a matter of adjusting certain parameters that enter the construction of wavelet bases on bounded domains Ω. However, for a significantly large set of examples this problem has been solved in [15, 21, 33]. We will focus on this later in Section 4.

Assumption 1 has some important immediate consequences, which will be used in the sequel: For each $\lambda \in \nabla$, we obtain by (3.1)

$$\partial_i \psi_\lambda^{(\gamma)} = \sum_{\mu \in \nabla_{|\lambda|}} D^{(i)}_{\lambda,\mu}\, \psi_\mu^{(\gamma-\delta_i)} = [D^{(i)}_{|\lambda|}]^\lambda \Psi^{(\gamma-\delta_i)}, \tag{3.3}$$

as well as $\partial_{i_\lambda} \psi_\lambda^{(\gamma)} = [D^{(i_\lambda)}_{i_\lambda}]^\lambda \Psi^{(\gamma-\delta_{i_\lambda})}$, where $[A]^\lambda$ denotes the λ-th row of A. Moreover, for the biorthogonal system $\tilde\Psi^{(\gamma)}$ we have

$$\partial_i \tilde\Psi^{(\gamma-\delta_i)} = -(D^{(i)})^T \tilde\Psi^{(\gamma)}, \tag{3.4}$$

which is easily seen.[1] Hence, the corresponding biorthogonal projectors fulfill

$$\partial_i Q^{(\gamma)} v = Q^{(\gamma-\delta_i)}(\partial_i v), \qquad \partial_i Q_j^{(\gamma)} v = Q_j^{(\gamma-\delta_i)}(\partial_i v), \quad j \geq 0. \tag{3.5}$$

3.2. **Norm equivalences.** We can use Assumption 1 to derive a different way to estimate the $H^1(\Omega)$-norm of a function than the "standard" one given in (2.20) for $s = 1$. It will turn out later that this new estimate is in certain cases better suited for characterizing the spaces $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$.

Denoting by $\ell(\nabla)$ the space of all series that are labelled by the set of indices ∇, we define the operator $\mathsf{d}^{(i)} : \ell(\nabla) \to \ell(\nabla)$, $i = 1, \dots, n$, by

$$(\mathsf{d}^{(i)} \boldsymbol{c})_\lambda := \sum_{\mu \in \nabla_{|\lambda|}} D^{(i)}_{\mu,\lambda}\, c_\mu = [D^{(i)}_{|\lambda|}]_\lambda\, \boldsymbol{c}, \qquad \boldsymbol{c} \in \ell(\nabla), \tag{3.6}$$

where $[A]_\lambda$ denotes the λ-th column of A. Moreover, $\mathsf{d}^{(i_\lambda)}$ is defined in a straightforward manner. Then, we obtain

Remark 3.2. Under the hypotheses of Assumption 1, we have for each function $f = \sum_{\lambda\in\nabla} c_\lambda\, \psi_\lambda^{(\mathbf{1})} \in H^1(\Omega)$ the estimate

$$\|f\|_{H^1(\Omega)} \sim \|\boldsymbol{c}\|_{h^1(\nabla)} := (\|\boldsymbol{c}\|^2_{\ell^2(\nabla)} + \|\mathsf{grad}\, \boldsymbol{c}\|^2_{\ell^2(\boldsymbol{\nabla})})^{1/2}, \tag{3.7}$$

where

$$\mathsf{grad}\, \boldsymbol{c} := (\mathsf{d}^{(1)}\, \boldsymbol{c}, \dots, \mathsf{d}^{(n)}\, \boldsymbol{c})^T \in \ell(\boldsymbol{\nabla}) \tag{3.8}$$

and hence

$$\|\mathsf{grad}\, \boldsymbol{c}\|^2_{\ell^2(\boldsymbol{\nabla})} = \sum_{i=1}^n \|\mathsf{d}^{(i)}\, \boldsymbol{c}\|^2_{\ell^2(\nabla)}.$$

[1] In fact, using the notation $\langle \Theta, \Phi \rangle := \big((\theta,\phi)_{L^2(\Omega)}\big)_{\theta\in\Theta,\phi\in\Phi}$ for two systems of functions, we obtain by integrating by parts and taking the compact support of $\psi_\lambda^{(\gamma)}$ into account

$$D^{(i)} = D^{(i)} \langle \Psi^{(\gamma-\delta_i)}, \tilde\Psi^{(\gamma-\delta_i)} \rangle = \langle \partial_i \Psi^{(\gamma)}, \tilde\Psi^{(\gamma-\delta_i)} \rangle = (-1) \langle \Psi^{(\gamma)}, \partial_i \tilde\Psi^{(\gamma-\delta_i)} \rangle.$$

Now, due to (3.1) and the biorthogonality, we know that $\partial_i \tilde\Psi^{(\gamma-\delta_i)}$ is some linear combination of $\tilde\Psi^{(\gamma)}$, which shows (3.4).

In fact,

$$\begin{aligned} \partial_i f &= \sum_{\lambda\in\nabla}\sum_{\mu\in\nabla_{|\lambda|}} c_\lambda\, D^{(i)}_{\lambda,\mu}\,\psi^{(\mathbf{1}-\boldsymbol{\delta}_i)}_\mu = \sum_{\lambda\in\nabla}\Big(\sum_{\mu\in\nabla_{|\lambda|}} D^{(i)}_{\mu,\lambda}\, c_\mu\Big)\psi^{(\mathbf{1}-\boldsymbol{\delta}_i)}_\lambda \\ &= \sum_{\lambda\in\nabla} (\mathsf{d}^{(i)}\, \boldsymbol{c})_\lambda\, \psi^{(\mathbf{1}-\boldsymbol{\delta}_i)}_\lambda. \end{aligned} \tag{3.9}$$

Note that we may change the ordering of the sums in the second equality above due to Assumption 1 (c) and (e). Using the norm equivalence (2.16) for $\Psi^{(\mathbf{1})}$ and $\boldsymbol{\Psi}^{(\mathbf{1}-\boldsymbol{\delta}_i)}$, $1\le i\le n$, yields

$$\begin{aligned} \|f\|^2_{H^1(\Omega)} &= \|f\|^2_{L^2(\Omega)} + \|\mathbf{grad}\, f\|^2_{\boldsymbol{L}^2(\Omega)} \\ &\sim \|\boldsymbol{c}\|^2_{\ell^2(\nabla)} + \sum_{i=1}^n \|\mathsf{d}^{(i)}\,\boldsymbol{c}\|^2_{\ell^2(\nabla)} = \|\boldsymbol{c}\|^2_{h^1(\nabla)}, \end{aligned}$$

which proves (3.7). □

We compare the estimates (3.7) and (2.20) by the following

Example 3.3. Let us consider the univariate case $\Omega=(0,1)$ and choose a wavelet $\psi^{(1)}_\lambda$, $\lambda=(j,k)$, such that

$$\psi^{(1)}_\lambda(x) = 2^{j/2}\,\psi^{(1)}(2^j x-k),\ x\in[0,1], \qquad \|\psi^{(1)}_\lambda\|_{L^2(0,1)}=1.$$

Then (2.20) gives an estimate of $\|\psi^{(1)}_\lambda\|_{H^1(0,1)}$ by the term $K_{\mathrm{old}} := 2^j$, whereas (3.7) results, in view of Theorem 3.1, in an estimate by $K_{\mathrm{new}} := \sqrt{1+4^2\,2^{2j}}$. On the other hand, since $\|\psi^{(0)}_\lambda\|_{L^2(0,1)}=1$, again Theorem 3.1 implies

$$\|\psi^{(1)}_\lambda\|^2_{H^1(0,1)} = \|\psi^{(1)}_\lambda\|^2_{L^2(0,1)} + \|4\,2^j\,\psi^{(0)}_\lambda\|^2_{L^2(0,1)} = 1+2^{2(j+2)},$$

so that in this case $K_{\mathrm{new}} = \|\psi^{(1)}_\lambda\|_{H^1(0,1)}$, i.e., we obtain an *exact* estimate. □

3.3. **Wavelet systems for $\boldsymbol{H}_0(\mathbf{curl};\Omega)$.** Now, we define a wavelet basis for the space $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ in 3D. The definition is also valid for the 2D case, where one has to consider the space $\boldsymbol{H}_0(\mathrm{curl};\Omega)^2$ defined in an obvious fashion.

Definition 3.4. Let $\mathbf{1} := (1,\dots,1)^T\in E^n$ and define vector wavelet functions for $\boldsymbol{\lambda}=(i,\lambda)\in\boldsymbol{\nabla} := \{(i,\lambda) : i=1,\dots,n;\lambda\in\nabla\}$ by

$$\boldsymbol{\psi}^{\mathrm{curl}}_{\boldsymbol{\lambda}} := \psi^{(\mathbf{1}-\boldsymbol{\delta}_i)}_\lambda\,\boldsymbol{\delta}_i, \qquad \tilde{\boldsymbol{\psi}}^{\mathrm{curl}}_{\boldsymbol{\lambda}} := \tilde{\psi}^{(\mathbf{1}-\boldsymbol{\delta}_i)}_\lambda\,\boldsymbol{\delta}_i. \tag{3.10}$$

Accordingly, we define $\boldsymbol{\Psi}^{\mathrm{curl}} := \{\boldsymbol{\psi}^{\mathrm{curl}}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda}\in\boldsymbol{\nabla}\}$ and $\tilde{\boldsymbol{\Psi}}^{\mathrm{curl}} := \{\tilde{\boldsymbol{\psi}}^{\mathrm{curl}}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda}\in\boldsymbol{\nabla}\}$. □

For the above-defined basis, we want to derive a norm equivalence that allows us to estimate the $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$-norm of each function $\boldsymbol{\zeta}\in\boldsymbol{H}(\mathbf{curl};\Omega)$ by a (discrete) norm of the coefficients in the expansion of $\boldsymbol{\zeta}$ in terms of $\boldsymbol{\Psi}^{\mathrm{curl}}$. Let us now define this norm, where we restrict ourselves to the 3D case, while a similar definition applies to $n=2$. For a given sequence $\boldsymbol{c}\in\ell(\boldsymbol{\nabla})$, we define

$$\mathsf{curl}\,\boldsymbol{c} := (\mathsf{d}^{(2)}\,c^{(3)} - \mathsf{d}^{(3)}\,c^{(2)}, \mathsf{d}^{(3)}\,c^{(1)} - \mathsf{d}^{(1)}\,c^{(3)}, \mathsf{d}^{(1)}\,c^{(2)} - \mathsf{d}^{(2)}\,c^{(1)})^T, \tag{3.11}$$

where we have used the short-hand notation $c^{(i)} := \{c_{(i,\lambda)}\}_{\lambda\in\nabla}$. Then, we set

$$\|\boldsymbol{c}\|_{h(\mathsf{curl};\boldsymbol{\nabla})} := (\|\boldsymbol{c}\|^2_{\ell^2(\boldsymbol{\nabla})} + \|\mathsf{curl}\,\boldsymbol{c}\|^2_{\ell^2(\boldsymbol{\nabla})})^{1/2}. \tag{3.12}$$

Now, we are ready to prove

Theorem 3.5. *Under the hypotheses of Assumption 1, we have for $n = 2, 3$ that the systems $\boldsymbol{\Psi}^{\mathrm{curl}}$, $\tilde{\boldsymbol{\Psi}}^{\mathrm{curl}}$ form a stable biorthogonal wavelet basis for $\boldsymbol{H}_0(\mathbf{curl};\Omega)$. Moreover, for $\boldsymbol{\zeta} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) = \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} c_{\boldsymbol{\lambda}}\, \boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\mathrm{curl}}$ we have*

$$\|\boldsymbol{\zeta}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \sim \|\boldsymbol{c}\|_{h(\mathrm{curl};\boldsymbol{\nabla})}. \tag{3.13}$$

Proof. We will only give the proof for $n = 3$ and remark that the case $n = 2$ can be treated completely analogously. Let us first check that the functions $\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\mathrm{curl}}$, $\boldsymbol{\lambda} \in \boldsymbol{\nabla}$, are indeed in $\boldsymbol{H}(\mathbf{curl};\Omega)$. To this end, set again $\boldsymbol{\lambda} = (i,\lambda) \in \boldsymbol{\nabla}$. Then, we have $[\mathbf{curl}\,\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\mathrm{curl}}]_i = 0$, which is trivially in $L^2(\Omega)$. For $i \neq i'$ we obtain $\partial_{i'}[\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\mathrm{curl}}]_i = [D_{|\lambda|}]^{\lambda}\, \Psi_j^{(\mathbf{1}-\boldsymbol{\delta}_i-\boldsymbol{\delta}_{i'})}$ which, by assumption, is a function in $L^2(\Omega)$ and hence $\mathbf{curl}\,\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\mathrm{curl}} \in \boldsymbol{L}^2(\Omega)$.

The biorthogonality is readily seen:

$$(\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\mathrm{curl}}, \tilde{\boldsymbol{\psi}}_{\boldsymbol{\lambda}'}^{\mathrm{curl}})_{\boldsymbol{L}^2(\Omega)} = \delta_{i,i'}(\psi_{\lambda}^{(\mathbf{1}-\boldsymbol{\delta}_i)}, \tilde{\psi}_{\lambda'}^{(\mathbf{1}-\boldsymbol{\delta}_i)})_{L^2(\Omega)} = \delta_{i,i'}\,\delta_{\lambda,\lambda'} = \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}, \quad \boldsymbol{\lambda},\boldsymbol{\lambda}' \in \boldsymbol{\nabla}.$$

Next, we have to show that each function $\boldsymbol{\zeta} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ has a unique expansion in terms of $\boldsymbol{\Psi}^{\mathrm{curl}}$. By assumption, each $\Psi^{(\mathbf{1}-\boldsymbol{\delta}_i)}$, $1 \le i \le n$, forms a Riesz basis for $L^2(\Omega)$. Since $\boldsymbol{\zeta} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \subset \boldsymbol{L}^2(\Omega)$, this function can in fact be uniquely be expanded in terms of $\boldsymbol{\Psi}^{\mathrm{curl}}$.

Finally, we have to prove (3.13). The norm equivalences for $\Psi^{(\mathbf{1}-\boldsymbol{\delta}_i)}$ imply that $\|\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(\Omega)} \sim \|\boldsymbol{c}\|_{\ell^2(\boldsymbol{\nabla})}$. By (3.9), we obtain

$$(\mathbf{curl}\,\boldsymbol{\zeta})_i = \sum_{\lambda\in\nabla} (\mathsf{curl}\,\boldsymbol{c})_{(i,\lambda)}\, \psi_{\lambda}^{(\boldsymbol{\delta}_i)}, \quad 1 \le i \le n.$$

Using the norm equivalences for $\Psi^{(\boldsymbol{\delta}_i)}$ leads to

$$\|\mathbf{curl}\,\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(\Omega)}^2 \sim \sum_{i=1}^{3} \sum_{\lambda\in\nabla} |(\mathsf{curl}\,\boldsymbol{c})_{(i,\lambda)}|^2 = \|\mathsf{curl}\,\boldsymbol{c}\|_{\ell^2(\boldsymbol{\nabla})}^2, \tag{3.14}$$

so that we obtain (3.13). □

Remark 3.6. The estimate (3.14) shows that $\mathbf{curl}\,\boldsymbol{\zeta} = 0$ in $\boldsymbol{L}^2(\Omega)$ if and only if $\mathsf{curl}\,\boldsymbol{c} \equiv 0$. Hence, given a wavelet expansion of $\boldsymbol{\zeta} \in \boldsymbol{H}(\mathbf{curl};\Omega)$ in terms of $\boldsymbol{\Psi}^{\mathrm{curl}}$, we can check whether $\boldsymbol{\zeta}$ is **curl**-free or not by only considering the wavelet coefficients. This will turn out to be very useful in the sequel. Moreover, (3.14) implies for $\boldsymbol{\zeta} \in \boldsymbol{L}^2(\Omega)$ that $\boldsymbol{\zeta} \in \boldsymbol{H}(\mathbf{curl};\Omega)$ if and only if $\mathsf{curl}\,\boldsymbol{c} \in \ell^2(\boldsymbol{\nabla})$. □

Remark 3.7. When skipping hypothesis (b) of Assumption 1, it is also possible to study wavelet bases in $\boldsymbol{H}(\mathbf{curl};\Omega)$ instead of $\boldsymbol{H}_0(\mathbf{curl};\Omega)$. The above theorem remains valid for this case. □

Remark 3.8. For later purposes, we note the following estimate

$$\Big\| \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} c_{\boldsymbol{\lambda}}\, \boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\mathrm{curl}} \Big\|_{H^s(\Omega)}^2 \sim \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} 2^{2|\boldsymbol{\lambda}|s} |c_{\boldsymbol{\lambda}}|^2, \tag{3.15}$$

provided that all systems $\Psi^{(\mathbf{1}-\boldsymbol{\delta}_i)}$, $1 \le i \le n$, allow the characterization (2.20) for $\sigma = s \in \mathbb{R}$. In fact, (3.15) is an easy consequence of Definition 3.4 and (2.20). □

Let us finally mention that the functions

$$\check{\boldsymbol{\psi}}_{\boldsymbol{\lambda}}^{\mathrm{curl}} := \psi_{\lambda}^{[i]}\, \boldsymbol{\delta}_i, \qquad \check{\tilde{\boldsymbol{\psi}}}_{\boldsymbol{\lambda}}^{\mathrm{curl}} := \tilde{\psi}_{\lambda}^{[i]}\, \boldsymbol{\delta}_i$$

for *any* wavelet systems $\Psi^{[i]}$ in $L^2(\Omega)$, $1 \le i \le n$, give rise to a wavelet basis in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ provided that the functions $\psi_\lambda^{[i]}$ are sufficiently smooth (in order to ensure $\check{\boldsymbol{\psi}}_{\boldsymbol{\lambda}}^{\mathrm{curl}} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$. However, the corresponding norm equivalence will then in general not take such a nice structure as in (3.13). By "nice" we mean that $\mathsf{curl}\, \boldsymbol{c}$ vanishes for $\boldsymbol{\zeta} \in \boldsymbol{H}_0^0(\mathbf{curl};\Omega)$ (see Remark 3.6), which is due to the special choice of the basis functions. Such a property cannot be expected by using arbitrary systems. Moreover, it will become clear in the next subsection that this particular choice enables us to construct **curl**-free vector wavelets.

3.4. **Wavelet bases for $\boldsymbol{H}_0(\mathbf{curl};\Omega)$.** In this subsection we construct **curl**-free wavelet bases, which will be defined as follows. Let us again focus on the 3D case.

Definition 3.9. For $\lambda \in \nabla$ and $1 \le i \le n$, we define

$$(\boldsymbol{\psi}_\lambda^{\mathrm{cf}})_{|i} := [D_{|\lambda|}^{(i)}]^\lambda \, \Psi_{|\lambda|}^{(\mathbf{1}-\boldsymbol{\delta}_i)}, \qquad \tilde{\boldsymbol{\psi}}_\lambda^{\mathrm{cf}} := [(D_{i_\lambda}^{(i_\lambda)})^{-1}]_\lambda^T \, \tilde{\Psi}_{|\lambda|}^{(\mathbf{1}-\boldsymbol{\delta}_{i_\lambda})} \, \boldsymbol{\delta}_{i_\lambda} \tag{3.16}$$

as well as $\boldsymbol{\Psi}^{\mathrm{cf}} := \{\boldsymbol{\psi}_\lambda^{\mathrm{cf}} : \lambda \in \nabla\}$, $\tilde{\boldsymbol{\Psi}}^{\mathrm{cf}} := \{\tilde{\boldsymbol{\psi}}_\lambda^{\mathrm{cf}} : \lambda \in \nabla\}$. □

We remark that we have

$$\boldsymbol{\psi}_\lambda^{\mathrm{cf}} = \mathbf{grad}\, \psi_\lambda^{(\mathbf{1})}, \tag{3.17}$$

which is in the spirit of (2.14).

Theorem 3.10. *Under the hypotheses of Assumption 1, we have that for $n = 2, 3$, the systems $\boldsymbol{\Psi}^{\mathrm{cf}}$, $\tilde{\boldsymbol{\Psi}}^{\mathrm{cf}}$ form a stable biorthogonal wavelet basis for $\boldsymbol{H}_0^0(\mathbf{curl};\Omega)$.*

Proof. It can easily be seen that $\boldsymbol{\psi}_\lambda^{\mathrm{cf}}$ are in fact **curl**-free. Moreover, they are obviously linear combinations of $\boldsymbol{\Psi}^{\mathrm{curl}}$ and $\tilde{\boldsymbol{\Psi}}^{\mathrm{curl}}$, respectively, and hence in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$. Using (3.1), integration by parts, and the biorthogonality of the scalar systems, we obtain the biorthogonality of the vector fields:

$$\begin{aligned}
(\boldsymbol{\psi}_\lambda^{\mathrm{cf}}, \tilde{\boldsymbol{\psi}}_{\lambda'}^{\mathrm{cf}})_{\boldsymbol{L}^2(\Omega)} &= \left(\partial_{i_{\lambda'}} \psi_\lambda^{(\mathbf{1})}, [(D_{i_{\lambda'}}^{(i_{\lambda'})})^{-1}]_{\lambda'}^T \, \tilde{\Psi}_{|\lambda'|}^{(\mathbf{1}-\boldsymbol{\delta}_{i_{\lambda'}})}\right)_{L^2(\Omega)} \\
&= (-1)\left(\psi_\lambda^{(\mathbf{1})}, [(D_{i_{\lambda'}}^{(i_{\lambda'})})^{-1}]_{\lambda'}^T \, \partial_{i_{\lambda'}} \tilde{\Psi}_{|\lambda'|}^{(\mathbf{1}-\boldsymbol{\delta}_{i_{\lambda'}})}\right)_{L^2(\Omega)} \\
&= (\psi_\lambda^{(\mathbf{1})}, \tilde{\psi}_{\lambda'}^{(\mathbf{1})})_{L^2(\Omega)} = \delta_{\lambda,\lambda'}, \quad \lambda, \lambda' \in \nabla.
\end{aligned}$$

Now it remains to prove that the system $\boldsymbol{\Psi}^{\mathrm{cf}}$ is in fact a basis for $\boldsymbol{H}_0^0(\mathbf{curl};\Omega)$, i.e., that each vector field in $\boldsymbol{H}_0^0(\mathbf{curl};\Omega)$ has a unique expansion in terms of $\boldsymbol{\Psi}^{\mathrm{cf}}$. To this end, let $\boldsymbol{\zeta} \in \boldsymbol{H}_0^0(\mathbf{curl};\Omega)$. Since $\boldsymbol{H}_0^0(\mathbf{curl};\Omega) \subset \boldsymbol{L}^2(\Omega)$, Assumption 1 yields

$$\begin{aligned}
\zeta_i &= \sum_{\lambda\in\nabla} (\zeta_i, \tilde{\psi}_\lambda^{(\mathbf{1}-\boldsymbol{\delta}_i)})_{L^2(\Omega)} \, \psi_\lambda^{(\mathbf{1}-\boldsymbol{\delta}_i)}, \\
&= \left(\sum_{\substack{\lambda\in\nabla \\ i_\lambda = i}} + \sum_{\substack{\lambda\in\nabla \\ i_\lambda \neq i}} \right) (\zeta_i, \tilde{\psi}_\lambda^{(\mathbf{1}-\boldsymbol{\delta}_i)})_{L^2(\Omega)} \, \psi_\lambda^{(\mathbf{1}-\boldsymbol{\delta}_i)}
\end{aligned} \tag{3.18}$$

for $1 \le i \le n$. For the first sum in (3.18) we obtain

$$\begin{aligned}
&\sum_{\substack{\lambda\in\nabla\\ i_\lambda=i}} (\zeta_i, \tilde\psi_\lambda^{(\mathbf{1}-\boldsymbol{\delta}_i)})_{L^2(\Omega)}\, \psi_\lambda^{(\mathbf{1}-\boldsymbol{\delta}_i)} \\
&= \sum_{\substack{\lambda\in\nabla\\ i_\lambda=i}} \Big(\zeta_{i_\lambda}, [(D_{i_\lambda}^{(i_\lambda)})^{-1}]_\lambda^T D_{i_\lambda}^{(i_\lambda)}\, \tilde\Psi_{|\lambda|}^{(\mathbf{1}-\boldsymbol{\delta}_{i_\lambda})}\Big)_{L^2(\Omega)}\, \psi_\lambda^{(\mathbf{1}-\boldsymbol{\delta}_{i_\lambda})} \qquad (3.19)\\
&= \sum_{\substack{\lambda\in\nabla\\ i_\lambda=i}} \Big(\zeta_{i_\lambda}, [(D_{i_\lambda}^{(i_\lambda)})^{-1}]_\lambda^T\, \tilde\Psi_{|\lambda|}^{(\mathbf{1}-\boldsymbol{\delta}_i)}\Big)_{L^2(\Omega)}\, [D_{i_\lambda}^{(i_\lambda)}]^\lambda\, \Psi_{|\lambda|}^{(\mathbf{1}-\boldsymbol{\delta}_{i_\lambda})}.
\end{aligned}$$

For the second term in (3.18), we use the fact that $\partial_i\zeta_{i_\lambda} = \partial_{i_\lambda}\zeta_i$ for $i \ne i_\lambda$, which is a consequence of $\mathbf{curl}\,\boldsymbol{\zeta} = 0$. In this case ($i \ne i_\lambda$) we obtain

$$\begin{aligned}
(\zeta_i, \tilde\psi_\lambda^{(\mathbf{1}-\boldsymbol{\delta}_i)})_{L^2(\Omega)} &= (-1)[(D_{i_\lambda}^{(i_\lambda)})^{-1}]_\lambda^T\, (\zeta_i, \partial_{i_\lambda}\tilde\Psi_{|\lambda|}^{(\mathbf{1}-\boldsymbol{\delta}_i-\boldsymbol{\delta}_{i_\lambda})})_{L^2(\Omega)}\\
&= (-1)[(D_{i_\lambda}^{(i_\lambda)})^{-1}]_\lambda^T\, (\zeta_{i_\lambda}, \partial_i\tilde\Psi_{|\lambda|}^{(\mathbf{1}-\boldsymbol{\delta}_i-\boldsymbol{\delta}_{i_\lambda})})_{L^2(\Omega)}\\
&= [(D_{i_\lambda}^{(i_\lambda)})^{-1}]_\lambda^T\, (D^{(i)})^T\, (\zeta_{i_\lambda}, \tilde\Psi_{|\lambda|}^{(\mathbf{1}-\boldsymbol{\delta}_{i_\lambda})})_{L^2(\Omega)},
\end{aligned}$$

which, in turn, implies

$$\begin{aligned}
&\sum_{\substack{\lambda\in\nabla\\ i_\lambda\ne i}} (\zeta_i, \tilde\psi_\lambda^{(\mathbf{1}-\boldsymbol{\delta}_i)})_{L^2(\Omega)}\, \psi_\lambda^{(\mathbf{1}-\boldsymbol{\delta}_i)}\\
&= \sum_{\substack{\lambda\in\nabla\\ i_\lambda\ne i}} (\zeta_{i_\lambda}, [(D_{i_\lambda}^{(i_\lambda)})^{-1}]_\lambda^T\, \tilde\Psi_{|\lambda|}^{(\mathbf{1}-\boldsymbol{\delta}_{i_\lambda})})_{L^2(\Omega)}\, [D_{|\lambda|}^{(i)}]^\lambda\, \Psi_{|\lambda|}^{(\mathbf{1}-\boldsymbol{\delta}_i)}.
\end{aligned}$$

Finally, using the latter equation, (3.18) and (3.19) yields

$$\zeta_i = \sum_{\lambda\in\nabla} (\zeta_{i_\lambda}, [(D_{i_\lambda}^{(i_\lambda)})^{-1}]_\lambda^T\, \tilde\Psi_{|\lambda|}^{(\mathbf{1}-\boldsymbol{\delta}_{i_\lambda})})_{L^2(\Omega)}\, \partial_i\psi_\lambda^{(\mathbf{1})} = \sum_{\lambda\in\nabla} (\boldsymbol{\zeta}, \tilde{\boldsymbol{\psi}}_\lambda^{\mathrm{cf}})_{\boldsymbol{L}^2(\Omega)}\, (\boldsymbol{\psi}_\lambda^{\mathrm{cf}})_{|i}$$

for $1 \le i \le n$, which completes the proof. □

The above result also gives rise to a characterization of $\boldsymbol{H}_0^0(\mathbf{curl};\Omega)'$. Of course, since the space $\boldsymbol{H}_0^0(\mathbf{curl};\Omega)$ is a closed subspace of $\boldsymbol{L}^2(\Omega)$ (see, e.g., [25]), the dual space of $\boldsymbol{H}_0^0(\mathbf{curl};\Omega)$ is isomorphic to the space itself. However, it may be helpful to have a concrete representation at hand. We would like to point out first that the dual system $\tilde{\boldsymbol{\Psi}}^{\mathrm{cf}}$ is in general *not* in $\boldsymbol{H}_0^0(\mathbf{curl};\Omega)$. The biorthogonality between $\boldsymbol{\Psi}^{\mathrm{cf}}$ and $\tilde{\boldsymbol{\Psi}}^{\mathrm{cf}}$, however, ensures that $\tilde{\boldsymbol{\Psi}}^{\mathrm{cf}} \subset \boldsymbol{H}_0^0(\mathbf{curl};\Omega)'$. In view of Theorem 3.10, this already proves the following result.

Corollary 3.11. *A vector field $\boldsymbol{\zeta} \in \boldsymbol{L}^2(\Omega)$ is in $\boldsymbol{H}_0^0(\mathbf{curl};\Omega)'$ if and only if*

$$\sum_{\lambda\in\nabla} |(\boldsymbol{\zeta}, \boldsymbol{\psi}_\lambda^{\mathrm{cf}})_{\boldsymbol{L}^2(\Omega)}|^2 < \infty. \qquad (3.20)$$

□

Finally, we derive some norm equivalences. If Ω is a bounded domain, (3.17) and the Poincaré–Friedrichs inequality imply the estimate

$$\Big\| \sum_{\lambda\in\nabla} c_\lambda\, \boldsymbol{\psi}_\lambda^{\mathrm{cf}} \Big\|_{\boldsymbol{H}^s(\Omega)}^2 \sim \sum_{\lambda\in\nabla} 2^{2(s+1)|\lambda|}\, |c_\lambda|^2, \qquad (3.21)$$

provided that the norm equivalence (2.20) holds for $\Psi^{(\mathbf{1})}$ and $\sigma = s+1$. Again, we obtain another estimate using the operator grad defined in (3.8).

Proposition 3.12. *Under the hypotheses of Assumption 1, we have*

$$\Big\| \sum_{\lambda\in\nabla} c_\lambda\, \boldsymbol{\psi}_\lambda^{\mathrm{cf}} \Big\|_{\boldsymbol{L}^2(\Omega)} \sim \|\mathsf{grad}\, \boldsymbol{c}\|_{\ell^2(\nabla)}.$$

Proof. For each component $(i = 1, \dots, n)$, we have by (3.9)

$$\sum_{\lambda\in\nabla} c_\lambda\, (\boldsymbol{\psi}_\lambda^{\mathrm{cf}})_{|i} = \sum_{\lambda\in\nabla} (\mathsf{grad}\, \boldsymbol{c})_{(i,\lambda)}\, \psi_\lambda^{(\mathbf{1}-\boldsymbol{\delta}_i)}.$$

Using the norm equivalence for $\Psi^{(\mathbf{1}-\boldsymbol{\delta}_i)}$ yields the desired result. □

3.5. **Wavelets in flux spaces.** All constructions and facts in this subsection are basically known and can be found in [17, 26, 32, 33]. However, we simplified and generalized them somewhat and recall them in order to show the interrelations to the wavelet bases in $\boldsymbol{H}(\mathbf{curl};\Omega)$.

3.5.1. *Wavelet bases in* $\boldsymbol{H}(\mathrm{div};\Omega)$. With the technology of Section 2 at hand, it is not hard to see that the wavelet systems $\boldsymbol{\Psi}^{\mathrm{div}}$, $\tilde{\boldsymbol{\Psi}}^{\mathrm{div}}$ defined by the functions

$$\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\mathrm{div}} := \psi_\lambda^{(\boldsymbol{\delta}_i)}\, \boldsymbol{\delta}_i, \quad \tilde{\boldsymbol{\psi}}_{\boldsymbol{\lambda}}^{\mathrm{div}} := \tilde{\psi}_\lambda^{(\boldsymbol{\delta}_i)}\, \boldsymbol{\delta}_i, \qquad \boldsymbol{\lambda} = (i,\lambda) \in \boldsymbol{\nabla}, \tag{3.22}$$

form a biorthogonal system in $\boldsymbol{H}(\mathrm{div};\Omega)$. Moreover, the norm equivalence

$$\|\boldsymbol{\zeta}\|_{\boldsymbol{H}(\mathrm{div};\Omega)} \sim \|\boldsymbol{c}\|_{h(\mathrm{div};\boldsymbol{\nabla})} \tag{3.23}$$

holds for any $\boldsymbol{\zeta} = \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} c_{\boldsymbol{\lambda}} \boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\mathrm{div}} \in \boldsymbol{H}(\mathrm{div};\Omega)$, where we have defined

$$\mathsf{div} : \ell(\boldsymbol{\nabla}) \to \ell(\nabla), \qquad \mathsf{div}\, \boldsymbol{c} := \sum_{i=1}^{n} \mathsf{d}^{(i)}\, c^{(i)},$$

and

$$\|\boldsymbol{c}\|_{h(\mathrm{div};\boldsymbol{\nabla})} := (\|\boldsymbol{c}\|_{\ell^2(\boldsymbol{\nabla})}^2 + \|\mathsf{div}\, \boldsymbol{c}\|_{\ell^2(\nabla)}^2)^{1/2}.$$

In fact, by using reasoning similar to (3.9), we obtain

$$\begin{aligned}
\|\boldsymbol{\zeta}\|_{\boldsymbol{H}(\mathrm{div};\Omega)}^2 &= \|\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(\Omega)}^2 + \Big\| \sum_{i=1}^{n} \sum_{\lambda\in\nabla} c_{\boldsymbol{\lambda}}\, \partial_i \psi_\lambda^{(\boldsymbol{\delta}_i)} \Big\|_{L^2(\Omega)}^2 \\
&= \|\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(\Omega)}^2 + \Big\| \sum_{\lambda\in\nabla} \sum_{i=1}^{n} (\mathsf{d}^{(i)} c^{(i)})_\lambda\, \psi_\lambda^{(\mathbf{0})} \Big\|_{L^2(\Omega)}^2 \\
&\sim \|\boldsymbol{c}\|_{\ell^2(\boldsymbol{\nabla})}^2 + \sum_{\lambda\in\nabla} |(\mathsf{div}\, \boldsymbol{c})_\lambda|^2 \;=\; \|\boldsymbol{c}\|_{h(\mathrm{div};\boldsymbol{\nabla})}^2,
\end{aligned}$$

which proves (3.23). Finally, by using the standard estimate (2.20), we obtain

$$\|\boldsymbol{\zeta}\|_{\boldsymbol{H}^s(\Omega)}^2 \sim \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} 2^{2|\boldsymbol{\lambda}|s}\, |c_{\boldsymbol{\lambda}}|^2 \tag{3.24}$$

provided that all $\Psi^{(\boldsymbol{\delta}_i)}$, $1 \le i \le n$, fulfill (2.20) for $\sigma = s$.

3.5.2. *Divergence-free wavelet bases.* Let us recall the construction of divergence-free wavelets. The construction of compactly supported divergence-free wavelets was initiated by P.G. Lemarié-Rieusset [26] (see also [32] for the generalization to nontensor product functions).

Definition 3.13. For

$$\boldsymbol{\lambda} \in \boldsymbol{\nabla}^{\mathrm{df}} := \{\boldsymbol{\lambda} = (i,\lambda) : \lambda \in \nabla,\, 1 \le i \le n,\, i \neq i_\lambda\},$$

we define

$$(\boldsymbol{\psi}^{\mathrm{df}}_{\boldsymbol{\lambda}})_{|\nu} := \begin{cases} 0, & \nu \notin \{i, i_\lambda\}, \\ \psi_\lambda^{(\boldsymbol{\delta}_i)}, & \nu = i, \\ -\partial_i (D_{i_\lambda}^{(i_\lambda)})^{-1} \Psi_{|\lambda|}^{(\boldsymbol{\delta}_i + \boldsymbol{\delta}_{i_\lambda})}, & \nu = i_\lambda, \end{cases} \tag{3.25}$$

as well as

$$\tilde{\boldsymbol{\psi}}^{\mathrm{df}}_{\boldsymbol{\lambda}} := \tilde{\psi}_\lambda^{(\boldsymbol{\delta}_i)}, \quad \boldsymbol{\lambda} \in \boldsymbol{\nabla}^{\mathrm{df}}. \tag{3.26}$$

The systems $\boldsymbol{\Psi}^{\mathrm{df}}$ and $\tilde{\boldsymbol{\Psi}}^{\mathrm{df}}$ are defined in a straightforward way. □

For convenience, let us briefly check that these functions are divergence-free. In fact, using (3.1) again, we obtain

$$\begin{aligned} \operatorname{div} \boldsymbol{\psi}^{\mathrm{df}}_{\boldsymbol{\lambda}} &= \partial_i \psi_\lambda^{(\boldsymbol{\delta}_i)} - \partial_{i_\lambda} \partial_i (D_{i_\lambda}^{(i_\lambda)})^{-1} \Psi_{|\lambda|}^{(\boldsymbol{\delta}_i + \boldsymbol{\delta}_{i_\lambda})} \\ &= \partial_{i_\lambda} \partial_i (D_{i_\lambda}^{(i_\lambda)})^{-1} \Psi_{|\lambda|}^{(\boldsymbol{\delta}_i + \boldsymbol{\delta}_{i_\lambda})} - \partial_{i_\lambda} \partial_i (D_{i_\lambda}^{(i_\lambda)})^{-1} \Psi_{|\lambda|}^{(\boldsymbol{\delta}_i + \boldsymbol{\delta}_{i_\lambda})} = 0. \end{aligned}$$

In summary, we have

Theorem 3.14. [26, 32, 33] *Under the hypotheses of Assumption 1, the following statements hold:*

(a) $\boldsymbol{\Psi}^{\mathrm{div}}$, $\tilde{\boldsymbol{\Psi}}^{\mathrm{div}}$ *form a biorthogonal system for* $\boldsymbol{H}_0(\mathrm{div};\Omega)$.

(b) $\boldsymbol{\Psi}^{\mathrm{df}}$ *forms a Riesz basis for* $\boldsymbol{H}^0_0(\mathrm{div};\Omega)$. □

Finally, we aim at deriving a norm equivalence in $\boldsymbol{H}^s(\Omega)$. Since this result is not included in the above-mentioned references, we will give it in detail. Let us consider a divergence-free vector field:

$$\boldsymbol{\zeta} = \sum_{\boldsymbol{\lambda} \in \boldsymbol{\nabla}^{\mathrm{df}}} c_{\boldsymbol{\lambda}}\, \boldsymbol{\psi}^{\mathrm{df}}_{\boldsymbol{\lambda}}, \qquad \boldsymbol{c} \in \ell^2(\boldsymbol{\nabla}^{\mathrm{df}}). \tag{3.27}$$

Proposition 3.15. *Under the hypotheses of Assumption 1, the following estimate holds for the vector field* $\boldsymbol{\zeta}$ *in* (3.27):

$$\|\boldsymbol{\zeta}\|^2_{\boldsymbol{H}^s(\Omega)} \sim \sum_{\boldsymbol{\lambda} \in \boldsymbol{\nabla}^{\mathrm{df}}} 2^{2s|\boldsymbol{\lambda}|} |c_{\boldsymbol{\lambda}}|^2.$$

Proof. Since $\boldsymbol{\zeta}$ is in particular in $\boldsymbol{H}_0(\mathrm{div};\Omega)$, it has a unique expansion in terms of Ψ^{div}

$$\boldsymbol{\zeta} = \sum_{\boldsymbol{\lambda} \in \boldsymbol{\nabla}} \bar{c}_{\boldsymbol{\lambda}}\, \boldsymbol{\psi}^{\mathrm{div}}_{\boldsymbol{\lambda}}.$$

By construction, we have $\bar{c}_{\boldsymbol{\lambda}} = c_{\boldsymbol{\lambda}}$ for all $\boldsymbol{\lambda} \in \boldsymbol{\nabla}^{\mathrm{df}}$. Since $\boldsymbol{\zeta}$ is divergence-free, its coefficients $\bar{\boldsymbol{c}}$ fulfill $\mathsf{div}\, \bar{\boldsymbol{c}} = 0$ (see (3.23)) and hence

$$\mathsf{d}^{(i_\lambda)}\, \bar{c}^{(n)} = -\sum_{i \neq i_\lambda} \mathsf{d}^{(i)}\, \bar{c}^{(i)} = -\sum_{i \neq i_\lambda} \mathsf{d}^{(i)}\, c^{(i)},$$

which implies

$$\bar{c}^{(i_\lambda)} = -\sum_{i\neq i_\lambda} (\mathsf{d}^{(i_\lambda)})^{-1}\,\mathsf{d}^{(i)}\,c^{(i)}. \tag{3.28}$$

Now, we use the norm equivalences for $\boldsymbol{\Psi}^{\mathrm{div}}$ and obtain

$$\|\boldsymbol{\zeta}\|^2_{\boldsymbol{H}^s(\Omega)} \sim \sum_{i=1}^{n} 2^{2s|\boldsymbol{\lambda}|}\,\|\bar{c}^{(i)}\|^2_{\ell^2(\nabla)}, \tag{3.29}$$

so that by (3.28)

$$\begin{aligned}\|\boldsymbol{\zeta}\|^2_{\boldsymbol{H}^s(\Omega)} &\lesssim \sum_{i\neq i_\lambda} 2^{2s|\boldsymbol{\lambda}|}\,\|c^{(i)}\|^2_{\ell^2(\nabla)} + \sum_{i\neq i_\lambda} 2^{2s|\boldsymbol{\lambda}|}\,\|(\mathsf{d}^{(i_\lambda)})^{-1}\,\mathsf{d}^{(i)}\,c^{(i)}\|^2_{\ell^2(\nabla)}\\ &\lesssim \sum_{i\neq i_\lambda} 2^{2s|\boldsymbol{\lambda}|}\,\|c^{(i)}\|^2_{\ell^2(\nabla)},\end{aligned}$$

where we have used Assumption 1, (d). On the other hand, again by (3.29),

$$\|\boldsymbol{\zeta}\|^2_{\boldsymbol{H}^s(\Omega)} \gtrsim \sum_{i=1}^{n} 2^{2s|\boldsymbol{\lambda}|}\,\|\bar{c}^{(i)}\|^2_{\ell^2(\nabla)} \geq \sum_{i\neq i_\lambda} 2^{2s|\boldsymbol{\lambda}|}\,\|c^{(i)}\|^2_{\ell^2(\nabla)},$$

which proves our assertion. □

3.5.3. *A biorthogonal Helmholtz decomposition.* The wavelets defined in (3.25) give rise to a Helmholtz decomposition of $\boldsymbol{H}(\mathrm{div};\Omega)$. This was introduced in [33] and may be summarized as follows. Using the same notation as above, we define

$$\boldsymbol{\psi}^{\Delta}_{\lambda} := \psi^{(\boldsymbol{\delta}_{i_\lambda})}_{\lambda}\,\boldsymbol{\delta}_{i_\lambda}, \quad \lambda\in\nabla. \tag{3.30}$$

The induced spaces

$$\boldsymbol{S}^{\Delta}_{j} := \mathrm{clos}_{\boldsymbol{H}(\mathrm{div};\Omega)}\ \mathrm{span}\ \boldsymbol{\Psi}^{\Delta}_{j}, \qquad \boldsymbol{S}^{\mathrm{df}}_{j} := \mathrm{clos}_{\boldsymbol{H}(\mathrm{div};\Omega)}\ \mathrm{span}\ \boldsymbol{\Psi}^{\mathrm{df}}_{j}$$

fulfill

$$\boldsymbol{S}^{\mathrm{div}}_{j} = \boldsymbol{S}^{\Delta}_{j} \oplus \boldsymbol{S}^{\mathrm{df}}_{j},$$

where $\boldsymbol{S}^{\mathrm{div}}_{j}$ is induced by $\boldsymbol{\Psi}^{\mathrm{div}}$ in (3.22) in the natural way. This results in a splitting

$$\boldsymbol{H}_0(\mathrm{div};\Omega) = \boldsymbol{H}^0_0(\mathrm{div};\Omega) \oplus \boldsymbol{H}^{\Delta}(\Omega),$$

where $\boldsymbol{H}^{\Delta}(\Omega) = S(\boldsymbol{\Psi}^{\Delta})$ (and also for $\boldsymbol{L}^2(\Omega)$ instead of $\boldsymbol{H}_0(\mathrm{div};\Omega)$). This decomposition is *not* orthogonal, but stable, which can be shown by introducing the dual functions

$$\tilde{\boldsymbol{\psi}}^{\mathrm{df}}_{\boldsymbol{\lambda}} := \tilde{\psi}^{(\boldsymbol{\delta}_i)}_{\lambda}\boldsymbol{\delta}_i, \quad \boldsymbol{\lambda} = (i,\lambda)\in\boldsymbol{\nabla}^{\mathrm{df}}$$

as well as

$$(\tilde{\boldsymbol{\psi}}^{\Delta}_{\lambda})_{|\nu} := \partial_\nu[-(D^{(i_\lambda)}_{i_\lambda})^{-1}]_\lambda\tilde{\psi}^{(\mathbf{0})}_{\lambda}, \quad 1\leq\nu\leq n.$$

Then, the following orthogonality relations hold [33]

$$(\boldsymbol{\psi}^{\mathrm{df}}_{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\psi}}^{\Delta}_{\lambda'})_{\boldsymbol{L}^2(\Omega)} = (\tilde{\boldsymbol{\psi}}^{\mathrm{df}}_{\boldsymbol{\lambda}}, \boldsymbol{\psi}^{\Delta}_{\lambda'})_{\boldsymbol{L}^2(\Omega)} = 0$$

for $\boldsymbol{\lambda}\in\boldsymbol{\nabla}^{\mathrm{df}}$, $\lambda\in\nabla$, and

$$(\boldsymbol{\psi}^{\Delta}_{\lambda}, \tilde{\boldsymbol{\psi}}^{\Delta}_{\lambda'})_{\boldsymbol{L}^2(\Omega)} = \delta_{\lambda,\lambda'}, \quad (\boldsymbol{\psi}^{\mathrm{df}}_{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\psi}}^{\mathrm{df}}_{\boldsymbol{\lambda}'})_{\boldsymbol{L}^2(\Omega)} = \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}$$

for all $\lambda,\lambda'\in\nabla$ and $\boldsymbol{\lambda},\boldsymbol{\lambda}'\in\boldsymbol{\nabla}^{\mathrm{df}}$.

3.6. **Interrelations and Hodge decompositions.** With the above constructed various wavelet bases for flux and stream function spaces at hand, we obtain some relationships between the bases (and induced spaces, of course) as well as Hodge decompositions. Firstly, we may reformulate the norm equivalence (3.13) for $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ in the following way:

$$\|\boldsymbol{\zeta}\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega)} \sim \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} \left\{ |(\boldsymbol{\zeta}, \boldsymbol{\psi}^{\mathrm{curl}}_{\boldsymbol{\lambda}})_{\boldsymbol{L}^2(\Omega)}|^2 + |(\mathbf{curl}\,\boldsymbol{\zeta}, \boldsymbol{\psi}^{\mathrm{div}}_{\boldsymbol{\lambda}})_{\boldsymbol{L}^2(\Omega)}|^2 \right\}$$

for $\boldsymbol{\zeta} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$. Let us now consider the two Hodge decompositions (2.12) and (2.13).

Theorem 3.16. *Under the hypotheses of Assumption 1 we have that for any subsets $\Lambda^{\mathrm{df}} \subset \boldsymbol{\nabla}^{\mathrm{df}}$, $\Lambda^{\mathrm{cf}} \subset \nabla$, the induced spaces*

$$\boldsymbol{S}^{\mathrm{df}}_{\Lambda} := \mathrm{clos}_{\boldsymbol{H}^0_0(\mathrm{div};\Omega)}\ \mathrm{span}\ \boldsymbol{\Psi}^{\mathrm{df}}_{\Lambda^{\mathrm{df}}}, \qquad \boldsymbol{S}^{\mathrm{cf}}_{\Lambda} := \mathrm{clos}_{\boldsymbol{H}(\mathbf{curl};\Omega)}\ \mathrm{span}\ \boldsymbol{\Psi}^{\mathrm{cf}}_{\Lambda^{\mathrm{cf}}}, \tag{3.31}$$

are orthogonal with respect to $(\cdot,\cdot)_{\boldsymbol{L}^2(\Omega)}$ and

$$a_{\mathrm{div}}(\boldsymbol{u},\boldsymbol{v}) := (\boldsymbol{u},\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} + (\mathrm{div}\ \boldsymbol{u}, \mathrm{div}\ \boldsymbol{v})_{L^2(\Omega)}, \tag{3.32}$$

as well as

$$a_{\mathrm{curl}}(\boldsymbol{u},\boldsymbol{v}) := (\boldsymbol{u},\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} + (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}. \tag{3.33}$$

Proof. Using integration by parts and (3.1), we obtain

$$\begin{aligned} (\boldsymbol{\psi}^{\mathrm{df}}_{\boldsymbol{\lambda}}, \boldsymbol{\psi}^{\mathrm{cf}}_{\lambda'})_{\boldsymbol{L}^2(\Omega)} &= (\boldsymbol{\psi}^{\mathrm{df}}_{\boldsymbol{\lambda}}, \mathbf{grad}\,\psi^{(\mathbf{1})}_{\lambda'})_{\boldsymbol{L}^2(\Omega)} \\ &= (-1)\,(\mathrm{div}\ \boldsymbol{\psi}^{\mathrm{df}}_{\boldsymbol{\lambda}}, \psi^{(\mathbf{1})}_{\lambda'})_{L^2(\Omega)} \ = \ 0, \qquad \boldsymbol{\lambda}\in\boldsymbol{\nabla}^{\mathrm{df}},\ \lambda'\in\nabla, \end{aligned}$$

since $\boldsymbol{\psi}^{\mathrm{df}}_{\boldsymbol{\lambda}}$ is divergence-free. The orthogonality w.r.t. $a_{\mathrm{div}}(\cdot,\cdot)$ and $a_{\mathrm{curl}}(\cdot,\cdot)$ automatically follows from the fact that $\boldsymbol{\psi}^{\mathrm{df}}_{\boldsymbol{\lambda}}$, $\boldsymbol{\psi}^{\mathrm{cf}}_{\lambda}$ are, respectively, divergence- and **curl**-free. □

The latter theorem obviously gives rise to a Hodge decomposition in the spirit of (2.12). Note that Theorem 3.16 holds for *any* subsets Λ^{df}, Λ^{cf}, so that these bases may also be used for an adaptive approach. Moreover, it should be noted that $\boldsymbol{\Psi}^{\mathrm{cf}}$ in the latter theorem need not have vanishing traces on the boundary of Ω. Recall that, due to Remark 3.7, this is also provided by our construction. The second Hodge decomposition (2.13) now involves the space $\boldsymbol{H}^0_0(\mathbf{curl};\Omega)$, i.e., the trace of these functions vanishes on the boundary of Ω. The price we have to pay for this is a higher smoothness of the divergence-free wavelet functions (and hence, in general, a larger support).

Theorem 3.17. *In addition to Assumption 1, assume that $\Psi^{(\mathbf{0})} \subset \boldsymbol{H}^1_0(\Omega)$. Then, the subsets $\boldsymbol{S}^{\mathrm{df}}_{\Lambda}$ and $\boldsymbol{S}^{\mathrm{cf}}_{\Lambda}$ give rise to a Hodge decomposition relative to* (2.13) *w.r.t. $(\cdot,\cdot)_{\boldsymbol{L}^2(\Omega)}$, $a_{\mathrm{div}}(\cdot,\cdot)$ and $a_{\mathrm{curl}}(\cdot,\cdot)$.*

Proof. Since $\Psi^{(\mathbf{0})} \subset \boldsymbol{H}^1_0(\Omega)$, we obtain $\boldsymbol{S}^{\mathrm{df}}_{\Lambda} \subset \boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}^0(\boldsymbol{H}\mathrm{div};\Omega) \cap \boldsymbol{H}_0(\mathbf{curl};\Omega)$ which is isomorphic to $\boldsymbol{H}^0_0(\mathbf{curl};\Omega)^{\perp}$. The orthogonality was already shown in the proof of Theorem 3.16 above. □

4. Examples

Our construction is so far based on the general hypotheses stated in Assumption 1. In this section, we give examples of wavelet bases on certain domains $\Omega \subset \mathbb{R}^n$ that fulfill this assumption.

The first example is related to tensor product domains. On these domains, wavelet bases can be used that are tensor products of corresponding functions on the interval $[0,1]$. Hence, we consider Assumption 1 for wavelet systems on the interval. Moreover, we show concrete examples for Daubechies' orthonormal wavelets as well as biorthogonal spline wavelets (adapted to the interval).

Of course, tensor product domains are of limited use when one thinks to applications in numerical analysis. Hence, we consider two types of generalizations. Both treat domains that are the image of $\hat{\Omega} = [0,1]^n$ under certain parametric mappings. The first example are affine mappings, the second one conformal mappings in 2D. Finally, we outline further extensions using domain decomposition ideas.

Scaling systems. Before going to these examples, let us recall the basic facts on scaling systems. They will be of some importance in checking the conditions of Assumption 1. In many cases, the (*primal*) wavelet system Ψ is constructed with the aid of a family of $L^2(\Omega)$-functions $\Phi_j := \{\varphi_{j,k} : k \in \Delta_j\}$, where again $j \in \mathbb{N}_0$ can be understood as the *scale* or the *level* and Δ_j is some (finite) set of indices. Thinking of Φ_j as the column vector with components $\varphi_{j,k}$, $k \in \Delta_j$, we say that $\{\Phi_j\}_{j\in\mathbb{N}_0}$ is *refinable* if there exists a matrix $M_{j,0} \in \mathbb{R}^{|\Delta_{j+1}|\times|\Delta_j|}$ such that

$$\Phi_j = M_{j,0}^T \Phi_{j+1}. \tag{4.1}$$

In other words, (4.1) means that each function on level j can be represented as a linear combination of the basis functions on level $j+1$. A refinable system $\{\Phi_j\}_{j\in\mathbb{N}_0}$ of linearly independent functions (where the constants in the corresponding norm equivalences do not depend on j) is also called a *(primal) single scale system*, *generator* or *scaling* system. The refinement equation (4.1) in particular implies that the induced spaces

$$S_j := S(\Phi_j) := \operatorname{span}(\Phi_j)$$

are *nested*: $S_j \subset S_{j+1}$.

The *biorthogonal* wavelet system $\tilde{\Psi}$ is generated by a (*dual*) single scale system formed by $\tilde{\Phi}_j = \{\tilde{\varphi}_{j,k};\ k \in \Delta_j\}$, i.e.,

$$(\varphi_{j,k}, \tilde{\varphi}_{j,k'})_{L^2(\Omega)} = \delta_{k,k'}, \tag{4.2}$$

for $j \in \mathbb{N}_0$ and $k, k' \in \Delta_j$.

Now, biorthogonal wavelet spaces $W_j, \tilde{W}_j$ are defined by

$$W_j := S_{j+1} \ominus S_j,\ \tilde{W}_j := \tilde{S}_{j+1} \ominus \tilde{S}_j, \qquad S_j \perp \tilde{W}_j,\ \tilde{S}_j \perp W_j, \tag{4.3}$$

where the orthogonality is to be understood with respect to the L^2-inner product. Constructing *biorthogonal wavelets* then amounts to finding bases

$$\Psi_j := \{\psi_{j,k} :\ k \in \nabla_j\},\ \tilde{\Psi}_j := \{\tilde{\psi}_{j,k} :\ k \in \nabla_j\}, \quad (\nabla_j := \Delta_{j+1} \setminus \Delta_j) \tag{4.4}$$

of W_j, $\tilde{W}_j$, respectively, such that

$$\Psi_j = M_{j,1}^T \Phi_{j+1}, \quad \tilde{\Psi}_j = \tilde{M}_{j,1}^T \tilde{\Phi}_{j+1}, \qquad M_{j,1}, \tilde{M}_{j,1} \in \mathbb{R}^{|\Delta_{j+1}|\times|\nabla_j|},$$

as well as

$$(\psi_{j,k}, \tilde{\psi}_{j,k'})_{L^2(\Omega)} = \delta_{k,k'}, \qquad j \in \mathbb{N}_0,\ k, k' \in \nabla_j, \tag{4.5}$$

and the collections

$$\Psi := \{\psi_\lambda : \lambda \in \nabla\}, \qquad \tilde{\Psi} := \{\tilde{\psi}_\lambda : \lambda \in \nabla\}, \qquad \nabla := \bigcup_{j \geq -1} \nabla_j \tag{4.6}$$

($\Psi_{-1} := \Phi_0, \tilde{\Psi}_{-1} := \tilde{\Phi}_0, \nabla_{-1} := \Delta_0$) form Riesz bases for $L^2(\Omega)$.

One may also view the construction of wavelet systems as follows (see [16]). Given a rectangular matrix $M_{j,0}$, find a sparse completion $M_{j,1}$ such that the composed matrix $M_j := (M_{j,0}\ M_{j,1})$ is invertible, its inverse is also sparse, and one has

$$\|M_j\|, \|M_j^{-1}\| = \mathcal{O}(1), \quad j \to \infty.$$

4.1. **Tensor product domains.** Now we are ready to consider wavelet bases on tensor product domains that fulfill the conditions of Assumption 1. We start by the univariate functions on the interval.

4.1.1. *Univariate functions.* Since we will be concerned with tensor products of univariate scaling and wavelet systems, it will be convenient to use a different notation for the systems in 1D. We will denote by

$$\Xi_j := \{\xi_{j,k} : k \in \mathcal{I}_j\}, \qquad \tilde{\Xi}_j := \{\tilde{\xi}_{j,k} : k \in \mathcal{I}_j\} \tag{4.7}$$

the univariate dual scaling systems, and by

$$\Upsilon_j := \{\eta_{j,k} : k \in \mathcal{J}_j\}, \qquad \tilde{\Upsilon}_j := \{\tilde{\eta}_{j,k} : k \in \mathcal{J}_j\} \tag{4.8}$$

the univariate biorthogonal wavelet systems ($\mathcal{I}_j$, $\mathcal{J}_j$ being appropriate sets of indices). Using, as above $\Upsilon_{-1} := \Xi_0$, $\tilde{\Upsilon}_{-1} := \tilde{\Xi}_0$, we obtain the full set of wavelets

$$\Upsilon := \bigcup_{j \geq -1} \Upsilon_j, \qquad \tilde{\Upsilon} := \bigcup_{j \geq -1} \tilde{\Upsilon}_j, \tag{4.9}$$

and the corresponding set of indices

$$\mathcal{J} := \bigcup_{j \geq -1} \mathcal{J}_j. \tag{4.10}$$

Finally, we introduce the abbreviation

$$\mathcal{J}_{j,e} := \begin{cases} \mathcal{J}_j, & e = 1, \\ \mathcal{I}_j & e = 0. \end{cases} \tag{4.11}$$

4.1.2. *Differentiation and integration.* In order to check the conditions of Assumption 1, we first need the following extension of Theorem 3.1 (which was stated for the real line) to systems on the interval $[0, 1]$. The proof can be found e.g., in [21, 28, 33]. So far, to our knowledge, the result has been proved for three examples of wavelet systems on $[0, 1]$:

(a) Orthonormal wavelets on $[0, 1]$, see [13]. In this case, all what is said below holds for $\tilde{\Upsilon} = \Upsilon$.
(b) Systems arising by iteratively applying Theorem 4.1 below to the systems in (a) and the arising results. I.e., these are biorthogonal systems arising from orthonormal ones by differentiation and integration.
(c) Biorthogonal spline wavelets on $[0, 1]$, see [18].

Theorem 4.1. *Let $\Xi_j^{(\mathbf{1})}$, $\tilde{\Xi}_j^{(\mathbf{1})}$ be one of the above listed systems of univariate scaling functions and $\Upsilon_j^{(\mathbf{1})}$, $\tilde{\Upsilon}_j^{(\mathbf{1})}$ be the induced biorthogonal wavelet system on $\Omega \in \{[0,1], \mathbb{R}, \mathbb{T}$ (the torus)$\}$ such that $\Upsilon^{(\mathbf{1})} \subset H_0^1(\Omega)$. Then, there exists a second system of dual scaling functions $\Xi_j^{(\mathbf{0})}$, $\tilde{\Xi}_j^{(\mathbf{0})}$ and induced biorthogonal wavelets $\Upsilon_j^{(\mathbf{0})}$, $\tilde{\Upsilon}_j^{(\mathbf{0})}$ (w.r.t. the* same *set of indices $\mathcal{I}_j$, $\mathcal{J}_j$, respectively) such that*

$$
\begin{aligned}
\tfrac{d}{dx}\Xi_j^{(\mathbf{1})} &= D_{j,0}\,\Xi_j^{(\mathbf{0})}, & \tfrac{d}{dx}\tilde{\Xi}_j^{(\mathbf{0})} &= -D_{j,0}^T\,\tilde{\Xi}_j^{(\mathbf{1})},\\
\tfrac{d}{dx}\Upsilon_j^{(\mathbf{1})} &= D_{j,1}\,\Upsilon_j^{(\mathbf{0})}, & \tfrac{d}{dx}\tilde{\Upsilon}_j^{(\mathbf{0})} &= -D_{j,1}^T\,\tilde{\Upsilon}_j^{(\mathbf{1})},
\end{aligned}
\tag{4.12}
$$

where $D_{j,e} \in \mathbb{R}^{|\mathcal{J}_{j,e}|\times|\mathcal{J}_{j,e}|}$, $e = 0,1$, are sparse matrices such that $D_{j,1}$ is regular and $\|D_{j,1}^{-1} D_{j,0}\| \lesssim 1$ independently of j. □

Obviously, Theorem 4.1 implies the validity of Assumption 1 in 1D and $\Omega \in \{[0,1], \mathbb{R}, \mathbb{T}\}$. Using again the abbreviations $\lambda = (j,k)$ and $[D_j]^\lambda$, $[D_j]_\lambda$ for the λ-th row and column of D_j, respectively, we can also reformulate (4.12) for each wavelet function:

$$
\begin{aligned}
\tfrac{d}{dx}\xi_\lambda^{(\mathbf{1})} &= [D_{j,0}]^\lambda\,\Xi_j^{(\mathbf{0})}, & \tilde{\xi}_\lambda^{(\mathbf{1})} &= -[D_{j,0}]_\lambda^T\,\tfrac{d}{dx}\tilde{\Xi}_j^{(\mathbf{0})},\\
\tfrac{d}{dx}\eta_\lambda^{(\mathbf{1})} &= [D_{j,1}]^\lambda\,\Upsilon_j^{(\mathbf{0})}, & \tilde{\eta}_\lambda^{(\mathbf{1})} &= -[D_{j,1}]_\lambda^T\,\tfrac{d}{dx}\tilde{\Upsilon}_j^{(\mathbf{0})}.
\end{aligned}
\tag{4.13}
$$

4.1.3. *Tensor product wavelets.* Next, we build tensor products of the univariate systems in the following way. For any index vectors $e = (e_1, \dots, e_n)^T \in E^* := \{0,1\}^n \setminus \{\mathbf{0}\}$ and $k = (k_1, \dots, k_n)^T$, we define

$$
\psi_{j,e,k}(x) := \prod_{\nu=1}^{n} \vartheta_{j,e_\nu,k_\nu}(x_\nu), \qquad \vartheta_{j,e_\nu,k_\nu} := \begin{cases} \eta_{j,k_\nu}, & e_\nu = 1,\\ \xi_{j,k_\nu}, & e_\nu = 0. \end{cases}
\tag{4.14}
$$

Here, k ranges over the set of indices

$$
\nabla_j := \bigcup_{e\in E^*} \nabla_{j,e},
$$

where $\nabla_{j,e} := \mathcal{J}_{j,e_1} \times \cdots \times \mathcal{J}_{j,e_n}$, see (4.11).

Moreover, we define the scaling functions by

$$
\varphi_{j,k}(x) := \prod_{\nu=1}^{n} \xi_{j,k_\nu}(x_\nu), \qquad k \in \Delta_j := \mathcal{I}_j^n.
$$

The wavelet spaces are then defined by

$$
\Psi_j := \bigcup_{e\in E^*} \Psi_{j,e},
$$

and all definitions hold similarly for the dual/biorthogonal systems of functions.

4.1.4. *Differentiation and integration continued.* Now, we use the systems induced by $\Xi^{(\mathbf{1})}$, $\tilde{\Xi}^{(\mathbf{1})}$ as well as $\Xi^{(\mathbf{0})}$, $\tilde{\Xi}^{(\mathbf{0})}$ within a tensor product framework. For $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)^T \in \{0,1\}^n =: E^n$, we define

(4.15)
$$\psi_\lambda^{(\boldsymbol{\gamma})}(x) := \psi_{(j,e,k)}^{(\boldsymbol{\gamma})}(x) := \prod_{\nu=1}^{n} \vartheta_{j,e_\nu,k_\nu}^{\gamma_\nu}(x_\nu), \quad \vartheta_{j,e_\nu,k_\nu}^{\gamma_\nu} := \begin{cases} \eta_{j,k_\nu}^{(\mathbf{1})}, & \gamma_\nu = 1, \quad e_\nu = 1, \\ \eta_{j,k_\nu}^{(\mathbf{0})}, & \gamma_\nu = 0, \quad e_\nu = 1, \\ \xi_{j,k_\nu}^{(\mathbf{1})}, & \gamma_\nu = 1, \quad e_\nu = 0, \\ \xi_{j,k_\nu}^{(\mathbf{0})}, & \gamma_\nu = 0, \quad e_\nu = 0. \end{cases}$$

All systems of functions $\Psi_j^{(\boldsymbol{\gamma})}$ as well as its duals are defined in a straightforward manner. Now, we may collect all what is said in this subsection to obtain:

Proposition 4.2. *The biorthogonal systems $\Psi^{(\boldsymbol{\gamma})}$, $\tilde{\Psi}^{(\boldsymbol{\gamma})}$, $\boldsymbol{\gamma} \in E^n$ fulfill Assumption 1 for $\Omega = [0,1]^n$.* □

4.2. **Parametric mappings.** In this section, we consider domains $\Omega \subset \mathbb{R}^n$ that are images under certain parametric mappings $F : \mathbb{R}^n \to \mathbb{R}^n$, i.e., $\Omega = F(\hat{\Omega})$, where $\hat{\Omega} := (0,1)^n$ denotes the reference cube. Of course, F has to be bijective.

4.2.1. *Affine images.* If Ω is a parallelepiped in $\mathbb{R}^n$, it is known that there exists an affine mapping F such that $\Omega = F(\hat{\Omega})$. In particular, there exists some $A \in GL(\mathbb{R}^n, \mathbb{R}^n)$ and some $b \in \mathbb{R}^n$ such that $F(\hat{x}) = A\hat{x} + b$, $\hat{x} \in \hat{\Omega}$. Hence, the linear tangent mapping DF as well as the Jacobian J of F are constant with respect to $\hat{x}$. Moreover, we denote by $|J|$ the determinant of J.

Now, define for any $\hat{\boldsymbol{v}} \in \boldsymbol{H}(\mathrm{div}; \hat{\Omega})$

$$\boldsymbol{v}(x) := \frac{1}{|J|} DF\, \hat{\boldsymbol{v}}(\hat{x}), \quad x = F(\hat{x}) \in \Omega. \tag{4.16}$$

It is well known (see for example [30]) that the mapping $\hat{\boldsymbol{v}} \mapsto \boldsymbol{v}$ is a bijective affine transformation from $\boldsymbol{H}(\mathrm{div}; \Omega)$ onto $\boldsymbol{H}(\mathrm{div}; \hat{\Omega})$. Moreover, one has

$$\int_\Omega q(x)\ \mathrm{div}\ \boldsymbol{v}(x)\, dx = \int_{\hat{\Omega}} \hat{q}(\hat{x})\ \mathrm{div}\ \hat{\boldsymbol{v}}(\hat{x})\, d\hat{x} \tag{4.17}$$

for all $\hat{q} \in L^2(\hat{\Omega})$, where

$$q(x) := \hat{q}(\hat{x}), \quad x = F(\hat{x}) \in \Omega. \tag{4.18}$$

Let us now consider wavelet bases $\hat{\boldsymbol{\Psi}}^{\mathrm{div}}$ and $\hat{\boldsymbol{\Psi}}^{\mathrm{df}}$ on $\hat{\Omega}$ as defined in (3.22) and Definition 3.13 above. Then, we define $\boldsymbol{\Psi}^{\mathrm{div}}$ and $\boldsymbol{\Psi}^{\mathrm{df}}$ by using (4.16). Since $\hat{\boldsymbol{v}} \mapsto \boldsymbol{v}$ is a mapping onto and because of (4.17), the latter wavelet systems are in fact wavelet bases for $\boldsymbol{H}_0(\mathrm{div}; \Omega)$ and $\boldsymbol{H}_0^0(\mathrm{div}; \Omega)$, respectively. Finally, concerning $\boldsymbol{H}_0^0(\mathbf{curl}; \Omega)$, we remark that (4.17) implies

$$\int_\Omega \mathbf{grad}\, q(x) \cdot \boldsymbol{v}(x)\, dx = \int_{\hat{\Omega}} \mathbf{grad}\, \hat{q}(\hat{x}) \cdot \hat{\boldsymbol{v}}(\hat{x})\, d\hat{x}. \tag{4.19}$$

Now, define $\Psi^{(\mathbf{1})}$ starting from $\hat{\Psi}^{(\mathbf{1})}$ using (4.18). Then, we obtain that $\mathbf{grad}\,\psi_\lambda^{(\mathbf{1})}$ is orthogonal to $\mathbf{\Psi}^{\mathrm{df}}$ for all $\lambda \in \nabla$ in view of (4.19). Hence, (2.14) implies that $\mathbf{grad}\,\Psi^{(\mathbf{1})}$ is in fact a wavelet basis for $\boldsymbol{H}_0^0(\mathbf{curl};\Omega)$. Note that the duals are defined by

$$\tilde{\boldsymbol{v}}(x) := DF^{-T}\,\hat{\tilde{\boldsymbol{v}}}(\hat{x}), \quad x = F(\hat{x}) \in \Omega.$$

4.2.2. *Conformal images.* Conformal mappings have been used extensively both for the analysis and the numerical solution of incompressible flow problems. This is due to the fact that mappings that are conformal (and orthogonal) preserve certain simple differential operators as div and **grad**, see [8, 23]. On the other hand, these mappings are restricted to the 2D case.

In contrary to (4.16), we define in this case for any $\hat{\boldsymbol{v}} \in \boldsymbol{H}(\mathrm{div};\hat{\Omega})$

$$\boldsymbol{v}(x) := \hat{\boldsymbol{v}}(\hat{x}), \qquad x = F(\hat{x}) \in \Omega.$$

Since div and **grad** are preserved, it is obvious that this gives wavelet systems in $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$ as well as, respectively, divergence and curl-free wavelet bases (in 2D using (2.1)). In this case, one may define the dual system by

$$\tilde{\boldsymbol{v}}(x) := \frac{1}{|J(\hat{x})|}\hat{\tilde{\boldsymbol{v}}}(\hat{x}), \qquad x = F(\hat{x}) \in \Omega.$$

4.3. **Domain decomposition.** The construction of wavelet bases on more general domains is still a field of very active research, see [9, 10, 15, 19, 20]. The basic idea in all those references is to decompose the domain Ω of interest into nonoverlapping subdomains Ω_i, $i = 1, \ldots, N$. Each of the subdomains is then mapped to the reference cube $\hat{\Omega}$ with the aid of parametric mappings. Finally, additional care has to be taken with respect to the interfaces between the subdomains in order to obtain global continuity, for example.

Now, one could try to use the bases created above within this framework in order to obtain wavelets in $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$ on more complex domains. However, since we used integration by parts, the traces of the divergence- and **curl**-free wavelets vanish on the boundary. Hence, one has to add certain functions near the boundary in order to generate the full spaces $\boldsymbol{H}^0(\mathrm{div};\Omega)$ and $\boldsymbol{H}^0(\mathbf{curl};\Omega)$, respectively. These functions then have to be combined across the interelement boundaries while preserving their div and **curl**, respectively, which might be somewhat unsatisfactory as well as very technical.

A different approach was proposed in [7]. Here, the global continuity is enforced by means of Lagrange multipliers and hence one does not need to take care explicitly on the interelement boundaries. However, on each of the subdomains Ω_i one still has to have a *full* basis, i.e., one in $\boldsymbol{H}^0(\mathrm{div};\Omega_i)$ and $\boldsymbol{H}^0(\mathbf{curl};\Omega_i)$ rather than in $\boldsymbol{H}_0^0(\mathrm{div};\Omega_i)$ and $\boldsymbol{H}_0^0(\mathbf{curl};\Omega_i)$. Hence, one is still left with the problem of adding appropriate functions near the boundary. This problem as well as possible further extensions to more complicated domains will be treated in a forthcomming paper.

4.4. **Examples.** In this section, we show concrete examples for our construction. We would like to mention, that all computations needed to produce the figures below have been made using the *Multiscale Library* that is documented in [3].

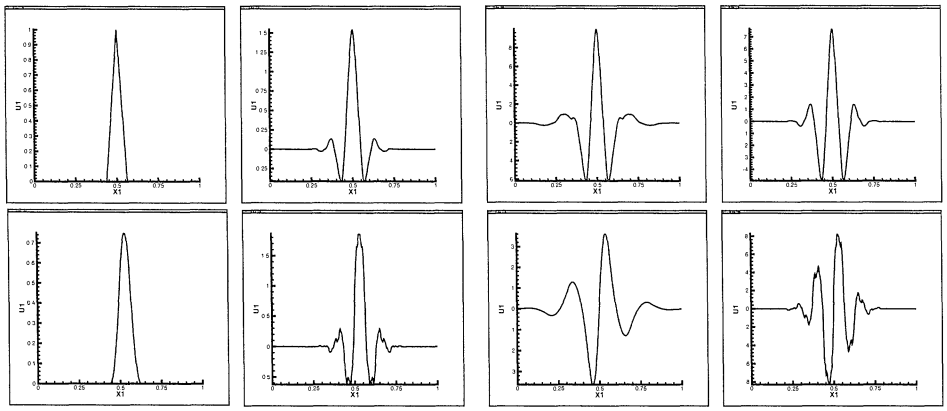

FIGURE 2. B–spline generators (left) N_2, $\tilde{N}_{2,8}$ (top) and N_3, $\tilde{N}_{3,7}$ (bottom) and biorthogonal wavelets (right) generated by these functions.

Let us start with some remarks on the structure of the matrices $D_{j,i}$ and the support of the wavelets. For biorthogonal systems on $\mathbb{R}$, it was proven in [26] that

$$(D_{j,0})_{k,k'} = 2^j(\delta_{k,k'} - \delta_{k,k'-1}), \qquad (D_{j,1})_{k,k'} = 2^{j+2}\delta_{k,k'}. \tag{4.20}$$

Note that $D_{j,1}$ is a diagonal matrix. Adapting these systems of functions to $[0,1]$ as in [18], leads to matrices of the form

$$D_{j,i} = \begin{array}{|c|c|c|} \hline D^L & & \\ \hline & D^I & \\ \hline & & D^R \\ \hline \end{array} \in \mathbb{R}^{|\nabla_{j,i}|\times|\nabla_{j,i}|}, \ i = 0,1. \tag{4.21}$$

The large central block D^I is of the form (4.20) and the two blocks D^L, D^R that correspond to the modifications of those functions living near the boundary, are small. For $D_{j,1}$, it is obvious that the inverse is (block) diagonal. Hence also those dual functions involving the inverses (such as $\tilde{\mathbf{\Psi}}^{\mathrm{cf}}$) are locally supported. As can be seen in Definition 3.13, this is important for the locality of the divergence-free wavelets as well as for the norm equivalences.

4.4.1. *Biothogonal B-spline wavelets.* Let us denote by N_L the cardinal B-spline of order $L \geq 1$. It is well known that N_L is refinable. In [12], a whole family of dual scaling functions $\tilde{N}_{L,\tilde{L}}$ for $L + \tilde{L}$ even, was constructed. Here $\tilde{L}$ is related to the smoothness (and hence also to the size of the support) of $\tilde{N}_{L,\tilde{L}}$ in the sense that for growing $\tilde{L}$ also the smoothness grows. It can be shown [26, 32] that for $\varphi^{(\mathbf{0})} = N_L$, $\tilde{\varphi}^{(\mathbf{0})} = \tilde{N}_{L,\tilde{L}}$ one has $\varphi^{(\mathbf{1})} = N_{L+1}$ and $\tilde{\varphi}^{(\mathbf{1})} = \tilde{N}_{L+1,\tilde{L}-1}$. We display these functions and the associated biorthogonal wavelets in Figure 2 for $L = 2$ and $\tilde{L} = 8$.

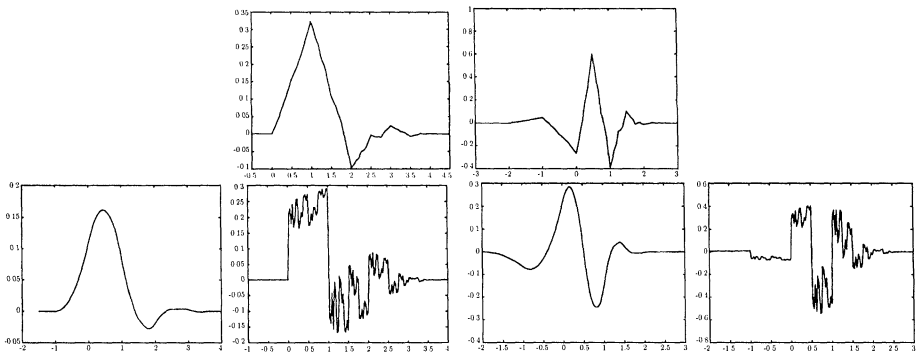

FIGURE 3. Daubechies' orthogonal scaling function (left) and wavelet (right) for $N = 3$ (up) and arising functions ${}_N\varphi^{(\mathbf{1})}$, ${}_N\tilde{\varphi}^{(\mathbf{1})}$, ${}_N\psi^{(\mathbf{1})}$, ${}_N\tilde{\psi}^{(\mathbf{1})}$.

4.4.2. *Orthogonal wavelets.* One may wish to have an orthonormal wavelet basis for $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$. As already stated, orthonormal wavelets are a special case of biorthogonal ones where primal and dual functions coincide. Hence one can, in principle, apply our construction. However, a closer look to (4.12) shows that even if $\Xi_j^{(\mathbf{1})}$ gives rise to orthonormal wavelets $\Upsilon_j^{(\mathbf{1})}$, the second system $\Xi_j^{(\mathbf{0})}$, $\Upsilon_j^{(\mathbf{0})}$ is *not* orthogonal, but biorthogonal. Hence, all constructed wavelet bases are *not* orthogonal even when Daubechies' wavelets are used on $[0,1]$, see [22]. For divergence-free wavelets it is moreover known that no basis for $\boldsymbol{H}^0(\mathrm{div};\Omega)$ consisting of compactly supported orthogonal divergence-free wavelets exists, see [27].

However, as already mentioned, we may apply our construction to Daubechies' orthogonal wavelets. In Figure 3, we display the orthonormal scaling functions $\varphi^{(\mathbf{0})} = {}_N\varphi$ and $\psi^{(\mathbf{0})} = {}_N\psi$ for $N = 3$ (i.e., the mask has $2N$ nonvanishing coefficients). Moreover, the functions ${}_N\varphi^{(\mathbf{1})}$, ${}_N\psi^{(\mathbf{1})}$ as well as their duals from (4.12) are shown. The duals show strange graphs, but note that these functions are *only* used for the analysis. They do not enter in a numerical scheme.

4.4.3. *Bivariate* curl*-free wavelets.* For graphical reasons we display only 2D functions here. Then, we end up with three types of wavelets (corresponding to $e \in E^2$)

$$\begin{aligned}
\boldsymbol{\psi}_1^{\mathrm{cf}}(x,y) &= \begin{pmatrix} (\varphi^{(\mathbf{0})}(x) - \varphi^{(\mathbf{0})}(x-1))\psi^{(\mathbf{1})}(y) \\ 4\varphi^{(\mathbf{1})}(x)\psi^{(\mathbf{0})}(y) \end{pmatrix}, \\
\boldsymbol{\psi}_2^{\mathrm{cf}}(x,y) &= \begin{pmatrix} 4\psi^{(\mathbf{0})}(x)\varphi^{(\mathbf{1})}(y) \\ \psi^{(\mathbf{1})}(x)(\varphi^{(\mathbf{0})}(y) - \varphi^{(\mathbf{0})}(y-1)) \end{pmatrix}, \\
\boldsymbol{\psi}_3^{\mathrm{cf}}(x,y) &= \begin{pmatrix} 4\psi^{(\mathbf{0})}(x)\psi^{(\mathbf{1})}(y) \\ 4\psi^{(\mathbf{1})}(x)\psi^{(\mathbf{0})}(y) \end{pmatrix},
\end{aligned}$$

that are displayed for using biorthogonal B-spline wavelets with $L = 2$, $\tilde{L} = 8$ in Figure 4.

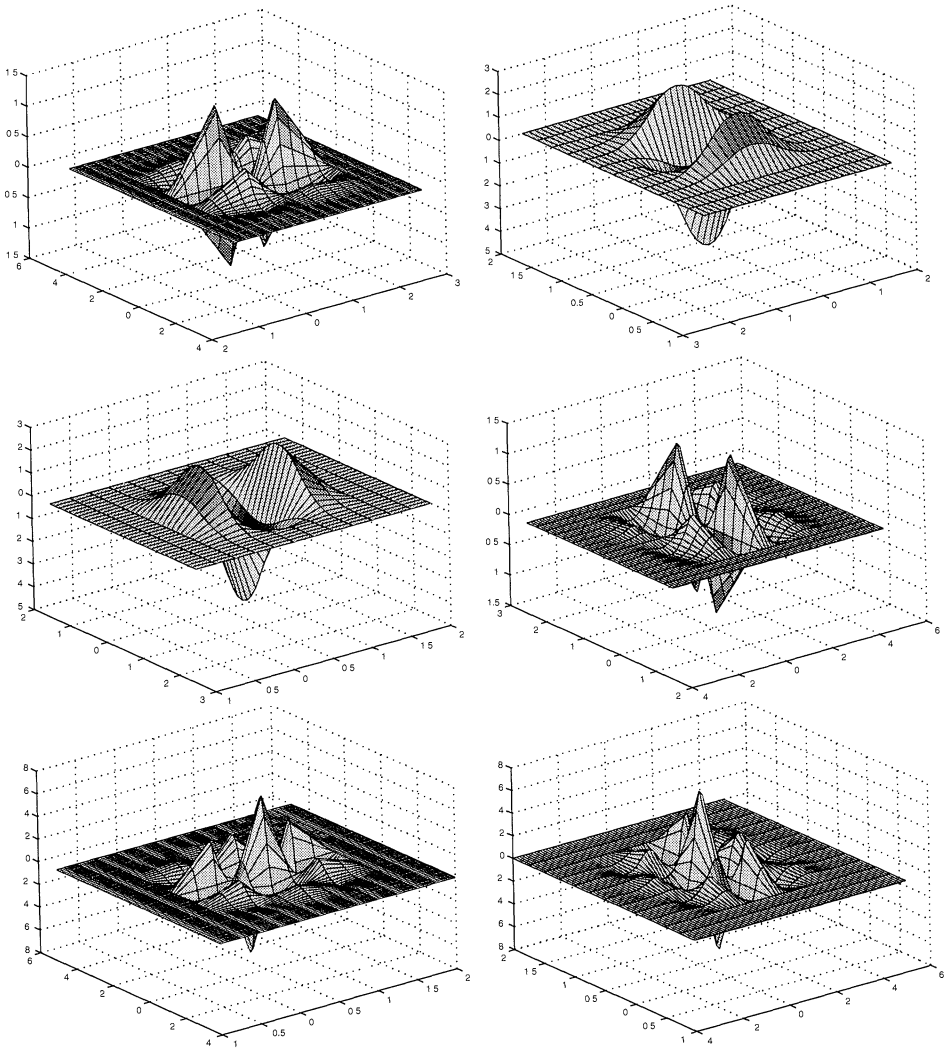

FIGURE 4. Components of the three different types of curl-free wavelets in 2D for biorthogonal B-spline wavelets with $L = 2$, $\tilde{L} = 8$ (first component left, second right).

5. One application: Multilevel preconditioning in $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$

Let us finally consider the following two variational problems for given data $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$:

$$\boldsymbol{u} \in \boldsymbol{H}_0(\mathrm{div};\Omega): \qquad a_{\mathrm{div}}(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}, \quad \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div};\Omega), \tag{5.1}$$

$$\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega): \qquad a_{\mathrm{curl}}(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}, \quad \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega), \tag{5.2}$$

where the bilinear forms are defined in (3.32) and (3.33), respectively. The aim of this section is to provide asymptotically optimal multilevel preconditioners for the

corresponding matrices

$$\boldsymbol{A}_{\boldsymbol{\Lambda}}^{\mathrm{div}} := \big(a_{\mathrm{div}}(\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\mathrm{div}}, \boldsymbol{\psi}_{\boldsymbol{\lambda}'}^{\mathrm{div}})\big)_{\boldsymbol{\lambda},\boldsymbol{\lambda}'\in\boldsymbol{\Lambda}^{\mathrm{div}}\subset\boldsymbol{\nabla}}, \tag{5.3}$$

$$\boldsymbol{A}_{\boldsymbol{\Lambda}}^{\mathrm{curl}} := \big(a_{\mathrm{curl}}(\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\mathrm{curl}}, \boldsymbol{\psi}_{\boldsymbol{\lambda}'}^{\mathrm{curl}})\big)_{\boldsymbol{\lambda},\boldsymbol{\lambda}'\in\boldsymbol{\Lambda}^{\mathrm{curl}}\subset\boldsymbol{\nabla}}. \tag{5.4}$$

The corresponding discrete formulations arise in many problems of practical interest such as time-discretizations of Maxwell's equations, the sequential regularization method for the nonstationary incompressible Navier–Stokes equations and also certain plate problems, see [1, 6] and the references therein. In the context of finite element discretizations, this has recently been studied in [1, 2, 4, 24]. We want to use the discrete Hodge decompositions for developing preconditioners. Then, we apply the following result which is well known (see [16] and the references therein):

Theorem 5.1. [16] *Let the wavelet system Ψ be stable in $H^s(\Omega)$, i.e.,*

$$\|f\|_{H^s(\Omega)}^2 \sim \sum_{\lambda\in\nabla} 2^{2s|\lambda|}|\langle f, \tilde{\psi}_\lambda\rangle|^2, \qquad s\in(\tilde{\gamma},\gamma), \tag{5.5}$$

*for some constants $\tilde{\gamma},\gamma>0$. Let $\mathcal{L}: H^t(\Omega)\to H^{-t}(\Omega)$ be a boundedly invertible operator, i.e., $\|\mathcal{L}v\|_{H^{-t}(\Omega)} \sim \|v\|_{H^t(\Omega)}$, and assume moreover that the Galerkin method $\tilde{Q}_\Lambda\mathcal{L}Q_\Lambda u_\Lambda = \tilde{Q}_\Lambda f$ is t-*stable *in the sense*

$$\|\tilde{Q}_\Lambda\mathcal{L}Q_\Lambda v_\Lambda\|_{H^{-t}(\Omega)} \sim \|v_\Lambda\|_{H^t(\Omega)}, \qquad v_\Lambda\in S_\Lambda, \tag{5.6}$$

such that $|t|<\tilde{\gamma},\gamma$. Then, defining the matrices

$$\boldsymbol{D}_\Lambda^s := \mathrm{diag}(2^{s|\lambda|})_{\lambda\in\Lambda}, \qquad \boldsymbol{A}_\Lambda := (\langle\mathcal{L}\psi_\lambda,\psi_{\lambda'}\rangle)_{\lambda,\lambda'\in\Lambda},$$

the matrices

$$\boldsymbol{B}_\Lambda := \boldsymbol{D}_\Lambda^{-t}\boldsymbol{A}_\Lambda\boldsymbol{D}_\Lambda^{-t}$$

have uniformly bounded spectral condition numbers

$$\|\boldsymbol{B}_\Lambda\|\,\|\boldsymbol{B}_\Lambda^{-1}\| = \mathcal{O}(1), \quad \Lambda\subset\nabla,\ \#\Lambda\to\infty. \quad \square$$

Let us start by considering (5.3). Using the Hodge decomposition in Theorem 3.16, we may subdivide the set of indices as $\boldsymbol{\Lambda}^{\mathrm{div}} = \boldsymbol{\Lambda}^{\mathrm{df}}\cup\Lambda^{\mathrm{cf}}$. It is readily seen that

$$a_{\mathrm{div}}(\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\mathrm{df}}, \boldsymbol{\psi}_{\boldsymbol{\lambda}'}^{\mathrm{df}}) = (\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\mathrm{df}}, \boldsymbol{\psi}_{\boldsymbol{\lambda}'}^{\mathrm{df}})_{\boldsymbol{L}^2(\Omega)}, \qquad \boldsymbol{\lambda},\boldsymbol{\lambda}'\in\boldsymbol{\Lambda}^{\mathrm{df}}\subset\boldsymbol{\nabla}^{\mathrm{df}}, \tag{5.7}$$

as well as

$$\begin{aligned} a_{\mathrm{div}}(\boldsymbol{\psi}_\lambda^{\mathrm{cf}}, \boldsymbol{\psi}_{\lambda'}^{\mathrm{cf}}) &= (\boldsymbol{\psi}_\lambda^{\mathrm{cf}}, \boldsymbol{\psi}_{\lambda'}^{\mathrm{cf}})_{\boldsymbol{L}^2(\Omega)} + (\mathrm{div}\ \boldsymbol{\psi}_\lambda^{\mathrm{cf}}, \mathrm{div}\ \boldsymbol{\psi}_{\lambda'}^{\mathrm{cf}})_{\boldsymbol{L}^2(\Omega)} \\ &= (\Delta\,\psi_\lambda^{(\mathbf{1})}, \Delta\,\psi_{\lambda'}^{(\mathbf{1})})_{L^2(\Omega)} + (\mathbf{grad}\,\psi_\lambda^{(\mathbf{1})}, \mathbf{grad}\,\psi_{\lambda'}^{(\mathbf{1})})_{\boldsymbol{L}^2(\Omega)} \end{aligned} \tag{5.8}$$

for $\lambda,\lambda'\in\Lambda^{\mathrm{cf}}\subset\nabla$. Hence, (5.1) behaves like an elliptic fourth-order problem.

Corollary 5.2. *The matrices*

$$\boldsymbol{B}_{\boldsymbol{\Lambda}} := \begin{bmatrix} \boldsymbol{D}_{\Lambda^{\mathrm{cf}}}^{-2} & \\ & \boldsymbol{I}_{\boldsymbol{\Lambda}^{\mathrm{df}}} \end{bmatrix} \begin{bmatrix} (\boldsymbol{A}_{\boldsymbol{\Lambda}}^{\mathrm{div}})_{|\Lambda^{\mathrm{cf}}} & \\ & (\boldsymbol{A}_{\boldsymbol{\Lambda}}^{\mathrm{div}})_{|\boldsymbol{\Lambda}^{\mathrm{df}}} \end{bmatrix} \begin{bmatrix} \boldsymbol{D}_{\Lambda^{\mathrm{cf}}}^{-2} & \\ & \boldsymbol{I}_{\boldsymbol{\Lambda}^{\mathrm{df}}} \end{bmatrix}$$

have uniformly bounded condition numbers, where $\boldsymbol{I}_{\Lambda^X}$ denotes the identity matrix w.r.t. some index set $\Lambda^X\subset\boldsymbol{\nabla}$.

Proof. With the above preparations at hand, we use the standard norm equivalence (2.20) for $\Psi^{(1)}$ in order to precondition $(\boldsymbol{A}_{\boldsymbol{\Lambda}}^{\mathrm{div}})_{|\Lambda^{\mathrm{cf}}}$, and Proposition 3.15 with $s = 0$ for $(\boldsymbol{A}_{\boldsymbol{\Lambda}}^{\mathrm{div}})_{|\boldsymbol{\Lambda}^{\mathrm{df}}}$. □

Finally, consider (5.4). Again, we may subdivide the set of indices as above. Note that in this case, we have to pose the additional condition $\Psi^{(\mathbf{0})} \subset H_0^1(\Omega)$ according to Theorem 3.17. We obtain

$$a_{\mathrm{curl}}(\boldsymbol{\psi}_\lambda^{\mathrm{cf}}, \boldsymbol{\psi}_{\lambda'}^{\mathrm{cf}}) = (\boldsymbol{\psi}_\lambda^{\mathrm{cf}}, \boldsymbol{\psi}_{\lambda'}^{\mathrm{cf}})_{\boldsymbol{L}^2(\Omega)}, \qquad \lambda, \lambda' \in \Lambda^{\mathrm{cf}} \subset \nabla. \tag{5.9}$$

For $\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\mathrm{df}}$, we cannot simplify the bilinear form any more. This shows

Corollary 5.3. *The matrices*

$$\boldsymbol{B}_{\boldsymbol{\Lambda}} := \begin{bmatrix} \boldsymbol{D}_{\boldsymbol{\Lambda}^{\mathrm{df}}}^{-1} & \\ & \boldsymbol{D}_{\Lambda^{\mathrm{cf}}}^{-1} \end{bmatrix} \begin{bmatrix} (\boldsymbol{A}_{\boldsymbol{\Lambda}}^{\mathrm{curl}})_{|\boldsymbol{\Lambda}^{\mathrm{df}}} & \\ & (\boldsymbol{A}_{\boldsymbol{\Lambda}}^{\mathrm{curl}})_{|\Lambda^{\mathrm{cf}}} \end{bmatrix} \begin{bmatrix} \boldsymbol{D}_{\boldsymbol{\Lambda}^{\mathrm{df}}}^{-1} & \\ & \boldsymbol{D}_{\Lambda^{\mathrm{cf}}}^{-1} \end{bmatrix}$$

have uniformly bounded condition numbers.

Proof. The preconditioning of $(\boldsymbol{A}_{\boldsymbol{\Lambda}}^{\mathrm{curl}})_{|\Lambda^{\mathrm{cf}}}$ follows by (5.9), Theorem 5.1 and (3.21) for $s = 0$. For the remaining block $(\boldsymbol{A}_{\boldsymbol{\Lambda}}^{\mathrm{curl}})_{|\boldsymbol{\Lambda}^{\mathrm{df}}}$, we use the well-known fact

$$(\mathbf{grad}\, \boldsymbol{u}, \mathbf{grad}\, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} = (\operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{v})_{L^2(\Omega)} + (\mathbf{curl}\, \boldsymbol{u}, \mathbf{curl}\, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)},$$

i.e.,

$$a_{\mathrm{curl}}(\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\mathrm{df}}, \boldsymbol{\psi}_{\boldsymbol{\lambda}'}^{\mathrm{df}}) = (\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\mathrm{df}}, \boldsymbol{\psi}_{\boldsymbol{\lambda}'}^{\mathrm{df}})_{\boldsymbol{L}^2(\Omega)} + (\mathbf{grad}\, \boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\mathrm{df}}, \mathbf{grad}\, \boldsymbol{\psi}_{\boldsymbol{\lambda}'}^{\mathrm{df}})_{\boldsymbol{L}^2(\Omega)},$$

so that the claim follows by using Theorem 5.1 and Proposition 3.15 for $s = 1$. □

A few comments on the results obtained in Corollaries 5.2 and 5.3 are in order. It is clear that the Hodge decompositions enable us to separate the two parts of the operators in (5.3) and (5.4) in such a way that for both parts standard multilevel preconditioners are available. Here, the orthogonality of the decomposition is really crucial. Consequently, the *biorthogonal* decomposition described in subsection 3.5.3 would not give rise to the same results.

On the first look it seems that the preconditioning in Corollary 5.3 may be better since one has to deal with a second-order problem, whereas Corollary 5.2 deals with a fourth-order problem. Hence, one expects worse condition numbers in the latter one. On the other hand, one should note that for the decomposition used in Corollary 5.3 one has to use smoother functions which also may have negative influence on the constant in $\mathcal{O}(1)$.

Finally, we would like to mention that numerical experiments in [14] have shown that the multilevel preconditioning simply using the diagonal matrix as proposed in Theorem 5.1 by far do not give optimal condition numbers. In [14] *spy preconditioners* based on local inverses have been tested and gave very good condition numbers (for systems on the interval). It is obvious that we could also insert such preconditioners in the above results. Since the Hodge decompositions decouple the problems, we may use *any* appropriate preconditioners for the remaining subproblems.

References

[1] D.N. Arnold, R.S. Falk, and R. Winther, *Preconditioning in $H(\mathrm{div})$ and applications*, Math. Comput. **66**, No. 219 (1997), 957–984. MR **97i:**65177

[2] D.N. Arnold, R.S. Falk, and R. Winther, *Multigrid in $H(\mathrm{div})$ and $H(\mathbf{curl})$*, Dept. of Math., Penn State Univ., Preprint, 1998.

[3] A. Barinka, T. Barsch, K. Urban, and J. Vorloeper, *The Multilevel Library: Software Tools for Multiscale Methods and Wavelets, Version 1.0, Documentation*, RWTH Aachen, IGPM Preprint 156, 1998.

[4] R. Beck, P. Deuflhard, R. Hiptmair, R.H.W. Hoppe, and B. Wohlmuth, *Adaptive multilevel methods for edge element discretizations of Maxwell's equations*, ZIB Berlin, Report SC-97-66, 1997, to appear in Surveys of Mathematics in Industry.

[5] G. Beylkin, R. Coifman, and V. Rokhlin, *Fast wavelet transforms and numerical algorithms*, Comm. Pure Appl. Math. **44** (1991), 141–183. MR **92c:**65061

[6] A. Bossavit, Computational Electromagnetism, Academic Press, San Diego, 1998. MR **99m:**78001

[7] F. Brezzi and L.D. Marini, *A three–field domain decomposition method*, Contemp. Math. **157** (1994), 27–34. MR **95a:**65202

[8] C. Canuto, M.Y. Hussaini, A. Quarteroni, and T.A. Zang, Spectral Methods in Fluid Dynamics, Springer–Verlag, New York, 1988. MR **89m:**76004

[9] C. Canuto, A. Tabacco, and K. Urban, *The wavelet element method, part I: Construction and analysis*, Appl. Comp. Harm. Anal. **6** (1999), 1–52. MR **99k:**42055

[10] C. Canuto, A. Tabacco, and K. Urban, *The wavelet element method, part II: Realization and additional features in 2d and 3d*, Preprint 1052, Instituto di Analisi Numerica del C.N.R., Pavia, 1997. To appear in Appl. Comp. Harm. Anal.

[11] A. Cohen, *Wavelet methods in Numerical Analysis*, in: Handbook of Numerical Analysis, North Holland, Amsterdam, to appear.

[12] A. Cohen, I. Daubechies, and J.–C. Feauveau, *Biorthogonal bases of compactly supported wavelets*, Comm. Pure Appl. Math. **45** (1992), 485–560. MR **93e:**42044

[13] A. Cohen, I. Daubechies, and P. Vial, *Wavelets on the interval and fast wavelet transforms*, Appl. Comp. Harm. Anal. **1** (1993), 54–81. MR **94m:**42074

[14] A. Cohen and R. Masson, *Wavelet methods for second order elliptic problems — preconditioning and adaptivity*, Preprint, Univ. P. et M. Curie, Paris, 1997.

[15] A. Cohen and R. Masson, *Wavelet adaptive method for second order elliptic problems — boundary conditions and domain decomposition*, Preprint, Univ. P. et M. Curie, Paris, 1997.

[16] W. Dahmen, *Wavelet and multiscale methods for operator equations*, Acta Numerica **6** (1997), 55–228. MR **98m:**65102

[17] W. Dahmen, A. Kunoth, and K. Urban, *A wavelet Galerkin method for the Stokes problem*, Computing **56** (1996), 259–301. MR **97g:**65228

[18] W. Dahmen, A. Kunoth, and K. Urban, *Biorthogonal spline wavelets on the interval— stability and moment conditions*, Appl. Comp. Harm. Anal. **6** (1999), 132–196. MR **99m:**42046

[19] W. Dahmen and R. Schneider, *Composite wavelet bases for operator equations*, Math. Comput. **68** (1999), 1533–1567. MR **99m:**65122

[20] W. Dahmen and R. Schneider, *Wavelets on Manifolds I: Construction and Domain Decomposition*, RWTH Aachen, IGPM Preprint 149, 1998. To appear in SIAM J. Math. Anal.

[21] W. Dahmen and R. Schneider, *Wavelets with complementary boundary conditions — Functions spaces on the cube*, Results Math. **34** (1998), 255–293. MR **99h:**42057

[22] I. Daubechies, Ten Lectures on Wavelets, CBMS–NSF Regional Conference Series in Applied Mathematics **61**, 1992. MR **93e:**42045

[23] C.A.J. Fletcher, Computational Techniques for Fluid Dynamics, Volume II, Springer–Verlag, Berlin, 1988. MR **90c:**76006b

[24] R. Hiptmair, *Multilevel Preconditioning for Mixed Problems in Three Dimensions*, PhD–Thesis, Univ. of Augsburg, 1996.

[25] V. Girault and P.-A. Raviart, Finite Element Methods for Navier–Stokes–Equations, Springer–Verlag, Berlin, 2nd edition, 1986. MR **88b:**65129

[26] P.G. Lemarié-Rieusset, *Analyses multi-résolutions non orthogonales, commutation entre projecteurs et derivation et ondelettes vecteurs à divergence nulle* (in French), Rev. Mat. Iberoamericana **8** (1992), 221–237. MR **94d:**42044

[27] P.G. Lemarié-Rieusset, *Un théorème d'inexistence pour les ondelettes vecteurs à divergence nulle* (in french), C.R. Acad. Sci. Paris Sér. I Math. **319** (1994), 811–813. MR **95h:**42028
[28] R. Masson, *Wavelet discretizations of the Stokes problem in velocity–pressure variables*, Preprint, Univ. P. et M. Curie, Paris, 1998.
[29] Y. Meyer, Ondelettes et Opèrateurs, vol. I (in french), Hermann, Paris, 1990.
[30] J.E. Roberts and J.-M. Thomas, *Mixed and hybrid methods*, in: Handbook of Numerical Analysis, Vol. II, P.G. Ciarlet and J.L. Lions (eds.), Elsevier Science Publishers, North–Holland, 1991, 523–640. CMP 91:14
[31] G. Strang and T. Nguyen, Wavelets and Filter Banks, Wellesley Cambridge Press, Cambridge, 1996. MR **98b:**94003
[32] K. Urban, *On divergence-free wavelets*, Adv. Comput. Math. **4** (1995), 51–82. MR **96e:**42035
[33] K. Urban, Multiskalenverfahren für das Stokes-Problem und angepaßte Wavelet-Basen (in german), PhD thesis, Verlag der Augustinus-Buchhandlung, Aachen, 1995.
[34] M.V. Wickerhauser, Adapted Wavelet Analysis from Theory to Software, AK Peters, Wellesley, 1994. MR **95j:**94005

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