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A DUAL-DUAL MIXED FORMULATION FOR NONLINEAR EXTERIOR TRANSMISSION PROBLEMS

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Dedicated to Professor Dr. George C. Hsiao on the occasion of his 65th birthday

ABSTRACT. We combine a dual-mixed finite element method with a Dirichlet-to-Neumann mapping (derived by the boundary integral equation method) to study the solvability and Galerkin approximations of a class of exterior nonlinear transmission problems in the plane. As a model problem, we consider a nonlinear elliptic equation in divergence form coupled with the Laplace equation in an unbounded region of the plane. Our combined approach leads to what we call a *dual-dual* mixed variational formulation since the main operator involved has itself a dual-type structure. We establish existence and uniqueness of solution for the continuous and discrete formulations, and provide the corresponding error analysis by using Raviart-Thomas elements. The main tool of our analysis is given by a generalization of the usual Babuska-Brezzi theory to a class of nonlinear variational problems with constraints.

1. INTRODUCTION

The numerical solution of interior and exterior nonlinear-linear transmission problems usually combines the finite element method (FEM) in the nonlinear region with the boundary integral equation method (BIM) in the linear and homogeneous domain. This method, which is known as the coupling of FEM and BIM, has been applied successfully during the last decades using traditional finite elements and, more recently, using mixed finite elements as well (see, e.g., [3], [6], [16], [17], [18], [20], [21], [31], [34], and the references therein).

An alternative procedure for dealing with exterior problems consists of employing Dirichlet-to-Neumann mappings. This means that one first introduces a sufficiently large circle Γ (in $\mathbf{R}^2$) or a sphere (in $\mathbf{R}^3$), so that the linear domain is divided into a bounded annular region and an unbounded one. Next, one derives an explicit formula for the Neumann data on Γ in terms of the Dirichlet data on the same curve, which is known as the Dirichlet-to-Neumann mapping. This has been done for several elliptic operators, including the Lamé system for elasticity, by using Fourier-type series developments (see, e.g., [9], [23], [24], [25]). Then, in [11] we utilized the mapping obtained in [24] together with our mixed finite element approach from [21] to study the weak solvability of an exterior hyperelastic interface problem.

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Nevertheless, up to now, all the works on the combined use of mixed-FEM with either BIM or Dirichlet-to-Neumann mappings for nonlinear transmission problems have provided satisfactory results only at the continuous level. The associated Galerkin schemes still require some open questions to be solved. Indeed, in order to prove the unique solvability of the resulting variational formulations, one needs to introduce certain quotient spaces for which it is not clear how to define explicit finite element subspaces satisfying the corresponding discrete compatibility conditions. This drawback has motivated either the use of alternative mixed formulations (see, e.g., [1]) or the search of new tools from analysis to deal with the usual mixed formulations.

The purpose of the present paper is, precisely, to show some advances in the direction of the latter approach. In fact, we combine the dual-mixed finite element method from [20, 21] with a Dirichlet-to-Neumann mapping (derived by the BIM) to study the solvability and Galerkin approximation of a class of nonlinear exterior transmission problems in the plane. The resulting variational formulation can be written as what we call a dual-dual type operator equation, which, thanks to an extension of the usual Babuska-Brezzi theory, allows us to obtain satisfactory results for both the continuous and discrete schemes.

The rest of the paper is presented as follows. In Section 2, we describe the exterior transmission problem and transform it, using the Dirichlet-to-Neumann mapping, into a nonlocal boundary value problem on a bounded domain. The corresponding dual-dual mixed formulation is derived in Section 3. In Section 4, we recall the main results from a recent work concerning a generalization of the classical Babuska-Brezzi theory to a family of nonlinear variational problems with constraints. Finally, in Section 5 we apply the theorems from Section 4 and provide the existence and uniqueness of solution for the continuous and Galerkin dual-dual formulations by using Raviart-Thomas elements of lowest order. In addition, we prove the Cea estimate and provide, under usual regularity assumptions, an error bound of $O(h)$.

2. The exterior nonlinear transmission problem

Let Ω_0 be a bounded and simply connected domain in $\mathbf{R}^2$ with Lipschitz-continuous boundary Γ_0. Also, let Ω_1 be the annular domain bounded by Γ_0 and another Lipschitz-continuous closed curve Γ_1 whose interior region contains Ω_0. In addition, let $a_i : \Omega_1 \times \mathbf{R}^2 \to \mathbf{R}$, $i = 1, 2$, be nonlinear mappings satisfying certain conditions to be specified later on. Then, given $f_1 \in L^2(\Omega_1)$, we consider the exterior nonlinear transmission problem: *Find* $u_1 \in H^1(\Omega_1)$ *and* $u_2 \in H^1_{\text{loc}}(\mathbf{R}^2 - \overline{\Omega}_0 \cup \overline{\Omega}_1)$ *such that*

$$
\begin{aligned}
&u_1 = 0 \quad \text{on} \quad \Gamma_0\,,\\
&-\sum_{i=1}^{2} \frac{\partial}{\partial x_i} a_i(\cdot, \nabla u_1(\cdot)) = f_1 \quad \text{in} \quad \Omega_1\,,\\
&u_1 = u_2 \quad \text{and} \quad \sum_{i=1}^{2} a_i(\cdot, \nabla u_1(\cdot))\, n_i - \frac{\partial u_2}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \Gamma_1\,,\\
&-\Delta u_2 = 0 \quad \text{in} \quad \mathbf{R}^2 - \overline{\Omega}_0 \cup \overline{\Omega}_1\,,\\
&u_2(x) = O(1) \quad \text{as} \quad ||x|| \to +\infty\,,
\end{aligned}
\tag{2.1}
$$

where $\mathbf{n} := (n_1, n_2)$ denotes the unit outward normal to $\partial\Omega_1$. This kind of problem appears in the computation of magnetic fields of electromagnetic devices (see, e.g., [26, 27]), in some subsonic flow and fluid mechanics problems (see, e.g., [7, 8]), and also in steady heat conduction. For instance, in the latter case, one has $a_i(x, \nabla u(x)) = k(x, \nabla u(x)) \frac{\partial u}{\partial x_i}$, where u is the temperature and k is the heat conductivity. In all these problems, and in many others from physics and engineering sciences, the fluxes become variables of much interest and are required, therefore, to be approximated directly. This fact motivates the use of mixed finite element formulations.

According to the above comment, in what follows we apply a dual-mixed finite element method in Ω_1 and a Dirichlet-to-Neumann mapping (arising from the boundary integral equation method) in the exterior region $\mathbf{R}^2 - \overline{\Omega}_0 \cup \overline{\Omega}_1$. To this end, we first introduce a sufficiently large circle Γ with center at the origin and radius r such that its interior region contains $\overline{\Omega}_0 \cup \overline{\Omega}_1$. We denote by Ω_2 the annular region bounded by Γ_1 and Γ and put $\Omega := \Omega_1 \cup \Gamma_1 \cup \Omega_2$. Next, we define

$$u := \begin{cases} u_1 & \text{in} \quad \Omega_1, \\ u_2 & \text{in} \quad \Omega_2, \end{cases}$$

the flux variable

$$\boldsymbol{\sigma} := \begin{cases} (a_1(\cdot, \nabla u), a_2(\cdot, \nabla u))^T & \text{in} \quad \Omega_1, \\ \nabla u \quad \text{in} \quad \Omega_2, \end{cases}$$

the global data

$$f := \begin{cases} f_1 & \text{in} \quad \Omega_1, \\ 0 & \text{in} \quad \Omega_2, \end{cases}$$

and introduce the auxiliary unknowns

$$\boldsymbol{\theta} := \nabla u \quad \text{in} \quad \Omega \quad \text{and} \quad \xi := u|_\Gamma \,. \tag{2.2}$$

On the other hand, by applying the boundary integral equation method in the region exterior to the circle Γ, and according to the analysis from [28] (see also [19]), we obtain the Dirichlet-to-Neumann mapping

$$\frac{\partial u}{\partial \nu} = -2\,\mathbf{W}(u|_\Gamma) \quad \text{on} \quad \Gamma\,, \quad \text{or equivalently,} \quad \boldsymbol{\sigma} \cdot \nu = -2\,\mathbf{W}\xi \quad \text{on} \quad \Gamma\,, \tag{2.3}$$

where ν is the unit outward normal to Γ and $\mathbf{W}$ is the hypersingular boundary integral operator associated with the Laplacian. Denoting by $\nu(z)$ the unit outward normal to $z \in \Gamma$, we have

$$(\mathbf{W}\lambda)(x) := -\frac{\partial}{\partial \nu(x)} \int_\Gamma \left\{ \frac{\partial}{\partial \nu(y)}\, E(x,y) \right\} \lambda(y)\, ds_y \quad \forall\, x \in \Gamma, \forall\, \lambda \in H^{1/2}(\Gamma)\,,$$

where $E(x,y) := -\frac{1}{2\pi} \log ||x-y||$ is the two-dimensional fundamental solution of the Laplace operator. It is well known (see, e.g., [5]) that $\mathbf{W} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is linear and bounded and that there exists $C_0 > 0$ such that

$$\langle \lambda, \mathbf{W}\lambda \rangle \;\geq\; C_0\, ||\lambda||^2_{H^{1/2}(\Gamma)} \quad \forall\, \lambda \in H^{1/2}_0(\Gamma)\,, \tag{2.4}$$

where, hereafter, $\langle \cdot, \cdot \rangle$ denotes the duality pairing of $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ with respect to the $L^2(\Gamma)$-inner product, and

$$H^{1/2}_0(\Gamma) \;:=\; \{ \lambda \in H^{1/2}(\Gamma) : \quad \langle \lambda, 1 \rangle = 0 \}\,.$$

Also, since $\mathbf{W}\,1 \equiv 0$ on Γ, we note that it suffices to look for the unknown ξ in the space $H_0^{1/2}(\Gamma)$.

Now, the asymptotic behaviour of u at ∞ implies that $\displaystyle\int_\Gamma \frac{\partial u}{\partial \nu}\, ds = 0$, which means that $\boldsymbol{\sigma} \in H_0(\mathrm{div};\Omega)$, where

$$H_0(\mathrm{div};\Omega) \,:=\, \{\, \boldsymbol{\tau} \in H(\mathrm{div};\Omega) : \quad \langle 1, \boldsymbol{\tau}\cdot\nu\rangle = 0 \,\}\,.$$

We recall here that $H(\mathrm{div};\Omega)$ is the space of functions $\boldsymbol{\tau} \in [L^2(\Omega)]^2$ such that $\mathrm{div}\,\boldsymbol{\tau} \in L^2(\Omega)$. Provided with the inner product $\langle \boldsymbol{\tau}, \boldsymbol{\sigma}\rangle_{H(\mathrm{div};\Omega)} := \langle \boldsymbol{\tau}, \boldsymbol{\sigma}\rangle_{[L^2(\Omega)]^2} + \langle \mathrm{div}\,\boldsymbol{\tau}, \mathrm{div}\,\boldsymbol{\sigma}\rangle_{L^2(\Omega)}$, $H(\mathrm{div};\Omega)$ is a Hilbert space. Moreover, for all $\boldsymbol{\tau} \in H(\mathrm{div};\Omega)$, $\boldsymbol{\tau}\cdot\nu \in H^{-1/2}(\Gamma)$ and $||\boldsymbol{\tau}\cdot\nu||_{H^{-1/2}(\Gamma)} \le ||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)}$ (see [22] for the proof of these results).

By virtue of the above analysis, the exterior transmission problem (2.1) can be reformulated as the following nonlocal boundary value problem in $\overline{\Omega}$: *Find* $(\boldsymbol{\theta}, \xi, \boldsymbol{\sigma}, u) \in [L^2(\Omega)]^2 \times H_0^{1/2}(\Gamma) \times H_0(\mathrm{div};\Omega) \times L^2(\Omega)$ *such that*

$$u = 0 \quad \text{on} \quad \Gamma_0, \quad \boldsymbol{\theta} = \nabla u \quad \text{in} \quad \Omega\,,$$

$$\boldsymbol{\sigma} = \begin{cases} a(\cdot, \boldsymbol{\theta}) & \text{in} \quad \Omega_1 \\ \boldsymbol{\theta} & \text{in} \quad \Omega_2 \end{cases} \qquad \text{and} \quad \mathrm{div}\,\boldsymbol{\sigma} = -f \quad \text{in} \quad \Omega\,, \tag{2.5}$$

$$\boldsymbol{\sigma}\cdot\nu = -2\,W\xi \quad \text{and} \quad u = \xi \quad \text{on} \quad \Gamma\,,$$

where we have adopted the notation $a(\cdot, \boldsymbol{\theta}) := (a_1(\cdot, \boldsymbol{\theta}), a_2(\cdot, \boldsymbol{\theta}))^T$, and the second and fourth equations of (2.5) must be taken in the distributional sense.

The previous procedure induces a dual-mixed finite element approach in Ω, which, up to now, is very close to the one employed in [20, 21] and [11]. However, the main difference will arise later on when we derive the corresponding variational formulation. Indeed, instead of using the complicated quotient spaces introduced in [20], we will rewrite the formulation in such a way that only the spaces indicated in (2.5) will be required in our subsequent analysis.

3. The dual-dual mixed formulation

From now on, we assume that the nonlinear mappings a_i satisfy the following conditions:

(A.1) Carathéodory condition. *The function $a_i(\cdot, \hat{\boldsymbol{\theta}})$, $i = 1, 2$, is measurable in Ω_1 for all $\hat{\boldsymbol{\theta}} \in \mathbf{R}^2$, and $a_i(x, \cdot)$ is continuous in $\mathbf{R}^2$ for almost all $x \in \Omega_1$.*

(A.2) Growth condition. *There exist functions $\phi_i \in L^2(\Omega_1)$, $i = 1, 2$, such that*

$$|a_i(x, \hat{\boldsymbol{\theta}})| \,\le\, C\,\{1 + |\hat{\boldsymbol{\theta}}|\} + |\phi_i(x)|\,,$$

for all $\hat{\boldsymbol{\theta}} \in \mathbf{R}^2$ and for almost all $x \in \Omega_1$.

(A.3) *The functions $a_i(x, \cdot)$, $i = 1, 2$, have continuous first order partial derivatives in $\mathbf{R}^2$ for almost all $x \in \Omega_1$. In addition, there exists $C > 0$ such that*

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial \hat{\theta}_j}\, a_i(x, \hat{\boldsymbol{\theta}})\, \hat{\zeta}_i\, \hat{\zeta}_j \,\ge\, C \sum_{i=1}^{2} \hat{\zeta}_i^2\,,$$

for all $\hat{\boldsymbol{\theta}} := (\hat{\theta}_1, \hat{\theta}_2)$, $\hat{\boldsymbol{\zeta}} := (\hat{\zeta}_1, \hat{\zeta}_2) \in \mathbf{R}^2$ and for almost all $x \in \Omega_1$.

(A.4) *The functions $a_i(x,\cdot)$ have continuous first order partial derivatives in $\mathbf{R}^2$ for almost all $x \in \Omega_1$. In addition, there exists $C > 0$ such that for each $i, j \in \{1,2\}$, $\frac{\partial}{\partial \hat{\theta}_j} a_i(x,\hat{\boldsymbol{\theta}})$ satisfies the Carathéodory condition* (A.1)*, and $\left|\frac{\partial}{\partial \hat{\theta}_j} a_i(x,\hat{\boldsymbol{\theta}})\right| \le C$, for all $\hat{\boldsymbol{\theta}} \in \mathbf{R}^2$ and for almost all $x \in \Omega_1$.*

For specific examples of coefficients a_i satisfying the above conditions, we refer to [4], [34] and [35].

As a consequence of (A.1) and (A.2), one can prove (see, e.g., Theorem 2.8 in [10]) that the Nemytsky operator $\mathcal{A}_i : [L^2(\Omega_1)]^2 \to L^2(\Omega_1)$, defined by $(\mathcal{A}_i\,\boldsymbol{\theta})(x) := a_i(x,\boldsymbol{\theta}(x))$ for all $\boldsymbol{\theta} \in [L^2(\Omega_1)]^2$ and for almost all $x \in \Omega_1$, is continuous and bounded.

Now, for the weak formulation, we first multiply the second equation in (2.5) by a function $\boldsymbol{\tau} \in H_0(\mathrm{div};\Omega)$, integrate by parts in Ω, and use that $u = 0$ on Γ_0 and that $u = \xi$ on Γ, to obtain

$$-\int_\Omega \boldsymbol{\theta}\cdot\boldsymbol{\tau}\,dx + \langle \xi, \boldsymbol{\tau}\cdot\nu\rangle - \int_\Omega u \,\mathrm{div}\,\boldsymbol{\tau}\,dx = 0\,. \tag{3.1}$$

Next, the third equation in (2.5) is tested against $\boldsymbol{\zeta} \in [L^2(\Omega)]^2$, which gives

$$\int_{\Omega_1} a(\cdot,\boldsymbol{\theta})\cdot\boldsymbol{\zeta}\,dx + \int_{\Omega_2} \boldsymbol{\theta}\cdot\boldsymbol{\zeta}\,dx - \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\zeta}\,dx = 0\,. \tag{3.2}$$

Finally, the fourth and fifth equations in (2.5) are tested against $v \in L^2(\Omega)$ and $\lambda \in H_0^{1/2}(\Gamma)$, respectively, which yields

$$-\int_\Omega v\,\mathrm{div}\,\boldsymbol{\sigma}\,dx = \int_\Omega f v\,dx \tag{3.3}$$

and

$$2\,\langle \lambda, W\xi\rangle + \langle \lambda, \boldsymbol{\sigma}\cdot\nu\rangle \;=\; 0\,. \tag{3.4}$$

Thus, collecting appropriately (3.1), (3.2), (3.3) and (3.4), we arrive at the following variational formulation of (2.5): *Find* $((\boldsymbol{\theta},\xi),\boldsymbol{\sigma},u) \in ([L^2(\Omega)]^2 \times H_0^{1/2}(\Gamma)) \times H_0(\mathrm{div};\Omega) \times L^2(\Omega)$ *such that*

$$\begin{aligned}
\int_{\Omega_1} a(\cdot,\boldsymbol{\theta})\cdot\boldsymbol{\zeta}\,dx + \int_{\Omega_2} \boldsymbol{\theta}\cdot\boldsymbol{\zeta}\,dx + 2\,\langle \lambda, \mathbf{W}\xi\rangle & \\
- \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\zeta}\,dx + \langle \lambda, \boldsymbol{\sigma}\cdot\nu\rangle &= 0\,,\\
- \int_\Omega \boldsymbol{\theta}\cdot\boldsymbol{\tau}\,dx + \langle \xi, \boldsymbol{\tau}\cdot\nu\rangle - \int_\Omega u\,\mathrm{div}\,\boldsymbol{\tau}\,dx &= 0\,,\\
- \int_\Omega v\,\mathrm{div}\,\boldsymbol{\sigma}\,dx &= \int_\Omega f v\,dx\,,
\end{aligned} \tag{3.5}$$

for all $((\boldsymbol{\zeta},\lambda),\boldsymbol{\tau},v) \in ([L^2(\Omega)]^2 \times H_0^{1/2}(\Gamma)) \times H_0(\mathrm{div};\Omega) \times L^2(\Omega)$.

We show next that (3.5) can be rewritten in the form of a nonlinear variational problem with linear constraints. For this purpose, we put $X_1 := [L^2(\Omega)]^2 \times H_0^{1/2}(\Gamma)$, $M_1 := H_0(\mathrm{div};\Omega)$, $X := X_1 \times M_1$, $M := L^2(\Omega)$, denote $\mathbf{t} := (\boldsymbol{\theta},\xi)$, $\mathbf{s} := (\boldsymbol{\zeta},\lambda) \in X_1$, and define the operators $\mathbf{A}_1 : X_1 \to X_1'$, $\mathbf{B}_1 : X_1 \to M_1'$, $\mathbf{A} : X \to X'$, $\mathbf{B} : X \to M'$, and the functional $\mathbf{G} \in M'$, as follows:

$$[\mathbf{A}_1(\mathbf{t}), \mathbf{s}] := \int_{\Omega_1} a(\cdot, \boldsymbol{\theta}) \cdot \zeta \, dx + \int_{\Omega_2} \boldsymbol{\theta} \cdot \zeta \, dx + 2 \langle \lambda, \mathbf{W}\xi \rangle , \tag{3.6}$$

$$[\mathbf{B}_1(\mathbf{t}), \boldsymbol{\tau}] := - \int_{\Omega} \boldsymbol{\theta} \cdot \boldsymbol{\tau} \, dx + \langle \xi, \boldsymbol{\tau} \cdot \nu \rangle , \tag{3.7}$$

$$[\mathbf{A}(\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{s}, \boldsymbol{\tau})] := [\mathbf{A}_1(\mathbf{t}), \mathbf{s}] + [\mathbf{B}_1(\mathbf{s}), \boldsymbol{\sigma}] + [\mathbf{B}_1(\mathbf{t}), \boldsymbol{\tau}] , \tag{3.8}$$

$$[\mathbf{B}(\mathbf{t}, \boldsymbol{\sigma}), v] := - \int_{\Omega} v \operatorname{div} \boldsymbol{\sigma} \, dx \tag{3.9}$$

and

$$[\mathbf{G}, v] := \int_{\Omega} f v \, dx \tag{3.10}$$

for all $(\mathbf{t}, \boldsymbol{\sigma})$, $(\mathbf{s}, \boldsymbol{\tau}) \in X$ and for all $v \in M$, where $[\cdot, \cdot]$ stands for the duality pairing induced by the operators appearing in each case.

Further, let $\mathbf{B}_1^* : M_1 \to X_1'$ and $\mathbf{B}^* : M \to X'$ be the transposes of $\mathbf{B}_1$ and $\mathbf{B}$, respectively, and let $\mathbf{O}$ denote both the null functional and the null operator.

It is worth remarking that $\mathbf{B}_1$ and $\mathbf{B}$ are linear and bounded operators, and that $\mathbf{A}_1$, and hence $\mathbf{A}$, are nonlinear. Moreover, $\mathbf{A}$ can be defined, equivalently, as:

$$\mathbf{A}(\mathbf{t}, \boldsymbol{\sigma}) := \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1^* \\ \mathbf{B}_1 & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \boldsymbol{\sigma} \end{bmatrix} \in X' := X_1' \times M_1' . \tag{3.11}$$

Therefore, the system (3.5) can be reformulated as the following operator equation: *Find* $((\mathbf{t}, \boldsymbol{\sigma}), u) \in X \times M$ *such that*

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}^* \\ \mathbf{B} & \mathbf{O} \end{bmatrix} \begin{bmatrix} (\mathbf{t}, \boldsymbol{\sigma}) \\ u \end{bmatrix} = \begin{bmatrix} \mathbf{O} \\ \mathbf{G} \end{bmatrix} . \tag{3.12}$$

The equation (3.12), which can be viewed as a nonlinear variational problem with linear constraints, constitutes our so-called *dual-dual mixed formulation* of (2.5) since the operator $\mathbf{A}$ itself has the dual-type structure given by (3.11).

In order to establish the unique solvability of (3.12), study its Galerkin approximations, and derive the corresponding error analysis, we need an extension of the usual Babuska-Brezzi theory to the above class of nonlinear problems. This is, precisely, the subject of the next section. We will go back to our problem (3.12) in Section 5.

4. An extension of the Babuska-Brezzi theory

In the recent paper [12] we have generalized the classical Babuska-Brezzi theory to the class of nonlinear variational problems with constraints given by (3.12). The purpose of this section is to recall the main results from that work.

In order to set the abstract problem of interest, we let X_1, M_1, M be Hilbert spaces and define $X := X_1 \times M_1$. Then, we consider a nonlinear operator $\mathbf{A}_1 : X_1 \to X_1'$, and linear bounded operators $\mathbf{B}_1 : X_1 \to M_1'$ and $\mathbf{B} : X \to M'$, with transposes $\mathbf{B}_1^* : M_1 \to X_1'$ and $\mathbf{B}^* : M \to X'$, respectively. With $\mathbf{A}_1$, $\mathbf{B}_1$ and $\mathbf{B}_1^*$ we define a nonlinear operator $\mathbf{A} : X \to X'$ as in (3.11).

Then, we are interested in the following nonlinear variational problem: *Given* $(\mathbf{F},\mathbf{G}) \in X' \times M'$, *find* $((\mathbf{t},\boldsymbol{\sigma}),u) \in X \times M$ *such that*

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}^* \\ \mathbf{B} & \mathbf{O} \end{bmatrix} \begin{bmatrix} (\mathbf{t},\boldsymbol{\sigma}) \\ u \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix}. \tag{4.1}$$

Sufficient conditions for the unique solvability of (4.1) are provided in the following theorem.

Theorem 4.1. *Let* $V := \mathrm{Ker}\,(\mathbf{B})$ *such that* $V = \tilde{X}_1 \times \tilde{M}_1$, *with* $\tilde{X}_1 \subseteq X_1$ *and* $\tilde{M}_1 \subseteq M_1$. *Also, define* $V_1 := \{\mathbf{s} \in \tilde{X}_1 \colon [\mathbf{B}_1(\mathbf{s}),\boldsymbol{\tau}] = 0 \ \forall \boldsymbol{\tau} \in \tilde{M}_1\}$ *and let* $\Pi_1 : X_1' \to V_1'$ *be the canonical imbedding defined by* $\Pi_1(\mathbf{F}_1) = \mathbf{F}_1|_{V_1}$ *for all* $\mathbf{F}_1 \in X_1'$. *Assume that*

i) *there exists* $\beta > 0$ *such that for all* $v \in M$

$$\sup_{\substack{(\mathbf{s},\boldsymbol{\tau})\in X \\ (\mathbf{s},\boldsymbol{\tau})\neq 0}} \frac{[\mathbf{B}(\mathbf{s},\boldsymbol{\tau}),v]}{||(\mathbf{s},\boldsymbol{\tau})||_X} \;\geq\; \beta\,||v||_M;$$

ii) *there exists* $\beta_1 > 0$ *such that for all* $\boldsymbol{\tau} \in \tilde{M}_1$

$$\sup_{\substack{\mathbf{s}\in\tilde{X}_1 \\ \mathbf{s}\neq 0}} \frac{[\mathbf{B}_1(\mathbf{s}),\boldsymbol{\tau}]}{||\mathbf{s}||_{X_1}} \;\geq\; \beta_1\,||\boldsymbol{\tau}||_{M_1};$$

iii) *the nonlinear operator* $\mathbf{A}_1 : X_1 \to X_1'$ *is Lipschitz continuous with a Lipschitz constant* $\gamma > 0$, *and for any* $\tilde{\mathbf{t}} \in X_1$, *the nonlinear operator* $\Pi_1\mathbf{A}_1(\cdot + \tilde{\mathbf{t}}) : V_1 \to V_1'$ *is strongly monotone.*

Then, for each $(\mathbf{F},\mathbf{G}) \in X' \times M'$ *there exists a unique* $((\mathbf{t},\boldsymbol{\sigma}),u) \in X \times M$ *solution of* (4.1).

Proof. We adapt the analysis from [22] (Chapter I, Section 4) to the present situation. Thus, given $\mathbf{G} \in M'$ we set

$$V(\mathbf{G}) := \{\, (\mathbf{s},\boldsymbol{\tau}) \in X : \quad \mathbf{B}(\mathbf{s},\boldsymbol{\tau}) = \mathbf{G} \,\}$$

and observe that $V := \mathrm{Ker}\,(\mathbf{B}) = V(\mathbf{O})$.

Then, with (4.1) we associate the following problem: *Find* $(\mathbf{t},\boldsymbol{\sigma}) \in V(\mathbf{G})$ *such that*

$$[\mathbf{A}(\mathbf{t},\boldsymbol{\sigma}),(\mathbf{s},\boldsymbol{\tau})] \;=\; [\mathbf{F},(\mathbf{s},\boldsymbol{\tau})] \quad \forall\,(\mathbf{s},\boldsymbol{\tau}) \in V\,. \tag{4.2}$$

Clearly, if $(\mathbf{t},\boldsymbol{\sigma}) \in V(\mathbf{G})$ is a solution of (4.2), then, due to the inf-sup condition Theorem 4.1 i) and Lemma 4.1 in Chapter I of [22], there exists a unique $u \in M$ such that $((\mathbf{t},\boldsymbol{\sigma}),u) \in X \times M$ is a solution of (4.1).

Conversely, if $((\mathbf{t},\boldsymbol{\sigma}),u) \in X \times M$ is a solution of (4.1), then $(\mathbf{t},\boldsymbol{\sigma}) \in V(\mathbf{G})$ and $(\mathbf{t},\boldsymbol{\sigma})$ is a solution of (4.2) since for all $(\mathbf{s},\boldsymbol{\tau}) \in V$, $[\mathbf{B}^*(u),(\mathbf{s},\boldsymbol{\tau})] = [\mathbf{B}(\mathbf{s},\boldsymbol{\tau}),u] = 0$.

Because of this equivalence, we now concentrate on problem (4.2). Again, by Lemma 4.1 in Chapter I of [22], there exists $(\mathbf{t}_0,\boldsymbol{\sigma}_0) \in X$ such that $\mathbf{B}(\mathbf{t}_0,\boldsymbol{\sigma}_0) = \mathbf{G}$. Thus, problem (4.2) can be replaced by: *Find* $(\tilde{\mathbf{t}},\tilde{\boldsymbol{\sigma}}) \in V$ *such that*

$$\begin{aligned} &[\mathbf{A}_1(\tilde{\mathbf{t}} + \mathbf{t}_0),\mathbf{s}] + [\mathbf{B}_1^*(\tilde{\boldsymbol{\sigma}}),\mathbf{s}] = [\tilde{\mathbf{F}}_1,\mathbf{s}] \\ &[\mathbf{B}_1(\tilde{\mathbf{t}}),\boldsymbol{\tau}] = [\tilde{\mathbf{G}}_1,\boldsymbol{\tau}] \end{aligned} \tag{4.3}$$

for all $(\mathbf{s}, \boldsymbol{\tau}) \in V$, where $\tilde{\mathbf{F}}_1 := \mathbf{F}_1 - \mathbf{B}_1^*(\boldsymbol{\sigma}_0) \in X_1'$ and $\tilde{\mathbf{G}}_1 := \mathbf{G}_1 - \mathbf{B}_1(\mathbf{t}_0) \in M_1'$, with $\mathbf{F} := (\mathbf{F}_1, \mathbf{G}_1) \in X_1' \times M_1' =: X'$.

Next, we set

$$V_1(\tilde{\mathbf{G}}_1) := \{\, \mathbf{s} \in \tilde{X}_1 : \quad [\mathbf{B}_1(\mathbf{s}), \boldsymbol{\tau}] = [\tilde{\mathbf{G}}_1, \boldsymbol{\tau}] \quad \forall\, \boldsymbol{\tau} \in \tilde{M}_1 \,\}$$

and observe that $V_1 = V_1(\mathbf{O})$.

Then we associate with (4.3) the following problem: *Find* $\tilde{\mathbf{t}} \in V_1(\tilde{\mathbf{G}}_1)$ *such that*

$$[\mathbf{A}_1(\tilde{\mathbf{t}} + \mathbf{t}_0), \mathbf{s}] \;=\; [\tilde{\mathbf{F}}_1, \mathbf{s}] \quad \forall\, \mathbf{s} \in V_1\,. \tag{4.4}$$

Using Theorem 4.1 ii) and Lemma 4.1 in Chapter I of [22], we deduce that there exists $\tilde{\mathbf{t}}_0 \in \tilde{X}_1$ such that $[\mathbf{B}_1(\tilde{\mathbf{t}}_0), \boldsymbol{\tau}] = [\tilde{\mathbf{G}}_1, \boldsymbol{\tau}]$ for all $\boldsymbol{\tau} \in \tilde{M}_1$. Therefore, problem (4.4) can be replaced by: *Find* $\hat{\mathbf{t}} \in V_1$ *such that*

$$[\mathbf{A}_1(\hat{\mathbf{t}} + \tilde{\mathbf{t}}_0 + \mathbf{t}_0), \mathbf{s}] \;=\; [\tilde{\mathbf{F}}_1, \mathbf{s}] \quad \forall\, \mathbf{s} \in V_1\,. \tag{4.5}$$

Thus, due to the hypotheses on $\mathbf{A}_1$ (see Theorem 4.1 iii)) and thanks to a well known result from nonlinear functional analysis (see, e.g., Theorem 3.3.23 in [32]) we conclude that (4.5) has a unique solution $\hat{\mathbf{t}} \in V_1$, and hence $\tilde{\mathbf{t}} := \hat{\mathbf{t}} + \tilde{\mathbf{t}}_0 \in V_1(\tilde{\mathbf{G}}_1)$ is the unique solution of (4.4). It follows, in virtue of Theorem 4.1 ii) and Lemma 4.1 in Chapter I of [22], that there exists $\tilde{\boldsymbol{\sigma}} \in M_1$ such that $(\tilde{\mathbf{t}}, \tilde{\boldsymbol{\sigma}}) \in V$ is the unique solution of (4.3). In this way, we deduce that $(\mathbf{t}, \boldsymbol{\sigma}) := (\tilde{\mathbf{t}} + \mathbf{t}_0, \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}_0) \in V(\mathbf{G})$ is the unique solution of (4.2). Finally, the equivalence between (4.1) and (4.2) completes the proof. □

Now, for the Galerkin approximation of (4.1), we let $X_{1,h}$, $M_{1,h}$ and M_h be finite dimensional subspaces of X_1, M_1 and M, respectively, and let $X_h := X_{1,h} \times M_{1,h}$ be the corresponding subspace of X. Here, we assume that the index h is taken in a numerable family $\mathcal{I} := \{h_j\}_{j \in \mathbf{N}}$ such that $h_j \geq h_{j+1}$ for all $j \in \mathbf{N}$.

Thus, the Galerkin scheme associated with (4.1) reads as follows: *Given* $(\mathbf{F}, \mathbf{G}) \in X' \times M'$, *find* $((\mathbf{t}_h, \boldsymbol{\sigma}_h), u_h) \in X_h \times M_h$ *such that*

$$\begin{aligned} [\mathbf{A}(\mathbf{t}_h, \boldsymbol{\sigma}_h), (\mathbf{s}_h, \boldsymbol{\tau}_h)] + [\mathbf{B}^*(u_h), (\mathbf{s}_h, \boldsymbol{\tau}_h)] \;&=\; [\mathbf{F}, (\mathbf{s}_h, \boldsymbol{\tau}_h)]\,, \\ [\mathbf{B}(\mathbf{t}_h, \boldsymbol{\sigma}_h), v_h] \;&=\; [\mathbf{G}, v_h]\,, \end{aligned} \tag{4.6}$$

for all $((\mathbf{s}_h, \boldsymbol{\tau}_h), v_h) \in X_h \times M_h$.

The discrete analogue of Theorem 4.1 is stated next.

Theorem 4.2. *Let* $V_h := \{(\mathbf{s}_h, \boldsymbol{\tau}_h) \in X_h \colon [\mathbf{B}(\mathbf{s}_h, \boldsymbol{\tau}_h), v_h] = 0 \;\forall\, v_h \in M_h\}$ *such that* $V_h := \tilde{X}_{1,h} \times \tilde{M}_{1,h}$, *with* $\tilde{X}_{1,h} \subseteq X_{1,h}$ *and* $\tilde{M}_{1,h} \subseteq M_{1,h}$. *Also, define* $V_{1,h} := \{\mathbf{s}_h \in \tilde{X}_{1,h} \colon [\mathbf{B}_1(\mathbf{s}_h), \boldsymbol{\tau}_h] = 0 \;\forall\, \boldsymbol{\tau}_h \in \tilde{M}_{1,h}\}$ *and let* $\Pi_{1,h} : X_{1,h}' \to V_{1,h}'$ *be the canonical imbedding. Further, let* $\mathbf{A}_{1,h} := p_h' \mathbf{A}_1 : X_1 \to X_{1,h}'$ *where* $p_h : X_{1,h} \to X_1$ *is the canonical injection with adjoint* $p_h' : X_1' \to X_{1,h}'$. *Assume that*

i) *there exists* $\beta^* > 0$, *independent of the subspaces involved, such that for all* $v_h \in M_h$

$$\sup_{\substack{(\mathbf{s}_h, \boldsymbol{\tau}_h) \in X_h \\ (\mathbf{s}_h, \boldsymbol{\tau}_h) \neq 0}} \frac{[\mathbf{B}(\mathbf{s}_h, \boldsymbol{\tau}_h), v_h]}{||(\mathbf{s}_h, \boldsymbol{\tau}_h)||_X} \;\geq\; \beta^* \, ||v_h||_M;$$

ii) *there exists* $\beta_1^* > 0$, *independent of the subspaces involved, such that for all* $\boldsymbol{\tau}_h \in \tilde{M}_{1,h}$

$$\sup_{\substack{\mathbf{s}_h \in \tilde{X}_{1,h} \\ \mathbf{s}_h \neq 0}} \frac{[\mathbf{B}_1(\mathbf{s}_h), \boldsymbol{\tau}_h]}{||\mathbf{s}_h||_{X_1}} \geq \beta_1^* ||\boldsymbol{\tau}_h||_{M_1};$$

iii) *the nonlinear operator* $\mathbf{A}_{1,h} : X_1 \to X'_{1,h}$ *is Lipschitz-continuous, and for any* $\tilde{\mathbf{t}} \in X_{1,h}$, *the nonlinear operator* $\Pi_{1,h}\mathbf{A}_{1,h}(\cdot + \tilde{\mathbf{t}}) : V_{1,h} \to V'_{1,h}$ *is strongly monotone with a monotonicity constant* $\alpha_h > 0$ *independent of* $\tilde{\mathbf{t}}$.

Then, for each $(\mathbf{F}, \mathbf{G}) \in X' \times M'$ *there exists a unique* $((\mathbf{t}_h, \boldsymbol{\sigma}_h), u_h) \in X_h \times M_h$ *solution of* (4.6).

Proof. It is similar to the proof of Theorem 4.1 and hence we omit further details. We refer the interested reader to Section 3 and Theorem 3.2 in [12]. □

Clearly, the Lipschitz-continuity of $\mathbf{A}_1$ yields the same property for $\mathbf{A}_{1,h}$, with the same Lipschitz constant γ, independent of h, given in Theorem 4.1.

Finally, concerning the error analysis, we recall the following result from [12].

Theorem 4.3. *Assume that all the hypotheses of both Theorem* 4.1 *and Theorem* 4.2 *are satisfied, and let* $((\mathbf{t}, \boldsymbol{\sigma}), u) \in X \times M$ *and* $((\mathbf{t}_h, \boldsymbol{\sigma}_h), u_h) \in X_h \times M_h$ *be the unique solutions of* (4.1) *and* (4.6), *respectively. Let* $\mathbf{F} := (\mathbf{F}_1, \mathbf{G}_1) \in X'$, *with* $\mathbf{F}_1 \in X'_1$ *and* $\mathbf{G}_1 \in M'_1$. *In addition, suppose that the family of nonlinear operators* $\{\Pi_{1,h}\mathbf{A}_{1,h}(\cdot + \tilde{\mathbf{t}}) : \quad \tilde{\mathbf{t}} \in X_{1,h}, \ h \in \mathcal{I}\}$ *is uniformly strongly monotone, i.e., there exists* $\alpha > 0$ *such that* $\alpha_h \geq \alpha$ *for all* $h \in \mathcal{I}$. *Then, there exists* $C > 0$, *depending only on* α, γ, $||\mathbf{B}_1||$, β_1^*, $||\mathbf{B}||$ *and* β^*, *such that the following Strang-type error estimate holds for all* $h \in \mathcal{I}$:

$$\begin{aligned} &||((\mathbf{t}, \boldsymbol{\sigma}), u) - ((\mathbf{t}_h, \boldsymbol{\sigma}_h), u_h)|| \\ &\quad \leq C \left\{ \inf_{((\mathbf{s}_h, \boldsymbol{\tau}_h), v_h) \in X_h \times M_h} ||((\mathbf{t}, \boldsymbol{\sigma}), u) - ((\mathbf{s}_h, \boldsymbol{\tau}_h), v_h)|| \right. \\ &\qquad + \sup_{\substack{\tilde{\mathbf{s}}_h \in \tilde{X}_{1,h} \\ \tilde{\mathbf{s}}_h \neq 0}} \left\{ \frac{[\mathbf{F}_1 - \mathbf{A}_1(\mathbf{t}) - \mathbf{B}_1^*(\boldsymbol{\sigma}), \tilde{\mathbf{s}}_h]}{||\tilde{\mathbf{s}}_h||} \right\} \\ &\qquad \left. + \sup_{\substack{\tilde{\boldsymbol{\tau}}_h \in \tilde{M}_{1,h} \\ \tilde{\boldsymbol{\tau}}_h \neq 0}} \left\{ \frac{[\mathbf{G}_1 - \mathbf{B}_1(\mathbf{t}), \tilde{\boldsymbol{\tau}}_h]}{||\tilde{\boldsymbol{\tau}}_h||} \right\} \right\}. \end{aligned} \tag{4.7}$$

Proof. We do not give full details here, but just sketch the main ideas. For the whole proof, we refer to Section 4 in [12].

First, the discrete inf-sup condition satisfied by $\mathbf{B}$ (cf. Theorem 4.2 i)) guarantees the existence of $(\mathbf{t}_{0,h}, \boldsymbol{\sigma}_{0,h}) \in X_{1,h} \times M_{1,h}$ such that $[\mathbf{B}(\mathbf{t}_{0,h}, \boldsymbol{\sigma}_{0,h}), v_h] = [\mathbf{G}, v_h]$ for all $v_h \in M_h$.

Then, by using the properties of the operators $\mathbf{A}_1$ and $\mathbf{B}_1$, one proves that for all $h \in \mathcal{I}$

$$
\begin{aligned}
||\mathbf{t} - \mathbf{t}_h|| \leq \frac{1}{\alpha} \sup_{\substack{\tilde{\mathbf{s}}_h \in \tilde{X}_{1,h} \\ \tilde{\mathbf{s}}_h \neq 0}} & \left\{ \frac{[\mathbf{F}_1 - \mathbf{A}_1(\mathbf{t}) - \mathbf{B}_1^*(\boldsymbol{\sigma}), \tilde{\mathbf{s}}_h]}{||\tilde{\mathbf{s}}_h||} \right\} \\
& + \left(1 + \frac{\gamma}{\alpha}\right) \inf_{\mathbf{s}_h \in V_{1,h}(\mathbf{G}_1)} ||\mathbf{t} - \mathbf{s}_h|| \\
& + \frac{||\mathbf{B}_1||}{\alpha} \inf_{\tilde{\boldsymbol{\tau}}_h \in \tilde{M}_{1,h}} ||\boldsymbol{\sigma} - (\tilde{\boldsymbol{\tau}}_h + \boldsymbol{\sigma}_{0,h})||,
\end{aligned}
\tag{4.8}
$$

where

$$
V_{1,h}(\mathbf{G}_1) := \{\mathbf{s}_h \in X_{1,h} : \quad [\mathbf{B}_1(\mathbf{s}_h), \boldsymbol{\tau}_h] = [\mathbf{G}_1, \boldsymbol{\tau}_h] \; \forall \, \boldsymbol{\tau}_h \in \tilde{M}_{1,h}\}.
$$

Now, the discrete inf-sup condition for $\mathbf{B}_1$ (cf. Theorem 4.2 ii)) allows us to improve the bound provided by the second term on the right hand side of inequality (4.8). Indeed, we show that the following estimate holds for all $h \in \mathcal{I}$:

$$
\begin{aligned}
\inf_{\mathbf{s}_h \in V_{1,h}(\mathbf{G}_1)} ||\mathbf{t} - \mathbf{s}_h|| \leq & \left(1 + \frac{||\mathbf{B}_1||}{\beta_1^*}\right) \inf_{\tilde{\mathbf{s}}_h \in \tilde{X}_{1,h}} ||\mathbf{t} - (\tilde{\mathbf{s}}_h + \mathbf{t}_{0,h})|| \\
& + \frac{1}{\beta_1^*} \sup_{\substack{\tilde{\boldsymbol{\tau}}_h \in \tilde{M}_{1,h} \\ \tilde{\boldsymbol{\tau}}_h \neq 0}} \left\{ \frac{[\mathbf{G}_1 - \mathbf{B}_1(\mathbf{t}), \tilde{\boldsymbol{\tau}}_h]}{||\tilde{\boldsymbol{\tau}}_h||} \right\}.
\end{aligned}
\tag{4.9}
$$

Next, by applying again the properties of the operators $\mathbf{A}_1$ and $\mathbf{B}_1$, and also the discrete inf-sup condition for $\mathbf{B}_1$, we obtain the following upper bound for the error $||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||$:

$$
\begin{aligned}
||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h|| \leq \frac{1}{\beta_1^*} \sup_{\substack{\tilde{\mathbf{s}}_h \in \tilde{X}_{1,h} \\ \tilde{\mathbf{s}}_h \neq 0}} & \left\{ \frac{[\mathbf{F}_1 - \mathbf{A}_1(\mathbf{t}) - \mathbf{B}_1^*(\boldsymbol{\sigma}), \tilde{\mathbf{s}}_h]}{||\tilde{\mathbf{s}}_h||} \right\} + \frac{\gamma}{\beta_1^*} ||\mathbf{t} - \mathbf{t}_h|| \\
& + \left(1 + \frac{||\mathbf{B}_1||}{\beta_1^*}\right) \inf_{\tilde{\boldsymbol{\tau}}_h \in \tilde{M}_{1,h}} ||\boldsymbol{\sigma} - (\tilde{\boldsymbol{\tau}}_h + \boldsymbol{\sigma}_{0,h})||.
\end{aligned}
\tag{4.10}
$$

Hence, as a consequence of (4.8), (4.9) and (4.10), and using also the discrete inf-sup condition for $\mathbf{B}$, we deduce that there exists $\tilde{C} > 0$, depending only on α, γ, $||\mathbf{B}_1||$, β_1^*, $||\mathbf{B}||$ and β^*, such that for all $h \in \mathcal{I}$:

$$
\begin{aligned}
||(\mathbf{t}, \boldsymbol{\sigma}) - (\mathbf{t}_h, \boldsymbol{\sigma}_h)|| \leq \tilde{C} \Bigg\{ & \inf_{(\mathbf{s}_h, \boldsymbol{\tau}_h) \in X_h} ||(\mathbf{t}, \boldsymbol{\sigma}) - (\mathbf{s}_h, \boldsymbol{\tau}_h)|| \\
& + \sup_{\substack{\tilde{\mathbf{s}}_h \in \tilde{X}_{1,h} \\ \tilde{\mathbf{s}}_h \neq 0}} \left\{ \frac{[\mathbf{F}_1 - \mathbf{A}_1(\mathbf{t}) - \mathbf{B}_1^*(\boldsymbol{\sigma}), \tilde{\mathbf{s}}_h]}{||\tilde{\mathbf{s}}_h||} \right\} \\
& + \sup_{\substack{\tilde{\boldsymbol{\tau}}_h \in \tilde{M}_{1,h} \\ \tilde{\boldsymbol{\tau}}_h \neq 0}} \left\{ \frac{[\mathbf{G}_1 - \mathbf{B}_1(\mathbf{t}), \tilde{\boldsymbol{\tau}}_h]}{||\tilde{\boldsymbol{\tau}}_h||} \right\} \Bigg\}.
\end{aligned}
\tag{4.11}
$$

On the other hand, following the usual approach from [22] and applying now the properties of the operators $\mathbf{A}$ and $\mathbf{B}$ one can prove that there exists $\bar{C} > 0$, depending only on γ, $||\mathbf{B}_1||$, $||\mathbf{B}||$ and β^*, such that for all $h \in \mathcal{I}$:

$$||u - u_h|| \;\leq\; \bar{C} \left\{ ||(\mathbf{t}, \boldsymbol{\sigma}) - (\mathbf{t}_h, \boldsymbol{\sigma}_h)|| \,+\, \inf_{v_h \in M_h} ||u - v_h|| \right\}. \tag{4.12}$$

Finally, (4.11) and (4.12) yield (4.7), thus completing the proof of the theorem. □

It is important to observe that if $\tilde{X}_{1,h} \subseteq \tilde{X}_1$, then

$$\sup_{\substack{\tilde{\mathbf{s}}_h \in \tilde{X}_{1,h} \\ \tilde{\mathbf{s}}_h \neq 0}} \left\{ \frac{[\mathbf{F}_1 - \mathbf{A}_1(\mathbf{t}) - \mathbf{B}_1^*(\boldsymbol{\sigma}), \tilde{\mathbf{s}}_h]}{||\tilde{\mathbf{s}}_h||} \right\} \;=\; 0\,.$$

Similarly, if $\tilde{M}_{1,h} \subseteq \tilde{M}_1$, then

$$\sup_{\substack{\tilde{\boldsymbol{\tau}}_h \in \tilde{M}_{1,h} \\ \tilde{\boldsymbol{\tau}}_h \neq 0}} \left\{ \frac{[\mathbf{G}_1 - \mathbf{B}_1(\mathbf{t}), \tilde{\boldsymbol{\tau}}_h]}{||\tilde{\boldsymbol{\tau}}_h||} \right\} \;=\; 0\,.$$

It follows that if $V_h \subseteq V$, then (4.7) becomes the usual Cea estimate for the Galerkin error. In other words, the second and third terms on the right hand side of (4.7) constitute the consistency error for the case in which V_h is not a subspace of V.

5. Existence, uniqueness and approximation results

5.1. **The continuous problem.** We now go back to our problem from Section 3. In the sequel, we show that (3.12) satisfies the hypotheses of Theorem 4.1.

To begin with, we state the continuous inf-sup condition for $\mathbf{B}$.

Lemma 5.1. *There exists $\beta > 0$ such that for all $v \in M$,*

$$\sup_{\substack{(\mathbf{s},\boldsymbol{\tau}) \in X \\ (\mathbf{s},\boldsymbol{\tau}) \neq 0}} \frac{[\mathbf{B}(\mathbf{s},\boldsymbol{\tau}), v]}{||(\mathbf{s},\boldsymbol{\tau})||_X} \;\geq\; \beta\, ||v||_M\,.$$

Proof. We only observe that

$$\sup_{\substack{(\mathbf{s},\boldsymbol{\tau}) \in X \\ (\mathbf{s},\boldsymbol{\tau}) \neq 0}} \frac{[\mathbf{B}(\mathbf{s},\boldsymbol{\tau}), v]}{||(\mathbf{s},\boldsymbol{\tau})||_X} \;\geq\; \sup_{\substack{\boldsymbol{\tau} \in M_1 \\ \boldsymbol{\tau} \neq 0}} \frac{-\int_\Omega v \operatorname{div} \boldsymbol{\tau}\, dx}{||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)}}\,.$$

The rest of the proof is quite standard and we refer the interested reader to [22], [20] or [31]. □

It is important to remark that, using classical regularity results, one can show (cf. Lemma 4.4 in [31]), that there exists $\beta > 0$ such that

$$\sup_{\substack{\boldsymbol{\tau} \in [H^1(\Omega)]^2 \cap H_0(\mathrm{div};\Omega) \\ \boldsymbol{\tau} \neq 0}} \frac{-\int_\Omega v \operatorname{div} \boldsymbol{\tau}\, dx}{||\boldsymbol{\tau}||_{[H^1(\Omega)]^2}} \;\geq\; \beta\; ||v||_{L^2(\Omega)} \quad \forall\, v \in L^2(\Omega)\,. \tag{5.1}$$

This stronger inf-sup condition will be needed in subsection 5.2 to prove the discrete inf-sup condition for $\mathbf{B}$.

On the other hand, it is straigthforward to see that $V := \mathrm{Ker}(\mathbf{B}) = \tilde{X}_1 \times \tilde{M}_1$, where

$$\tilde{X}_1 = X_1 \quad \text{and} \quad \tilde{M}_1 = \{\, \boldsymbol{\tau} \in M_1 : \quad \operatorname{div} \boldsymbol{\tau} = 0 \quad \text{in} \quad \Omega \,\}\,.$$

Then, the inf-sup condition for $\mathbf{B}_1$ is also easily established.

Lemma 5.2. *We have*

$$\sup_{\substack{\mathbf{s}\in\tilde{X}_1\\ \mathbf{s}\neq 0}} \frac{[\mathbf{B}_1(\mathbf{s}),\boldsymbol{\tau}]}{||\mathbf{s}||_{X_1}} \;\geq\; ||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)} \quad \forall\,\boldsymbol{\tau}\in\tilde{M}_1\,.$$

Proof. Since $[\mathbf{B}_1(\mathbf{s}),\boldsymbol{\tau}] = -\int_\Omega \boldsymbol{\zeta}\cdot\boldsymbol{\tau}\,dx + \langle\lambda,\boldsymbol{\tau}\cdot\nu\rangle$ for all $\mathbf{s} := (\boldsymbol{\zeta},\lambda)\in\tilde{X}_1$, it follows that

$$\sup_{\substack{\mathbf{s}\in\tilde{X}_1\\ \mathbf{s}\neq 0}} \frac{[\mathbf{B}_1(\mathbf{s}),\boldsymbol{\tau}]}{||\mathbf{s}||_{X_1}} \;\geq\; \sup_{\substack{(\boldsymbol{\zeta},0)\in\tilde{X}_1\\ \boldsymbol{\zeta}\neq 0}} \frac{-\int_\Omega \boldsymbol{\zeta}\cdot\boldsymbol{\tau}\,dx}{||\boldsymbol{\zeta}||_{[L^2(\Omega)]^2}} \;=\; ||\boldsymbol{\tau}||_{[L^2(\Omega)]^2} \;=\; ||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)}$$

for all $\boldsymbol{\tau}\in\tilde{M}_1$, which completes the proof. □

Let us now recall that the nonlinear coefficients a_i satisfy the assumptions (A.3) and (A.4) (cf. Section 3). Then, the following lemma establishes the strong monotonicity and Lipschitz-continuity of the nonlinear operator $\mathbf{A}_1 : X_1 \to X_1'$.

Lemma 5.3. *There exist positive constants* α, γ *such that*

$$[\mathbf{A}_1(\mathbf{t}) - \mathbf{A}_1(\mathbf{s}),\mathbf{t}-\mathbf{s}] \;\geq\; \alpha\,||\mathbf{t}-\mathbf{s}||^2_{X_1} \tag{5.2}$$

and

$$||\mathbf{A}_1(\mathbf{t}) - \mathbf{A}_1(\mathbf{s})||_{X_1'} \;\leq\; \gamma\,||\mathbf{t}-\mathbf{s}||_{X_1} \tag{5.3}$$

for all $\mathbf{t}$, $\mathbf{s}\in X_1$.

Proof. Let $\mathbf{t} := (\boldsymbol{\theta},\xi)$ and $\mathbf{s} := (\boldsymbol{\zeta},\lambda)\in X_1$. Then, we have

$$[\mathbf{A}_1(\mathbf{t}) - \mathbf{A}_1(\mathbf{s}),\mathbf{t}-\mathbf{s}] = \int_{\Omega_1} [a(\cdot,\boldsymbol{\theta}) - a(\cdot,\boldsymbol{\zeta})]\cdot[\boldsymbol{\theta}-\boldsymbol{\zeta}]\,dx$$
$$+||\boldsymbol{\theta}-\boldsymbol{\zeta}||^2_{[L^2(\Omega_2)]^2} + 2\,\langle\xi-\lambda,\mathbf{W}(\xi-\lambda)\rangle\,,$$

which, using the coerciveness property (2.4), yields

$$\begin{aligned}[\mathbf{A}_1(\mathbf{t}) - \mathbf{A}_1(\mathbf{s}),\mathbf{t}-\mathbf{s}] &\geq \int_{\Omega_1} [a(\cdot,\boldsymbol{\theta}) - a(\cdot,\boldsymbol{\zeta})]\cdot[\boldsymbol{\theta}-\boldsymbol{\zeta}]\,dx \\ &\quad +||\boldsymbol{\theta}-\boldsymbol{\zeta}||^2_{[L^2(\Omega_2)]^2} + 2\,C_0\,||\xi-\lambda||^2_{H^{1/2}(\Gamma)}\,.\end{aligned} \tag{5.4}$$

Now, proceeding as in Section 5 of [17], we find that

$$a_i(x,\boldsymbol{\theta}(x)) - a_i(x,\boldsymbol{\zeta}(x)) = \int_0^1 \sum_{j=1}^2 \frac{\partial}{\partial\hat{\theta}_j}\,a_i(x,\hat{\boldsymbol{\theta}}(x,t))\,[\theta_j(x)-\zeta_j(x)]\,dt\,, \tag{5.5}$$

where $\hat{\boldsymbol{\theta}}(x,t) := \boldsymbol{\zeta}(x) + t[\boldsymbol{\theta}(x)-\boldsymbol{\zeta}(x)]$, $\boldsymbol{\theta} = (\theta_1,\theta_2)$ and $\boldsymbol{\zeta} = (\zeta_1,\zeta_2)$.

Thus, applying (5.5) and (A.3), we deduce that

$$\begin{aligned}\int_{\Omega_1} [a(\cdot,\boldsymbol{\theta}) - a(\cdot,\boldsymbol{\zeta})]\cdot[\boldsymbol{\theta}-\boldsymbol{\zeta}]\,dx &= \sum_{i=1}^2 \int_{\Omega_1} [a_i(\cdot,\boldsymbol{\theta}) - a_i(\cdot,\boldsymbol{\zeta})]\,[\theta_i-\zeta_i]\,dx \\ &= \sum_{i=1}^2 \int_{\Omega_1}\int_0^1 \sum_{j=1}^2 \frac{\partial}{\partial\hat{\theta}_j}\,a_i(\cdot,\hat{\boldsymbol{\theta}}(\cdot,t))\,[\theta_j-\zeta_j]\,[\theta_i-\zeta_i]\,dt\,dx \\ &\geq\; C\sum_{i=1}^2 \int_{\Omega_1} [\theta_i-\zeta_i]^2\,dx \;=\; C\,||\boldsymbol{\theta}-\boldsymbol{\zeta}||^2_{[L^2(\Omega_1)]^2}\,.\end{aligned} \tag{5.6}$$

Therefore, replacing (5.6) back into (5.4) we obtain (5.2).

On the other hand, the proof of (5.3), which proceeds similarly to Section 6 of [17], again uses the relation (5.5) and applies now the assumption (A.4) and the continuity property of the boundary integral operator $\mathbf{W}$. Hence, we omit further details. □

Corollary 5.4. *Let* $V_1 := \{ \mathbf{s} \in \tilde{X}_1 : [\mathbf{B}_1(\mathbf{s}), \boldsymbol{\tau}] = 0 \ \forall \boldsymbol{\tau} \in \tilde{M}_1 \}$ *and let* $\Pi_1 : X_1' \to V_1'$ *be the canonical imbedding. Then, for any* $\tilde{\mathbf{t}} \in X_1$, *the nonlinear operator* $\Pi_1 \mathbf{A}_1(\cdot + \tilde{\mathbf{t}}) : V_1 \to V_1'$ *is strongly monotone.*

Proof. It follows straigthforwardly from the previous lemma and the fact that $V_1 \subseteq \tilde{X}_1 = X_1$. Indeed, given $\tilde{\mathbf{t}} \in X_1$, $\mathbf{t}, \mathbf{s} \in V_1$, we have

$$[\Pi_1 \mathbf{A}_1(\mathbf{t} + \tilde{\mathbf{t}}) - \Pi_1 \mathbf{A}_1(\mathbf{s} + \tilde{\mathbf{t}}), \mathbf{t} - \mathbf{s}]$$

$$= [\mathbf{A}_1(\mathbf{t} + \tilde{\mathbf{t}}) - \mathbf{A}_1(\mathbf{s} + \tilde{\mathbf{t}}), (\mathbf{t} + \tilde{\mathbf{t}}) - (\mathbf{s} + \tilde{\mathbf{t}})] \ \geq \ \alpha \, ||\mathbf{t} - \mathbf{s}||_{X_1}^2 ,$$

which ends the proof. □

We are now in position to provide our main result concerning the solvability of the continuous problem (3.12).

Theorem 5.5. *There exists a unique* $((\mathbf{t}, \boldsymbol{\sigma}), u) \in X \times M$ *solution of the dual-dual mixed formulation* (3.12).

Proof. By virtue of the previous results of this section, the proof follows from a direct application of the abstract Theorem 4.1. □

5.2. **A discrete Galerkin scheme.** Now, we introduce specific finite element subspaces, define the associated Galerkin scheme, and prove that the hypotheses of both Theorem 4.2 and Theorem 4.3 are satisfied.

First, given $N \in \mathbf{N}$, we let $0 = t_0 < t_1 < \cdots < t_N = 2\pi$ be a uniform partition of $[0, 2\pi]$ with $t_{j+1} - t_j = \tilde{h} = \frac{2\pi}{N}$ for $j \in \{0, 1, ..., N-1\}$. Also, let $\mathbf{z} : [0, 2\pi] \to \Gamma$ be the usual parametrization of the circle Γ given by $\mathbf{z}(t) := r\,(\cos(t), \sin(t))^T$ for all $t \in [0, 2\pi]$. We denote by $\Omega_{\tilde{h}}$ the annular domain bounded by Γ_0 and the polygonal line $\Gamma_{\tilde{h}}$ whose vertices are $\{\mathbf{z}(t_1), \mathbf{z}(t_2), ..., \mathbf{z}(t_N)\}$.

Let $\mathcal{T}_{\tilde{h}}$ be a regular triangulation of $\Omega_{\tilde{h}}$ by triangles T of diameter h_T such that $h := \sup_{T \in \mathcal{T}_{\tilde{h}}} h_T$. For simplicity, we assume that for each $T \in \mathcal{T}_{\tilde{h}}$, either $T \subseteq \overline{\Omega}_1$ or $T \subseteq \overline{\Omega}_2$. Then, we replace each triangle $T \in \mathcal{T}_{\tilde{h}}$ with one side along $\Gamma_{\tilde{h}}$, by the corresponding curved triangle with one side along Γ. In this way, we obtain from $\mathcal{T}_{\tilde{h}}$ a triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ made up of straight and curved triangles.

Next, we consider the canonical triangle with vertices $\hat{P}_1 = (0,0)^T$, $\hat{P}_2 = (1,0)^T$ and $\hat{P}_3 = (0,1)^T$ as a reference triangle $\hat{T}$, and introduce a family of bijective mappings $\{F_T\}_{T \in \mathcal{T}_h}$, such that $F_T(\hat{T}) = T$. In particular, if T is a straight triangle of $\mathcal{T}_h$, then F_T is the well known invertible affine mapping defined by $F_T(\hat{x}) = B_T\,\hat{x} + b_T$, where B_T, a square matrix of order 2, and $b_T \in \mathbf{R}^2$ depend on the vertices of T.

Now, if T is a curved triangle with vertices P_1, P_2 and P_3, such that $P_2 = \mathbf{z}(t_{j-1}) \in \Gamma$ and $P_3 = \mathbf{z}(t_j) \in \Gamma$, then $F_T(\hat{x}) = B_T\,\hat{x} + b_T + G_T(\hat{x})$ for all $\hat{x} := (\hat{x}_1, \hat{x}_2) \in \hat{T}$, where

$$G_T(\hat{x}) = \frac{\hat{x}_1}{1 - \hat{x}_2} \left\{ \mathbf{z}(t_{j-1} + \hat{x}_2(t_j - t_{j-1})) - [\mathbf{z}(t_{j-1}) + \hat{x}_2\,(\mathbf{z}(t_j) - \mathbf{z}(t_{j-1}))] \right\}.$$

It can be proved (see, e.g., Theorem 22.4 in [35]) that F_T is a diffeomorphism of class C^∞ that maps one-to-one $\hat{T}$ onto the curved triangle T in such a way that $F_T(\hat{P}_i) = P_i$ for $i \in \{1,2,3\}$. Also, the image of edge $\hat{P}_2\hat{P}_3$ is the curved side of T and, since $G_T(\hat{x}) = (0,0)^T$ for $\hat{x}_1 = 0$ and for $\hat{x}_2 = 0$, the two other edges of $\hat{T}$ are transformed linearly under F_T to the straight sides of T.

We now consider the lowest order Raviart-Thomas spaces. For this purpose, we first let

$$\mathcal{R}T_0(\hat{T}) := \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} \right\},$$

and for each triangle $T \in \mathcal{T}_h$, we put

$$\mathcal{R}T_0(T) := \{ \boldsymbol{\tau} : \quad \boldsymbol{\tau} = J(F_T)^{-1}\,(DF_T)\,\hat{\boldsymbol{\tau}} \circ F_T^{-1}\,,\ \hat{\boldsymbol{\tau}} \in \mathcal{R}T_0(\hat{T}) \}\,,$$

where $J(F_T)$ and $D(F_T)$ denote, respectively, the jacobian and the Frêchet differential of the mapping F_T.

Then, we define the finite element subspaces for the unknowns $\boldsymbol{\theta}$ and $\boldsymbol{\sigma}$, as follows:

$$X_{1,h}^{\boldsymbol{\theta}} := \{ \boldsymbol{\tau}_h \in [L^2(\Omega)]^2 : \quad \boldsymbol{\tau}_h|_T \in \mathcal{R}T_0(T) \quad \forall T \in \mathcal{T}_h \} \tag{5.7}$$

and

$$M_{1,h} := X_{1,h}^{\boldsymbol{\theta}} \cap H_0(\text{div};\Omega)\,. \tag{5.8}$$

Note that $X_{1,h}^{\boldsymbol{\theta}}$ does not require continuity of the normal components through the sides of each triangle T, while $M_{1,h}$ certainly does.

Next, we set

$$H_h(0,2\pi) := \Big\{ \tilde{\lambda}_h : [0,2\pi] \to \mathbf{R}\,, \quad \tilde{\lambda}_h \text{ is continuous and periodic of period } 2\pi$$
$$\tilde{\lambda}_h|_{[t_{j-1},t_j]} \in \mathbf{P}_1(t_{j-1},t_j) \quad \forall j \in \{1,...,N\}\,, \quad \int_0^{2\pi} \tilde{\lambda}_h(t)\,dt = 0 \Big\}$$

and define the finite element subspace for the unknown ξ:

$$X_{1,h}^{\xi} := \{ \lambda_h : \Gamma \to \mathbf{R}, \quad \lambda_h = \tilde{\lambda}_h \circ \mathbf{z}^{-1},\ \tilde{\lambda}_h \in H_h(0,2\pi) \}\,. \tag{5.9}$$

Hereafter, given a non-negative integer k and a subset S of $\mathbf{R}$ or $\mathbf{R}^2$, $\mathbf{P}_k(S)$ denotes the space of polynomials defined on S of degree $\leq k$. At this point we remark that the simplicity of the definition of $X_{1,h}^{\xi}$ is due to the fact that Γ is a circle, which yields a constant jacobian of the transformation $\mathbf{z}$. If this were not the case, then one should proceed differently (see, e.g., [30]).

Note that $H_h(0,2\pi) \subseteq H^{1/2}[0,2\pi]$, where for each $p > 0$, $H^p[0,2\pi]$ denotes the usual Sobolev space of 2π-periodic functions (see, e.g., Section 8.2 of [29]). Hence, according to the regularity of $\mathbf{z}$ and the definition of the Sobolev spaces on the boundary Γ (see, e.g., Section 8.3 in [29]), we deduce that $X_{1,h}^{\xi} \subseteq H_0^{1/2}(\Gamma)$. Finally, we put

$$X_{1,h} := X_{1,h}^{\boldsymbol{\theta}} \times X_{1,h}^{\xi}\,, \tag{5.10}$$

$$X_h := X_{1,h} \times M_{1,h}\,, \tag{5.11}$$

and consider the piecewise constant functions as the finite element subspace for the unknown u, that is

$$M_h := \{ v_h \in L^2(\Omega) : \quad v_h|_T \in \mathbf{P}_0(T) \quad \forall T \in \mathcal{T}_h \} . \tag{5.12}$$

In this way, the Galerkin scheme associated with the continuous problem (3.12) reads as follows: *Find* $((\mathbf{t}_h, \boldsymbol{\sigma}_h), u_h) \in X_h \times M_h$ *such that*

$$\begin{aligned} [\mathbf{A}(\mathbf{t}_h, \boldsymbol{\sigma}_h), (\mathbf{s}_h, \boldsymbol{\tau}_h)] + [\mathbf{B}^*(u_h), (\mathbf{s}_h, \boldsymbol{\tau}_h)] &= 0 , \\ [\mathbf{B}(\mathbf{t}_h, \boldsymbol{\sigma}_h), v_h] &= [\mathbf{G}, v_h] , \end{aligned} \tag{5.13}$$

for all $((\mathbf{s}_h, \boldsymbol{\tau}_h), v_h) \in X_h \times M_h$.

In what follows, we verify that the introduced finite element subspaces above satisfy the corresponding discrete inf-sup conditions.

Lemma 5.6. *There exists* $\beta^* > 0$*, independent of the subspaces involved, such that for all* $v_h \in M_h$

$$\sup_{\substack{(\mathbf{s}_h, \boldsymbol{\tau}_h) \in X_h \\ (\mathbf{s}_h, \boldsymbol{\tau}_h) \neq 0}} \frac{[\mathbf{B}(\mathbf{s}_h, \boldsymbol{\tau}_h), v_h]}{||(\mathbf{s}_h, \boldsymbol{\tau}_h)||_X} \geq \beta^* \, ||v_h||_M .$$

Proof. We proceed similarly as in Lemma 4.3 of [31]. First, we observe that

$$\sup_{\substack{(\mathbf{s}_h, \boldsymbol{\tau}_h) \in X_h \\ (\mathbf{s}_h, \boldsymbol{\tau}_h) \neq 0}} \frac{[\mathbf{B}(\mathbf{s}_h, \boldsymbol{\tau}_h), v_h]}{||(\mathbf{s}_h, \boldsymbol{\tau}_h)||_X} \geq \sup_{\substack{\boldsymbol{\tau}_h \in M_{1,h} \\ \boldsymbol{\tau}_h \neq 0}} \frac{-\int_\Omega v_h \operatorname{div} \boldsymbol{\tau}_h \, dx}{||\boldsymbol{\tau}_h||_{H(\operatorname{div};\Omega)}} . \tag{5.14}$$

Then, we introduce the equilibrium interpolation operator (cf. [2], [33]) $E_h : [H^1(\Omega)]^2 \to X_{1,h}^{\boldsymbol{\theta}} \cap H(\operatorname{div};\Omega)$, which is characterized on each $T \in \mathcal{T}_h$ by

$$\int_e (E_h \boldsymbol{\tau}) \cdot \nu \, ds = \int_e \boldsymbol{\tau} \cdot \nu \, ds \quad \text{for all edges } e \text{ of } T . \tag{5.15}$$

It is well known that E_h satisfies the commuting diagram property

$$\operatorname{div} (E_h \boldsymbol{\tau}) = \mathcal{P}_h (\operatorname{div} \boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in [H^1(\Omega)]^2 , \tag{5.16}$$

where $\mathcal{P}_h$ is the orthogonal projection from $L^2(\Omega)$ onto the subspace M_h. In other words,

$$\int_\Omega \operatorname{div} (E_h \boldsymbol{\tau}) \, v_h \, dx = \int_\Omega v_h \operatorname{div} \boldsymbol{\tau} \, dx \quad \forall \, v_h \in M_h , \tag{5.17}$$

and also, from (5.15) we get

$$\int_\Gamma (E_h \boldsymbol{\tau}) \cdot \nu \, ds = \int_\Gamma \boldsymbol{\tau} \cdot \nu \, ds \quad \forall \, \boldsymbol{\tau} \in [H^1(\Omega)]^2 .$$

It follows that $(E_h \boldsymbol{\tau}) \in X_{1,h}^{\boldsymbol{\theta}} \cap H_0(\operatorname{div};\Omega)$ for all $\boldsymbol{\tau} \in [H^1(\Omega)]^2 \cap H_0(\operatorname{div};\Omega)$. Moreover, using (5.16) and the approximation properties of E_h (cf. [2] or [33]), we conclude that the family of operators $E_h : [H^1(\Omega)]^2 \cap H_0(\operatorname{div};\Omega) \to M_{1,h}$ is uniformly bounded. This means that there exists $\tilde{C} > 0$, independent of $h \in \mathcal{I}$, such that

$$||E_h \, \boldsymbol{\tau}||_{H(\operatorname{div};\Omega)} \leq \tilde{C} \, ||\boldsymbol{\tau}||_{[H^1(\Omega)]^2} \quad \forall \, \boldsymbol{\tau} \in [H^1(\Omega)]^2 \cap H_0(\operatorname{div};\Omega) . \tag{5.18}$$

Now, going back to (5.14), using (5.17), (5.18) and the stronger continuous inf-sup condition (5.1), we can write

$$\sup_{\substack{\boldsymbol{\tau}_h \in M_{1,h} \\ \boldsymbol{\tau}_h \neq 0}} \frac{-\int_\Omega v_h \operatorname{div} \boldsymbol{\tau}_h \, dx}{||\boldsymbol{\tau}_h||_{H(\operatorname{div};\Omega)}} \geq \sup_{\substack{\boldsymbol{\tau} \in [H^1(\Omega)]^2 \cap H_0(\operatorname{div};\Omega) \\ \boldsymbol{\tau} \neq 0}} \frac{\int_\Omega v_h \operatorname{div} (E_h \boldsymbol{\tau}) \, dx}{||E_h \boldsymbol{\tau}||_{H(\operatorname{div};\Omega)}}$$

$$\geq \frac{1}{\tilde{\tilde{C}}} \sup_{\substack{\boldsymbol{\tau} \in [H^1(\Omega)]^2 \cap H_0(\operatorname{div};\Omega) \\ \boldsymbol{\tau} \neq 0}} \frac{\int_\Omega v_h \operatorname{div} \boldsymbol{\tau} \, dx}{||\boldsymbol{\tau}||_{[H^1(\Omega)]^2}} \geq \beta^* \, ||v_h||_M \,,$$

which finishes the proof. □

In order to continue our analysis, we need to characterize the discrete kernel V_h.

Lemma 5.7. *We have* $V_h = \tilde{X}_{1,h} \times \tilde{M}_{1,h}$, *where* $\tilde{X}_{1,h} = X_{1,h}$ *and*

$$\tilde{M}_{1,h} := \{ \boldsymbol{\tau}_h \in M_{1,h} : \quad \operatorname{div} \boldsymbol{\tau}_h = 0 \quad in \quad \Omega \} \,.$$

Proof. According to the definition of $\mathbf{B}$, we have

$$V_h = \{ (\mathbf{s}_h, \boldsymbol{\tau}_h) \in X_h : \quad \int_\Omega v_h \operatorname{div} \boldsymbol{\tau}_h \, dx = 0 \; \forall \, v_h \in M_h \} \,,$$

and then $V_h = \tilde{X}_{1,h} \times \tilde{M}_{1,h}$, with $\tilde{X}_{1,h} = X_{1,h}$ and

$$\tilde{M}_{1,h} = \{ \boldsymbol{\tau}_h \in M_{1,h} : \quad \int_\Omega v_h \operatorname{div} \boldsymbol{\tau}_h \, dx = 0 \; \forall \, v_h \in M_h \} \,.$$

Now, given $\boldsymbol{\tau}_h \in \tilde{M}_{1,h}$ and $T \in \mathcal{T}_h$, we may choose $v_h \in M_h$ such that $v_h|_T \equiv 1$ and $v_h \equiv 0$ in $\Omega - T$. It follows that $\int_T \operatorname{div} \boldsymbol{\tau}_h \, dx = 0$ for all $T \in \mathcal{T}_h$. But, using Lemma 1.5 (identity (1.49)) in Chapter III of [2], we can write

$$0 = \int_T \operatorname{div} \boldsymbol{\tau}_h \, dx = \int_{\hat{T}} \operatorname{div} \hat{\boldsymbol{\tau}}_{h,T} \, d\hat{x} \,, \tag{5.19}$$

where $\hat{\boldsymbol{\tau}}_{h,T} \in \mathcal{R}T_0(\hat{T})$ is such that $\boldsymbol{\tau}_h|_T = J(F_T)^{-1} \, (DF_T) \, \hat{\boldsymbol{\tau}}_{h,T} \circ F_T^{-1}$.

Since $\operatorname{div} \hat{\boldsymbol{\tau}}_{h,T}$ is constant in $\hat{T}$, we deduce from (5.19) that $\operatorname{div} \hat{\boldsymbol{\tau}}_{h,T} = 0$ for all $T \in \mathcal{T}_h$, and therefore, applying the identity (1.47) in Chapter III of [2], we conclude that

$$\operatorname{div} (\boldsymbol{\tau}_h|_T) \; = \; J(F_T)^{-1} \operatorname{div} \hat{\boldsymbol{\tau}}_{h,T} \; = \; 0 \quad \forall \, T \in \mathcal{T}_h \,.$$

This completes the proof. □

We now prove the discrete inf-sup condition for $\mathbf{B}_1$.

Lemma 5.8. *There exists* $\beta_1^* > 0$, *independent of the subspaces involved, such that for all* $\boldsymbol{\tau}_h \in \tilde{M}_{1,h}$

$$\sup_{\substack{\mathbf{s}_h \in \tilde{X}_{1,h} \\ \mathbf{s}_h \neq 0}} \frac{[\mathbf{B}_1(\mathbf{s}_h), \boldsymbol{\tau}_h]}{||\mathbf{s}_h||_{X_1}} \; \geq \; \beta_1^* \, ||\boldsymbol{\tau}_h||_{M_1} \,.$$

Proof. Since $\tilde{X}_{1,h} = X_{1,h} = X^{\boldsymbol{\theta}}_{1,h} \times X^{\xi}_{1,h}$, we have for all $\boldsymbol{\tau}_h \in \tilde{M}_{1,h}$

$$\sup_{\substack{\mathbf{s}_h \in \tilde{X}_{1,h} \\ \mathbf{s}_h \neq 0}} \frac{[\mathbf{B}_1(\mathbf{s}_h), \boldsymbol{\tau}_h]}{||\mathbf{s}_h||_{X_1}} \geq \sup_{\substack{\boldsymbol{\zeta}_h \in X^{\boldsymbol{\theta}}_{1,h} \\ \boldsymbol{\zeta}_h \neq 0}} \frac{-\int_\Omega \boldsymbol{\tau}_h \cdot \boldsymbol{\zeta}_h \, dx}{||\boldsymbol{\zeta}_h||_{[L^2(\Omega)]^2}} .$$

Then, using that $\tilde{M}_{1,h} \subseteq M_{1,h} \subseteq X^{\boldsymbol{\theta}}_{1,h}$, we deduce that

$$\sup_{\substack{\boldsymbol{\zeta}_h \in X^{\boldsymbol{\theta}}_{1,h} \\ \boldsymbol{\zeta}_h \neq 0}} \frac{-\int_\Omega \boldsymbol{\tau}_h \cdot \boldsymbol{\zeta}_h \, dx}{||\boldsymbol{\zeta}_h||_{[L^2(\Omega)]^2}} = ||\boldsymbol{\tau}_h||_{[L^2(\Omega)]^2} = ||\boldsymbol{\tau}_h||_{H(\mathrm{div};\Omega)} ,$$

where the last equality follows from the characterization of $\tilde{M}_{1,h}$ given in Lemma 5.7. This ends the proof. □

The unique solvability of the Galerkin scheme (5.13) and the corresponding error estimate can be established now.

Theorem 5.9. *There exists a unique* $((\mathbf{t}_h, \boldsymbol{\sigma}_h), u_h) \in X_h \times M_h$ *solution of the Galerkin scheme* (5.13). *In addition, there exists* $C > 0$, *independent of* h, *such that the following Cea estimate holds*

$$||((\mathbf{t}, \boldsymbol{\sigma}), u) - ((\mathbf{t}_h, \boldsymbol{\sigma}_h), u_h)||$$

$$\leq C \inf_{((\mathbf{s}_h, \boldsymbol{\tau}_h), v_h) \in X_h \times M_h} ||((\mathbf{t}, \boldsymbol{\sigma}), u) - ((\mathbf{s}_h, \boldsymbol{\tau}_h), v_h)|| .$$

Proof. We first observe that Lemma 5.3 and Corollary 5.4 guarantee that the discrete operator $\mathbf{A}_{1,h} : X_1 \to X'_{1,h}$ is also Lipschitz-continuous, and that the family of operators $\{ \Pi_{1,h} \mathbf{A}_{1,h}(\cdot + \tilde{\mathbf{t}}) : \tilde{\mathbf{t}} \in X_{1,h} , h \in \mathcal{I} \}$ is uniformly strongly monotone. Here, $\mathbf{A}_{1,h}$ and $\Pi_{1,h}$ are defined as in Theorem 4.2. In addition, from Lemma 5.7 and the definition of V given in subsection 5.1, we deduce that $V_h \subseteq V$.

Therefore, by virtue of Lemmas 5.6 and 5.8, a direct application of the abstract Theorems 4.2 and 4.3 finishes the proof. □

As a consequence of the Cea estimate given by the previous theorem, we deduce the following error bound.

Theorem 5.10. *Let* $((\mathbf{t}, \boldsymbol{\sigma}), u)$ *and* $((\mathbf{t}_h, \boldsymbol{\sigma}_h), u_h)$ *be the unique solutions of* (3.12) *and* (5.13), *respectively, with* $\mathbf{t} := (\boldsymbol{\theta}, \xi)$ *and* $\mathbf{t}_h := (\boldsymbol{\theta}_h, \xi_h)$. *In addition, assume that* $\boldsymbol{\theta}|_T \in [H^1(T)]^2 \; \forall T \in \mathcal{T}_h$, $\xi \in H^{3/2}(\Gamma)$, $\boldsymbol{\sigma} \in [H^1(\Omega)]^2$, $\mathrm{div}\, \boldsymbol{\sigma} \in H^1(\Omega)$ *and* $u \in H^1(\Omega)$. *Then, there exists* $\tilde{C} > 0$, *independent of* h, *such that the following estimate holds*

$$||(\boldsymbol{\theta}, \xi, \boldsymbol{\sigma}, u) - (\boldsymbol{\theta}_h, \xi_h, \boldsymbol{\sigma}_h, u_h)||$$
$$\leq \tilde{C}\, h \left\{ \sum_{T \in \mathcal{T}_h} ||\boldsymbol{\theta}||^2_{[H^1(T)]^2} + ||\xi||^2_{H^{3/2}(\Gamma)} \right.$$
$$\left. + ||\boldsymbol{\sigma}||^2_{[H^1(\Omega)]^2} + ||\mathrm{div}\, \boldsymbol{\sigma}||^2_{H^1(\Omega)} + ||u||^2_{H^1(\Omega)} \right\}^{1/2}$$

Proof. Let us denote by $E_{h,T}$ the restriction of the equilibrium operator E_h to a given triangle T of $\mathcal{T}_h$. We also introduce the Lagrange interpolation operator $\mathcal{L}_h$ from $H^{3/2}[0,2\pi]$ onto $H_h(0,2\pi)$. Further, as before, let $\mathcal{P}_h$ be the orthogonal projection from $L^2(\Omega)$ onto M_h.

Then from Theorem 5.9 and using again the commuting diagram property (5.16), we deduce that

$$\begin{aligned}
&||(\boldsymbol{\theta},\xi,\boldsymbol{\sigma},u)-(\boldsymbol{\theta}_h,\xi_h,\boldsymbol{\sigma}_h,u_h)||^2\\
&\quad\leq C\left\{\sum_{T\in\mathcal{T}_h}||\boldsymbol{\theta}-E_{h,T}\,\boldsymbol{\theta}||^2_{[L^2(T)]^2}+||\,[\,(\xi\circ\mathbf{z})-\mathcal{L}_h(\xi\circ\mathbf{z})\,]\circ\mathbf{z}^{-1}||^2_{H^{1/2}(\Gamma)}\right.\\
&\qquad\left.+\,||\boldsymbol{\sigma}-E_h\,\boldsymbol{\sigma}||^2_{[L^2(\Omega)]^2}+||\mathrm{div}\,\boldsymbol{\sigma}-\mathcal{P}_h\,(\mathrm{div}\,\boldsymbol{\sigma})||^2_{L^2(\Omega)}+||u-\mathcal{P}_hu||^2_{L^2(\Omega)}\right\}.
\end{aligned}$$

Thus, the result follows from classical error estimates for interpolation and projection operators in the corresponding Sobolev spaces. In particular, for the second term, and using again the definition of the Sobolev spaces on Γ through the parametrization $\mathbf{z}$ (see Section 8.3 in [29]), we obtain

$$\begin{aligned}
||\,[\,(\xi\circ\mathbf{z})-\mathcal{L}_h(\xi\circ\mathbf{z})\,]\circ\mathbf{z}^{-1}||_{H^{1/2}(\Gamma)}&\leq C\,||(\xi\circ\mathbf{z})-\mathcal{L}_h(\xi\circ\mathbf{z})||_{H^{1/2}[0,2\pi]}\\
&\leq C\,h\,||\xi\circ\mathbf{z}||_{H^{3/2}[0,2\pi]}\leq C\,h\,||\xi||_{H^{3/2}(\Gamma)}\,.
\end{aligned}$$

Since the other estimates are straightforward, we omit further details. □

We end this paper by remarking that efficient numerical algorithms for solving discrete schemes of dual-dual structure, which are based on minimum residual and conjugate gradient methods, are provided in [13], [14] and [15].

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