

JACOBI SUMS AND NEW FAMILIES OF IRREDUCIBLE POLYNOMIALS OF GAUSSIAN PERIODS

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ABSTRACT. Let $m > 2$, ζ_m an m -th primitive root of 1, $q \equiv 1 \pmod{2m}$ a prime number, $s = s_q$ a primitive root modulo q and $f = f_q = (q - 1)/m$. We study the Jacobi sums $J_{a,b} = -\sum_{k=2}^{q-1} \zeta_m^{a \operatorname{ind}_s(k) + b \operatorname{ind}_s(1-k)}$, $0 \leq a, b \leq m - 1$, where $\operatorname{ind}_s(k)$ is the least nonnegative integer such that $s^{\operatorname{ind}_s(k)} \equiv k \pmod{q}$. We exhibit a set of properties that characterize these sums, some congruences they satisfy, and a MAPLE program to calculate them. Then we use those results to show how one can construct families $P_q(x)$, $q \in \mathcal{P}$, of irreducible polynomials of Gaussian periods, $\eta_i = \sum_{j=0}^{f-1} \zeta_q^{s^{i+mj}}$, of degree m , where \mathcal{P} is a suitable set of primes $\equiv 1 \pmod{2m}$. We exhibit examples of such families for several small values of m , and give a MAPLE program to construct more of them.

INTRODUCTION

Let $m > 2$ be an integer and ζ_m an m -th primitive root of 1. For each prime $q \equiv 1 \pmod{2m}$ let ζ_q be a q -th primitive root of 1, $s = s_q$ a primitive root modulo q and $f = f_q = (q - 1)/m$ (we will assume that f is even for simplicity). Let S be the set of all primes $q \equiv 1 \pmod{2m}$. Given $q \in S$, define the Jacobi sums $J_{a,b}$, $0 \leq a, b \leq m - 1$, and the Gaussian periods η_i , $0 \leq i \leq m - 1$, of degree m in $\mathbb{Q}(\zeta_q)$, by

$$J_{a,b} = -\sum_{k=2}^{q-1} \zeta_m^{a \operatorname{ind}_s(k) + b \operatorname{ind}_s(1-k)},$$

where $\operatorname{ind}_s(k)$ is the least nonnegative integer such that $s^{\operatorname{ind}_s(k)} \equiv k \pmod{q}$, and

$$\eta_i = \sum_{j=0}^{f-1} \zeta_q^{s^{i+mj}}.$$

Define $P_q(x) = \prod_{i=0}^{m-1} (x - \eta_i)$, the irreducible polynomial, over \mathbb{Q} , of the periods η_i . In this article we study the numbers $J_{a,b}$, and use them to construct large families of polynomials $P_q(x)$, $q \in \mathcal{P}$, where \mathcal{P} is a subset of S . In principle the method shown here would allow us to construct a finite number of such families, whose indices put together include all the primes in S .

This research originated from a problem indicated to me by René Schoof. The first part of the problem was to find, for $m = 7$, or $m = 9$, or $m = 12$, families of

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irreducible polynomials of real Gaussian periods of degree m . The second part was to find families of irreducible polynomials of units of the number fields generated by those periods. I think we give here a complete answer to the first part (for arbitrary m). The second part seems to be an open problem, and a very interesting one in light of Schoof and Washington's work in [7].

For an account of previous work in this and related subjects see [1], [6] and [7]. The path that leads directly to this article is the following. For $m = 5$, H.W. Lloyd Tanner obtained, in [9], an expression for the family of polynomials $P_q(x)$, $q \in S$, in terms of coefficients of certain divisors of q in $\mathbb{Q}(\zeta_5)$. This result was used by Emma Lehmer, in [5], who gave a new expression for that family. In [6] Lehmer exhibited a family of polynomials of degree 5, which is obtained by a translation of a family of polynomials $P_q(x)$, and such that the roots of the polynomials in the family are units. This result has been used by Schoof and Washington in [7] to find some real cyclotomic fields with large class numbers. In [12], Section 1, we work with $m = p$, an odd prime, and show how to construct certain families of irreducible polynomials of Gaussian periods of degree p . In that article we were able to obtain, for general p , only some of the families our present method allows us to construct. We could give all the families only when $\mathbb{Z}[\zeta_p]$ was a principal ideal domain. In this article we work with general $m > 2$ and find all the families, thereby extending, in more than one way, the results of [12].

In Section 1 we use the well-known relations between Jacobi sums, Gauss sums, Gaussian periods and cyclotomic numbers to obtain a set of properties that characterize the numbers $J_{a,b}$ (Propositions 2 and 3). We write these numbers in the form

$$J_{a,b} = \sum_{k=0}^{m-1} d_{a,b,k} \zeta_m^k, \quad \text{with } d_{a,b,k} \in \mathbb{Z},$$

in such a way that we can give natural formulas for the coefficients $d_{a,b,k}$ (Propositions 1 and 4). This allows us to calculate Jacobi sums efficiently. We prove some congruences that the numbers $d_{a,b,k}$ satisfy (formula (13)) which allow us to distinguish the Jacobi sums $J_{a,b}$ among the other generators of the ideals $(J_{a,b})$ (a useful result when we apply the method of Section 2 to find families of polynomials $P_q(x)$). This generalizes some results of [11], where we considered only the case $m = p$, an odd prime number. We end Section 1 with a MAPLE program to calculate the Jacobi sums $J_{a,b}$.

In Section 2 we show how to construct families of irreducible polynomials of Gaussian periods in a very general situation. Let \mathcal{R} be an ideal of $\mathbb{Z}[\zeta_m]$ relatively prime with m . Suppose that we can calculate (for example using the MAPLE program of Section 1) the Jacobi sums corresponding to the prime ideals dividing \mathcal{R} (see formula (18)). Then we show a way to construct a family $P_q(x)$, $q \in \mathcal{P}$, of irreducible polynomials of Gaussian periods of degree m , where the elements q of \mathcal{P} are such that $q \in S$ and one of the prime ideals Q of $\mathbb{Z}[\zeta_m]$ above q is in the inverse of the ideal class of \mathcal{R} . We give examples for $m = 7$, $m = 9$, $m = 12$ and (partially) $m = 23$; in them the sets \mathcal{P} of indices are chosen so that there are simple descriptions of the families of polynomials $P_q(x)$. Examples 1-4 correspond to the case $\mathcal{R} = (1)$ (for $m = 7$, $m = 7$, $m = 9$ and $m = 12$, respectively). Examples 5 and 6 illustrate the use of the method in a general situation. A MAPLE program to carry out the calculations for our examples, and to search for more examples, is given at the end of the section.

1. JACOBI SUMS IN $\mathbb{Q}(\zeta_m)$

Let $m > 2$ be an integer and $q = mf + 1$ a prime number. For simplicity we assume that f is even. Let s be a primitive root modulo q , ζ_q a q -th primitive root of 1, and $\eta_0, \dots, \eta_{m-1}$ the Gaussian periods of degree m in $\mathbb{Q}(\zeta_q)$ defined by

$$(1) \quad \eta_i = \sum_{j=0}^{f-1} \zeta_q^{s^{i+mj}}.$$

The set $\{\eta_0, \dots, \eta_{m-1}\}$ is a normal integral basis of $\mathbb{Q}(\eta_0)/\mathbb{Q}$. Let $c_{i,j}$, $0 \leq i, j \leq m-1$, be the rational integers such that

$$(2) \quad \eta_0 \eta_i = \sum_{j=0}^{m-1} c_{i,j} \eta_j.$$

Define $C = [c_{i,j}]_{0 \leq i, j \leq m-1}$. It follows from (2) that the characteristic polynomial of the matrix C is the irreducible polynomial $P_q(x)$ of the Gaussian periods η_i ; that is,

$$(3) \quad P_q(x) = \prod_{i=0}^{m-1} (x - \eta_i) = \det(xI - C),$$

where I is the $m \times m$ identity matrix (see [2], formula 9, or [10], formula 19).

For $0 \leq i, j \leq m-1$, we denote by (i, j) the cyclotomic numbers of order m . Recall that (i, j) is defined as the number of ordered pairs of integers $\langle k, l \rangle$, $0 \leq k, l \leq f-1$, such that $1 + s^{km+i} \equiv s^{lm+j} \pmod q$ (see, for example, [1], §2.2, [2], or [8]). Define $\eta_{i+km} = \eta_i$, $c_{i+km, j+lm} = c_{i,j}$, and $(i+km, j+lm) = (i, j)$, for $0 \leq i, j \leq m-1$ and $k, l \in \mathbb{Z}$.

We use the following version of Kronecker's delta:

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i \equiv j \pmod m, \\ 0 & \text{if } i \not\equiv j \pmod m. \end{cases}$$

The cyclotomic numbers (i, j) are very close to the numbers $c_{i,j}$; we have

$$(4) \quad c_{i,j} = (i, j) - f\delta_{0,i},$$

for $i, j \in \mathbb{Z}$ (see [2], formula 6).

Let $G(x) = \sum_{k=0}^{q-2} x^k \zeta_q^{s^k}$, where x is an indeterminate. We have that $G(x) \equiv \sum_{k=0}^{m-1} \eta_k x^k \pmod{x^m - 1}$ and that $G(1) = -1$. Let ζ_m be an m -th primitive root of 1. If $m \nmid k$, then $G(\zeta_m^k)$ is a Gauss sum which satisfies $G(\zeta_m^k)G(\zeta_m^{-k}) = q$ (recall that since f is even the Gaussian periods η_i are real numbers).

For $a, b \in \mathbb{Z}$, define the Jacobi sums $J_{a,b}$ by

$$(5) \quad J_{a,b} = - \sum_{k=2}^{q-1} \zeta_m^{a \operatorname{ind}_s(k) + b \operatorname{ind}_s(1-k)},$$

where $\operatorname{ind}_s(k)$ is the least nonnegative integer such that $s^{\operatorname{ind}_s(k)} \equiv k \pmod q$. It follows directly from the definition that, for all $a, b \in \mathbb{Z}$,

$$(6) \quad J_{a+m,b} = J_{a,b+m} = J_{a,b}, \quad J_{a,b} = J_{b,a}, \quad \text{and} \quad J_{a,b} = J_{-a-b,b}.$$

For example,

$$\begin{aligned}
 J_{-a-b,b} &= - \sum_{k=2}^{q-1} \zeta_m^{(-a-b) \operatorname{ind}_s(k)+b \operatorname{ind}_s(1-k)} \\
 &= - \sum_{k=2}^{q-1} \zeta_m^{-a \operatorname{ind}_s(k)+b \operatorname{ind}_s(k^{-1}-1)} \\
 &= - \sum_{k=2}^{q-1} \zeta_m^{a \operatorname{ind}_s(k)+b \operatorname{ind}_s(k-1)} \\
 &= - \sum_{k=2}^{q-1} \zeta_m^{a \operatorname{ind}_s(k)+b \operatorname{ind}_s(1-k)} = J_{a,b},
 \end{aligned}$$

since f is even.

Suppose that $0 \leq a, b \leq m - 1$. If $a + b \not\equiv 0 \pmod m$, then

$$(7) \quad J_{a,b} = - \frac{G(\zeta_m^a)G(\zeta_m^b)}{G(\zeta_m^{a+b})};$$

also

$$(8) \quad J_{0,0} = -(q - 2), \text{ and } J_{a,b} = 1 \text{ if } a + b \equiv 0 \pmod m \text{ but } a \neq 0$$

(see, for example, [13], Lemma 6.2, or [4], page 4).

We show now a way to represent Jacobi sums as linear combinations, over \mathbb{Z} , of powers of ζ_m , which is very convenient for our purposes. For a and b nonnegative integers let $f_{a,b}(x)$ be the polynomial

$$f_{a,b}(x) = - \sum_{k=2}^{q-1} x^{a \operatorname{ind}_s(k)+b \operatorname{ind}_s(1-k)} + \frac{x^{q-1} - 1}{x - 1}.$$

Define $J_{a,b}(x) = \sum_{j=0}^{m-1} d_{a,b,j} x^j \in \mathbb{Z}[x]$ as the remainder of the division of $f_{a,b}(x)$ by $x^m - 1$; that is,

$$(9) \quad J_{a,b}(x) = \sum_{j=0}^{m-1} d_{a,b,j} x^j \equiv f_{a,b}(x) \pmod{x^m - 1}.$$

Clearly, for $a, b \geq 0$, we have

$$(10) \quad J_{a,b} = J_{a,b}(\zeta_m) = \sum_{j=0}^{m-1} d_{a,b,j} \zeta_m^j,$$

$$(11) \quad J_{a,b}(1) = \sum_{j=0}^{m-1} d_{a,b,j} = 1,$$

and, for $k \geq 0$ such that $k \not\equiv 0 \pmod m$,

$$(12) \quad J_{a,b}(\zeta_m^k) = J_{ka,kb}(\zeta_m) = J_{ka,kb}.$$

We also have

$$(13) \quad J'_{a,b}(1) = \sum_{j=1}^{m-1} j d_{a,b,j} \equiv 0 \pmod m.$$

In fact, by (9),

$$J_{a,b}(x) = - \sum_{k=2}^{q-1} x^{a \operatorname{ind}_s(k) + b \operatorname{ind}_s(1-k)} + (x^{q-1} - 1)/(x - 1) + (x^m - 1)g(x),$$

for some $g(x) \in \mathbb{Z}[x]$. Taking derivatives, we get

$$J'_{a,b}(x) = - \sum_{k=2}^{q-1} (a \operatorname{ind}_s(k) + b \operatorname{ind}_s(1 - k))x^{a \operatorname{ind}_s(k) + b \operatorname{ind}_s(1-k)-1} + (1 + 2x + \dots + (q - 2)x^{q-3}) + (x^m - 1)g'(x) + mx^{m-1}g(x).$$

Therefore

$$J'_{a,b}(1) = -a \sum_{k=2}^{q-1} \operatorname{ind}_s(k) - b \sum_{k=2}^{q-1} \operatorname{ind}_s(1 - k) + m(f/2)(q - 2) + mg(1) \equiv 0 \pmod{m}.$$

The following result will be useful in calculating Jacobi sums. We denote by $\bar{\alpha}$ the complex conjugate of the number α . Observe that, if we denote the Jacobi sums in (5) by $J_{a,b,m}$ and $c = \operatorname{g.c.d.}(a, b, m)$, then $J_{a,b,m} = J_{a/c,b/c,m/c}$, with $\operatorname{g.c.d.}(a/c, b/c, m/c) = 1$ (assume $c < m$ and choose $\zeta_{m/c} = \zeta_m^c$).

Proposition 1. *Let a and b be integers, $1 \leq a, b \leq m-1$, such that $\operatorname{g.c.d.}(a, b, m) = 1$. Let $v = \operatorname{g.c.d.}(a + b, m)$ and $u = m/v$. For $l \in \mathbb{Z}$ let*

$$\varepsilon(l) = \begin{cases} 1 & \text{if } v|l, \\ 0 & \text{if } v \nmid l. \end{cases}$$

Then, for $0 \leq l \leq m - 1$, we have

$$d_{a,b,l} = \frac{1}{m} \left(1 + \sum_{k=1}^{m-1} \zeta_m^{kl} \bar{J}_{ka,kb} \right) = \frac{1}{u} \varepsilon(l) + \frac{1}{m} \sum_{i=1}^{u-1} \zeta_m^{il} \sum_{k=0}^{v-1} \zeta_m^{ukl} \bar{J}_{(i+uk)a, (i+uk)b}.$$

Proof. Let $d_l = d_{a,b,l}$. For $0 \leq l \leq m - 1$, we have

$$\sum_{k=0}^{m-1} \zeta_m^{-kl} J_{a,b}(\zeta_m^k) = \sum_{k=0}^{m-1} \zeta_m^{-kl} \sum_{j=0}^{m-1} d_j \zeta_m^{kj} = \sum_{j=0}^{m-1} d_j \sum_{k=0}^{m-1} \zeta_m^{(j-l)k} = md_l;$$

so

$$d_l = \frac{1}{m} \sum_{k=0}^{m-1} \zeta_m^{kl} \overline{J_{a,b}(\zeta_m^k)} = \frac{1}{m} \left(1 + \sum_{k=1}^{m-1} \zeta_m^{kl} \bar{J}_{ka,kb} \right),$$

by (11) and (12). Therefore

$$\begin{aligned} d_l &= \frac{1}{m} \left(1 + \sum_{\substack{1 \leq k \leq m-1 \\ u|k}} \zeta_m^{kl} \bar{J}_{ka,kb} \right) + \frac{1}{m} \left(\sum_{\substack{1 \leq k \leq m-1 \\ u \nmid k}} \zeta_m^{kl} \bar{J}_{ka,kb} \right) \\ &= \frac{1}{m} \left(1 + \sum_{k=1}^{v-1} \zeta_m^{ukl} \right) + \frac{1}{m} \left(\sum_{i=1}^{u-1} \sum_{k=0}^{v-1} \zeta_m^{(i+uk)l} \bar{J}_{(i+uk)a, (i+uk)b} \right) \\ &= \frac{1}{u} \varepsilon(l) + \frac{1}{m} \left(\sum_{i=1}^{u-1} \zeta_m^{il} \sum_{k=0}^{v-1} \zeta_m^{ukl} \bar{J}_{(i+uk)a, (i+uk)b} \right), \end{aligned}$$

by (8), as we wanted to prove. □

We can express the Jacobi sums $J_{a,b}$ in terms of the cyclotomic numbers (i, j) , and vice versa, as follows:

For $a, b \in \mathbb{Z}$,

$$(14) \quad J_{a,b} = - \sum_{h=0}^{m-1} \sum_{k=0}^{m-1} \zeta_m^{ah+bk}(h, k).$$

In fact, for example, by [2], formula 26 (for the case where $m \nmid a$, $m \nmid b$ and $m \nmid (a + b)$), and a straightforward calculation using [2], formulas 14 and 17 (when $m \mid a$ or $m \mid b$ or $m \mid (a + b)$), we have

$$J_{a,b} = - \sum_{h=0}^{m-1} \sum_{k=0}^{m-1} \zeta_m^{bk-(a+b)h}(k, h).$$

So, by (6), and [2], formula 14,

$$- \sum_{h=0}^{m-1} \sum_{k=0}^{m-1} \zeta_m^{ah+bk}(h, k) = - \sum_{h=0}^{m-1} \sum_{k=0}^{m-1} \zeta_m^{ah+bk}(k, h) = J_{-a-b,b} = J_{a,b}.$$

For $i, j \in \mathbb{Z}$,

$$(15) \quad \begin{aligned} (i, j) &= -\frac{1}{m^2} \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \zeta_m^{-ia-jb} J_{a,b} \\ &= -\frac{1}{m^2} \left(m\delta_{0,i} + m\delta_{0,j} + m\delta_{i,j} - q - 1 + \sum_{\substack{1 \leq a,b \leq m-1 \\ a+b \neq m}} \zeta_m^{-ia-jb} J_{a,b} \right) \end{aligned}$$

(see, for example, [1], §2.5, or [12], Proposition 3, or formula (16) below).

Let P be the matrix $[\zeta_m^{ij}]_{0 \leq i,j \leq m-1}$. We have that $P^{-1} = \overline{P}/m$, and (14) is equivalent to

$$(16) \quad [J_{-a,b}]_{0 \leq a,b \leq m-1} = -mP^{-1}[(i, j)]_{0 \leq i,j \leq m-1}P.$$

In the next proposition we give a list of properties of the Jacobi sums $J_{a,b}$ that actually characterize these numbers, as will be proved later (see Proposition 3).

Proposition 2. For $a, b \in \mathbb{Z}$, the Jacobi sums $J_{a,b}$ are elements of $\mathbb{Z}[\zeta_m]$ which satisfy the following conditions:

1. $J_{a+m,b} = J_{a,b+m} = J_{a,b}$.
2. $J_{a,b} = J_{b,a}$.
3. $J_{a,b} = J_{-a-b,b}$.
4. $J_{0,0} = -(q - 2)$, and $J_{0,b} = 1$, if $m \nmid b$.
5. $J_{a,b}J_{-a,-b} = q$, if $m \nmid a$, $m \nmid b$ and $m \nmid (a + b)$.
6. $J_{a,b}J_{-a,-c} = J_{-(a+b+c),b}J_{a+b+c,-c}$, if $m \nmid (a + b)$, $m \nmid (a + c)$, $m \nmid a$ and $m \nmid (a + b + c)$.

7. For $i, j \in \mathbb{Z}$, the numbers

$$\begin{aligned} h_{i,j} &= -\frac{1}{m^2} \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \zeta_m^{-ai-bj} (J_{a,b} + (q-1)\delta_{0,b}) \\ &= -f\delta_{0,i} - \frac{1}{m^2} \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \zeta_m^{-ai-bj} J_{a,b} \end{aligned}$$

are rational integers. (Note that, by (4) and (15), the $h_{i,j}$ are in fact the numbers $c_{i,j}$.)

8. The characteristic polynomial of the matrix $[J_{-a,b} + (q-1)\delta_{0,b}]_{0 \leq a,b \leq m-1}$ (which, by 7, is equal to the characteristic polynomial of $[-mh_{i,j}]_{0 \leq i,j \leq m-1}$) is irreducible over \mathbb{Q} .

Proof. Properties 1-3 were shown in (6). Property 4 follows from (7) and (8). Property 5 follows from (7) and from the fact that $G(\zeta_m^k)G(\zeta_m^{-k}) = q$, if $m \nmid k$.

Suppose that $m \nmid (a+b)$, $m \nmid (a+c)$, $m \nmid a$ and $m \nmid (a+b+c)$. Then, by (7),

$$\begin{aligned} J_{a,b}J_{-a,-c} &= (G(\zeta_m^a)G(\zeta_m^b)/G(\zeta_m^{a+b}))(G(\zeta_m^{-a})G(\zeta_m^{-c})/G(\zeta_m^{-a-c})) \\ &= (G(\zeta_m^{-a-b-c})G(\zeta_m^b)/G(\zeta_m^{-a-c}))(G(\zeta_m^{a+b+c})G(\zeta_m^{-c})/G(\zeta_m^{a+b})) \\ &= J_{-(a+b+c),b}J_{a+b+c,-c}, \end{aligned}$$

since $G(\zeta_m^a)G(\zeta_m^{-a}) = q = G(\zeta_m^{-a-b-c})G(\zeta_m^{a+b+c})$. This proves property 6.

By (15) we have

$$h_{i,j} + f\delta_{0,i} = -\frac{1}{m^2} \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \zeta_m^{-ia-jb} J_{a,b} = (i, j).$$

So, $h_{i,j} = (i, j) - f\delta_{0,i} = c_{i,j} \in \mathbb{Z}$. This proves property 7.

To prove property 8, observe that, by (4), (16) and property 7, we have

$$[J_{-a,b} + (q-1)\delta_{0,b}]_{a,b} = P^{-1}[-mh_{i,j}]_{i,j}P = P^{-1}[-mc_{i,j}]_{i,j}P.$$

So, the characteristic polynomial of the matrix $[J_{-a,b} + (q-1)\delta_{0,b}]_{0 \leq a,b \leq m-1}$ is equal to the characteristic polynomial of the matrix $[-mc_{i,j}]_{0 \leq i,j \leq m-1}$, which is irreducible over \mathbb{Q} by (3). \square

Proposition 3. For $a, b \in \mathbb{Z}$, let $\mathcal{J}_{a,b}$ be elements in $\mathbb{Z}[\zeta_m]$ which satisfy conditions 1-8 of Proposition 2. Then, for some choice of the primitive root s modulo q , the $\mathcal{J}_{a,b}$ are the Jacobi sums $J_{a,b}$ defined in (5).

Observation. This proposition generalizes [11], Proposition 2, where we only considered the case $m = p$, a prime, and denoted $J_{1,n}$ by J_n .

Proof. Let $\mathcal{J}_{a,b}$, $a, b \in \mathbb{Z}$, be elements of $\mathbb{Z}[\zeta_m]$ satisfying conditions 1-8 of Proposition 2. We will prove that the integers $h_{i,j}$ of condition 7 are, for some choice of the primitive root s modulo q , the numbers $c_{i,j} = (i, j) - f\delta_{0,i}$. This will end the proof, since we can express the Jacobi sums $J_{a,b}$ in terms of the $c_{i,j}$ using (4) and (14), and, by condition 7, that expression must also give the numbers $\mathcal{J}_{a,b}$.

We showed in [10], Theorem 1 and the observation that follows it, that the numbers $c_{i,j}$, $i, j \in \mathbb{Z}$, are characterized (up to some reordering due to the choice of s) by the following conditions: The $c_{i,j}$ are integers such that $c_{i+m,j} = c_{i,j+m} = c_{i,j}$ and

- i) $\sum_{k=0}^{m-1} c_{i,k} = f - q\delta_{0,i}$,
- ii) $\sum_{k=0}^{m-1} c_{k,j} = -\delta_{0,j}$,
- iii) $c_{i,j} = c_{-i,j-i}$,
- iv) $\sum_{k=0}^{m-1} c_{i,k}c_{k-j,l-j} = \sum_{k=0}^{m-1} c_{j,k}c_{k-i,l-i}$,
- v) the characteristic polynomial of the matrix $[c_{i,j}]_{0 \leq i,j \leq m-1}$ is irreducible over \mathbb{Q} .

(See also [12], Proposition 2.)

We are going to prove that the integers

$$h_{i,j} = -f\delta_{0,i} - \frac{1}{m^2} \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \zeta_m^{-ai-bj} \mathcal{J}_{a,b}$$

satisfy the above conditions (with $c_{i,j}$ replaced by $h_{i,j}$). Clearly $h_{i+m,j} = h_{i,j+m} = h_{i,j}$, and condition 8 implies (v).

Define

$$[i, j] = h_{i,j} + f\delta_{0,i} = -\frac{1}{m^2} \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \zeta_m^{-ai-bj} \mathcal{J}_{a,b}.$$

By condition 2 we have $[i, j] = [j, i]$. By condition 4,

$$\begin{aligned} \sum_{k=0}^{m-1} [i, k] &= -\frac{1}{m^2} \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \zeta_m^{-ai} \mathcal{J}_{a,b} \sum_{k=0}^{m-1} \zeta_m^{-bk} \\ &= -\frac{1}{m} \sum_{a=0}^{m-1} \zeta_m^{-ai} \mathcal{J}_{a,0} \\ &= -\frac{1}{m} \left(-(q-2) + \sum_{a=1}^{m-1} \zeta_m^{-ai} \right) = f - \delta_{0,i}. \end{aligned}$$

Now (i) and (ii) follow at once.

By condition 3 we have

$$\begin{aligned} [-i, j - i] &= -\frac{1}{m^2} \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \zeta_m^{ai+b(i-j)} \mathcal{J}_{a,b} \\ &= -\frac{1}{m^2} \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \zeta_m^{(a+b)i-bj} \mathcal{J}_{a,b} \\ &= -\frac{1}{m^2} \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \zeta_m^{-ai-bj} \mathcal{J}_{-a-b,b} \\ &= -\frac{1}{m^2} \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \zeta_m^{-ai-bj} \mathcal{J}_{a,b} = [i, j]. \end{aligned}$$

Therefore $h_{-i,j-i} = [-i, j - i] - f\delta_{0,i} = [i, j] - f\delta_{0,i} = h_{i,j}$. This proves (iii).

Proof of (iv). It remains to prove that $\sum_{k=0}^{m-1} h_{i,k}h_{k-j,l-j} = \sum_{k=0}^{m-1} h_{j,k}h_{k-i,l-i}$. Since this proof requires a long calculation, to simplify matters we are going to use the following notation: If we have two expressions $U(i, j, l)$ and $V(i, j, l)$, we write $U(i, j, l) \sim V(i, j, l)$ if the difference $W(i, j, l) = U(i, j, l) - V(i, j, l)$ satisfies $W(i, j, l) = W(j, i, l)$. Define $H(i, j, l) = \sum_{k=0}^{m-1} h_{i,k}h_{k-j,l-j}$. We must prove that $H(i, j, l) \sim 0$.

We have

$$\begin{aligned} H(i, j, l) &= \sum_{k=0}^{m-1} ([i, k] - f\delta_{0,i})([k - j, l - j] - f\delta_{k,j}) \\ &= \sum_{k=0}^{m-1} [i, k][k - j, l - j] - f\delta_{0,i} \sum_{k=0}^{m-1} [k - j, l - j] \\ &\quad - f \sum_{k=0}^{m-1} [i, k]\delta_{k,j} + f^2\delta_{0,i} \sum_{k=0}^{m-1} \delta_{k,j} \\ &= \sum_{k=0}^{m-1} [i, k][k - j, l - j] - f\delta_{0,i}(f - \delta_{l,j}) - f[i, j] + f^2\delta_{0,i}. \end{aligned}$$

So,

$$(*) \quad H(i, j, l) \sim f\delta_{0,i}\delta_{l,j} + \sum_{k=0}^{m-1} [i, k][k - j, l - j].$$

Now, using conditions 2 and 3, we get

$$\begin{aligned} \sum_{k=0}^{m-1} [i, k][k - j, l - j] &= \frac{1}{m^4} \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \sum_{t=0}^{m-1} \sum_{w=0}^{m-1} \sum_{k=0}^{m-1} \zeta_m^{-ia-kb-(k-j)t-(l-j)w} \mathcal{J}_{a,b} \mathcal{J}_{t,w} \\ &= \frac{1}{m^4} \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \sum_{t=0}^{m-1} \sum_{w=0}^{m-1} \zeta_m^{-ia+jt-(l-j)w} \mathcal{J}_{a,b} \mathcal{J}_{t,w} \sum_{k=0}^{m-1} \zeta_m^{-k(b+t)} \\ &= \frac{1}{m^3} \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \sum_{w=0}^{m-1} \zeta_m^{-ia-jb-(l-j)w} \mathcal{J}_{a,b} \mathcal{J}_{-b,w} \\ &= \frac{1}{m^3} \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \sum_{w=0}^{m-1} \zeta_m^{-ia-j(b-w)-lw} \mathcal{J}_{a,b} \mathcal{J}_{-b,w} \\ &= \frac{1}{m^3} \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \sum_{w=0}^{m-1} \zeta_m^{-ia-jw-l(b-w)} \mathcal{J}_{a,b} \mathcal{J}_{-b,b-w} \\ &= \frac{1}{m^3} \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \sum_{w=0}^{m-1} \zeta_m^{-ia-jw-l(b-w)} \mathcal{J}_{a,b} \mathcal{J}_{-b,w} \\ &= \frac{1}{m^3} \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \sum_{w=0}^{m-1} \zeta_m^{-ib+jw-l(a+w)} \mathcal{J}_{a,b} \mathcal{J}_{-a,-w}. \end{aligned}$$

Now define

$$F(i, j, l) = m^2(q - 1)\delta_{0,i}\delta_{l,j} + \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \sum_{w=0}^{m-1} \zeta_m^{-ib+jw-l(a+w)} \mathcal{J}_{a,b} \mathcal{J}_{-a,-w}.$$

By (*), in order to prove (iv), it is enough to prove that $F(i, j, l) = F(j, i, l)$, i.e. that $F(i, j, l) \sim 0$. Define

$$A(i, j, l) = \sum_{\substack{0 \leq a, b, w \leq m-1 \\ m \nmid (a+b), (a+w), a, (a+b+w)}} \zeta_m^{-ib+jw-l(a+w)} \mathcal{J}_{a,b} \mathcal{J}_{-a,-w}$$

and

$$B(i, j, l) = m^2(q - 1)\delta_{0,i}\delta_{l,j} + \sum_{\substack{0 \leq a, b, w \leq m-1 \\ m \mid (a+b) \text{ or } (a+w) \text{ or } a \text{ or } (a+b+w)}} \zeta_m^{-ib+jw-l(a+w)} \mathcal{J}_{a,b} \mathcal{J}_{-a,-w}.$$

Since $F(i, j, l) = A(i, j, l) + B(i, j, l)$, it is enough to prove that $A(i, j, l) \sim 0 \sim B(i, j, l)$. By condition 6, we have

$$A(i, j, l) = \sum_{\substack{0 \leq a, b, w \leq m-1 \\ m \nmid (a+b), (a+w), a, (a+b+w)}} \zeta_m^{-ib+jw-l(a+w)} \mathcal{J}_{-(a+b+w),b} \mathcal{J}_{a+b+w,-w}.$$

Changing variables, $a \rightarrow -(a + b + w)$, we get

$$\begin{aligned} A(i, j, l) &= \sum_{\substack{0 \leq a, b, w \leq m-1 \\ m \nmid (a+b), (a+w), a, (a+b+w)}} \zeta_m^{-ib+jw+l(a+b)} \mathcal{J}_{a,b} \mathcal{J}_{-a,-w} \\ &= \sum_{\substack{0 \leq a, b, w \leq m-1 \\ m \nmid (a+b), (a+w), a, (a+b+w)}} \zeta_m^{-jw+ib-l(a+b)} \mathcal{J}_{a,w} \mathcal{J}_{-a,-b} \\ &= \sum_{\substack{0 \leq a, b, w \leq m-1 \\ m \nmid (a+b), (a+w), a, (a+b+w)}} \zeta_m^{-jb+iw-l(a+w)} \mathcal{J}_{a,b} \mathcal{J}_{-a,-w} \\ &= A(j, i, l). \end{aligned}$$

So, $A(i, j, l) \sim 0$.

It remains to prove that $B(i, j, l) \sim 0$. Write

$$B(i, j, l) = m^2(q - 1)\delta_{0,i}\delta_{l,j} + C(i, j, l) + D(i, j, l),$$

where

$$C(i, j, l) = \sum_{b=0}^{m-1} \sum_{w=0}^{m-1} \zeta_m^{-ib+jw-lw} \mathcal{J}_{0,b} \mathcal{J}_{0,-w},$$

$$D(i, j, l) = \sum_{\substack{1 \leq a \leq m-1 \\ 0 \leq b, w \leq m-1 \\ m|(a+b) \text{ or } (a+w) \text{ or } (a+b+w)}} \zeta_m^{-ib+jw-l(a+w)} \mathcal{J}_{a,b} \mathcal{J}_{-a,-w}.$$

By condition 4,

$$C(i, j, l) = \sum_{b=0}^{m-1} \sum_{w=0}^{m-1} \zeta_m^{-ib+(j-l)w} (-(q - 1)\delta_{0,b} + 1) (-(q - 1)\delta_{0,w} + 1)$$

$$= (q - 1)^2 - (q - 1) \sum_{w=0}^{m-1} \zeta_m^{(j-l)w} - (q - 1) \sum_{b=0}^{m-1} \zeta_m^{-ib} + \sum_{b=0}^{m-1} \zeta_m^{-ib} \sum_{w=0}^{m-1} \zeta_m^{(j-l)w}$$

$$= (q - 1)^2 - m(q - 1)\delta_{j,l} - m(q - 1)\delta_{0,i} + m^2\delta_{0,i}\delta_{j,l}.$$

So,

$$C(i, j, l) \sim -m(q - 1)\delta_{j,l} - m(q - 1)\delta_{0,i} + m^2\delta_{0,i}\delta_{j,l}.$$

Finally, write $D(i, j, l) = X(i, j, l) + Y(i, j, l)$, where

$$X(i, j, l) = \sum_{a=1}^{m-1} \sum_{b=0}^{m-1} \zeta_m^{-ib-ja} \mathcal{J}_{a,b} \mathcal{J}_{-a,a},$$

$$Y(i, j, l) = \sum_{\substack{1 \leq a \leq m-1 \\ 0 \leq b, w \leq m-1 \\ w \not\equiv -a \pmod m \\ m|(a+b) \text{ or } (a+b+w)}} \zeta_m^{-ib+jw-l(a+w)} \mathcal{J}_{a,b} \mathcal{J}_{-a,-w}.$$

If $m \nmid a$, by conditions 3 and 4, we have $\mathcal{J}_{-a,a} = \mathcal{J}_{0,a} = 1$. Therefore, by condition 2,

$$X(i, j, l) = \sum_{a=1}^{m-1} \sum_{b=0}^{m-1} \zeta_m^{-ib-ja} \mathcal{J}_{a,b} = - \sum_{b=0}^{m-1} \zeta_m^{-ib} \mathcal{J}_{0,b} - m^2[i, j]$$

$$= (q - 2) - \sum_{b=1}^{m-1} \zeta_m^{-ib} - m^2[i, j] = (q - 1) - m\delta_{0,i} - m^2[i, j].$$

So, $X(i, j, l) \sim -m\delta_{0,i}$. Also, by conditions 2, 3 and 5,

$$\begin{aligned}
 Y(i, j, l) &= \sum_{a=1}^{m-1} \sum_{\substack{0 \leq w \leq m-1 \\ w \not\equiv -a \pmod m}} \zeta_m^{ia+jw-l(a+w)} \mathcal{J}_{a,-a} \mathcal{J}_{-a,-w} \\
 &+ \sum_{\substack{1 \leq a \leq m-1 \\ 0 \leq b, w \leq m-1 \\ b \not\equiv -a, w \not\equiv -a \pmod m \\ m|a+b+w}} \zeta_m^{-ib+jw-l(a+w)} \mathcal{J}_{a,b} \mathcal{J}_{-a,-w} \\
 &= \sum_{a=1}^{m-1} \sum_{\substack{0 \leq w \leq m-1 \\ w \not\equiv -a \pmod m}} \zeta_m^{ia+jw-l(a+w)} \mathcal{J}_{-a,-w} \\
 &+ \sum_{a=1}^{m-1} \sum_{\substack{1 \leq b \leq m-1 \\ b \not\equiv -a \pmod m}} \zeta_m^{-ib-j(a+b)+lb} \mathcal{J}_{a,b} \mathcal{J}_{-a,a+b} \\
 &= \sum_{a=1}^{m-1} \sum_{\substack{0 \leq w \leq m-1 \\ w \not\equiv -a \pmod m}} \zeta_m^{-(i-l)a-(j-l)w} \mathcal{J}_{a,w} \\
 &+ \sum_{a=1}^{m-1} \sum_{\substack{1 \leq b \leq m-1 \\ b \not\equiv -a \pmod m}} \zeta_m^{-ib-j(a+b)+lb} \mathcal{J}_{a,b} \mathcal{J}_{-a,-b} \\
 &= -m^2[i-l, j-l] - \sum_{w=1}^{m-1} \zeta_m^{-(j-l)w} \mathcal{J}_{0,w} - \sum_{a=0}^{m-1} \zeta_m^{-(i-j)a} \mathcal{J}_{a,-a} \\
 &+ q \sum_{a=1}^{m-1} \zeta_m^{-ja} \sum_{b=1}^{m-1} \zeta_m^{(l-i-j)b} - q \sum_{a=1}^{m-1} \zeta_m^{-ja-(l-i-j)a} \\
 &= -m^2[i-l, j-l] - m\delta_{j,l} + 1 + 1 + (1-2) - (q-2) - m\delta_{i,j} + 1 \\
 &+ q(m\delta_{0,j} - 1)(m\delta_{l,i+j} - 1) - q(m\delta_{l,i} - 1).
 \end{aligned}$$

So, $Y(i, j, l) \sim -qm\delta_{0,j} - qm\delta_{l,i} - m\delta_{j,l} + qm^2\delta_{0,j}\delta_{l,i+j}$. Therefore,

$$\begin{aligned}
 B(i, j, l) &= m^2(q-1)\delta_{0,i}\delta_{l,j} + C(i, j, l) + X(i, j, l) + Y(i, j, l) \\
 &\sim m^2(q-1)\delta_{0,i}\delta_{l,j} - m(q-1)\delta_{j,l} - m(q-1)\delta_{0,i} \\
 &+ m^2\delta_{0,i}\delta_{j,l} - m\delta_{0,i} - qm\delta_{0,j} - qm\delta_{l,i} - m\delta_{j,l} + qm^2\delta_{0,j}\delta_{l,i+j} \\
 &= m^2q\delta_{0,i}\delta_{l,j} + m^2q\delta_{0,j}\delta_{l,i} - mq\delta_{l,i} - mq\delta_{l,j} - mq\delta_{0,i} - mq\delta_{0,j}.
 \end{aligned}$$

Therefore $B(i, j, l) \sim 0$. This ends the proof of (iv), and of Proposition 3. □

Let Q be the prime ideal of $\mathbb{Z}[\zeta_m]$ above q such that $s^f \equiv \zeta_m \pmod Q$. If $k \in \mathbb{Z}$ we denote by $|k|_m$ the least nonnegative integer such that $|k|_m \equiv k \pmod m$. We showed in [12], formula (27), that, for $0 \leq a, b \leq m-1$ with $a+b \not\equiv 0 \pmod m$,

$$(17) \quad \bar{J}_{a,b} \equiv \left(\begin{matrix} f|a+b|_m \\ f a \end{matrix} \right) \pmod Q.$$

This fact is a simple consequence of (7), and [4], Chapter 1, Theorem 2.1.

The MAPLE program to calculate Jacobi sums that ends this section is based on the following proposition.

Proposition 4. *Let a, b be integers, $1 \leq a, b \leq m - 1$, such that $\text{g.c.d.}(a, b, m) = 1$, and let $0 \leq l \leq m - 1$. Let u, v and $\varepsilon(l)$ be as in Proposition 1. Then*

$$d_{a,b,l} \equiv \frac{1}{u}\varepsilon(l) + \frac{1}{m} \sum_{i=1}^{u-1} \sum_{k=0}^{v-1} s^{f(i+uk)l} \begin{pmatrix} f|i(a+b)|_m \\ f|(i+uk)a|_m \end{pmatrix} \pmod q,$$

and $|d_{a,b,l}| < \sqrt{q} < q/2$.

Proof. The first assertion follows directly from Proposition 1 and (17). The second assertion follows from Proposition 1, the triangle inequality, and the fact that $|J_{a,b}| = \sqrt{q}$ if $m \nmid a$, $m \nmid b$ and $m \nmid (a + b)$. □

In the following program enter the values of $m > 2$, q a prime $\equiv 1 \pmod{2m}$, s a primitive root modulo q (the command: $s := \text{primroot}(q)$; will give to s the value of the smallest positive primitive root modulo q), a and b integers, $1 \leq a, b \leq m - 1$, such that $m \nmid a + b$, and such that $\text{g.c.d.}(a, b, m) = 1$ (see the observation preceding Proposition 1). The resulting matrix A is the row matrix $[d_{a,b,0}, d_{a,b,1}, \dots, d_{a,b,m-1}]$. The expression $F(x)$ is the Jacobi sum $J_{a,b} = \sum_{j=0}^{m-1} d_{a,b,j} \zeta_m^j$, if one replaces x by ζ_m . The expression $G(x)$, a polynomial of degree $< \varphi(m)$, is also equal to the Jacobi sum $J_{a,b}$, if one replaces x by ζ_m . The last two lines are to check that $J_{a,b}(1) = 1$ and that $J_{a,b} \bar{J}_{a,b} = q$.

A MAPLE program to calculate the Jacobi sums $J_{a,b}$ given m, q and s

```
with(linalg): with(numtheory):
m:=12; q:=73; s:=primroot(q); a:=2; b:=5;
f:=(q-1)/m: v:=igcd(a+b,m): u:=m/v:
for i from 0 to m-1 do;
ep(i):=floor(1-i/v+floor(i/v)); od:
C:=array(1..u,1..v):
for j1 from 1 to u do; for k1 from 1 to v do;
C[j1,k1]:=modp(binomial(f*modp((j1-1)*(a+b),m), f*modp(((j1-1)+u*(k1-1))*a,m)),q);
od: od:
A:=array(1..1,1..m):
for l from 1 to m do;
A[1,l]:=mods(ep(l-1)/u+(1/m)*sum(sum(s^((f*(j-1)+f*u*(k-1))*(l-1))*C[j,k],j=2..u),
k=1..v),q); od:
A:=evalm(A);
R:=cyclotomic(m,x);
F:=x->sum(A[1,t]*x^(t-1),t=1..m):
F(x):=F(x); G:=rem(F(x),R,x);
# check:
F(1);
rem(F(x)*F(x^(m-1)),R,x);
```

2. FAMILIES OF IRREDUCIBLE POLYNOMIALS
OF GAUSSIAN PERIODS OF DEGREE m

As in Section 1, let $m > 2$ be an integer and ζ_m an m -th primitive root of 1. Let S be the set of all prime numbers $q \equiv 1 \pmod{2m}$. If $q \in S$, s is a primitive root modulo q , and Q is the prime ideal of $\mathbb{Z}[\zeta_m]$ above q such that $s^{(q-1)/m} \equiv \zeta_m \pmod Q$, we write $J_{a,b} = J_{a,b}[Q]$ for the Jacobi sums defined in (5). In this section we

show how to construct families of irreducible polynomials of Gaussian periods of degree m . We first show how one can make this construction in a general situation, and then work out several examples with m small.

The first step in our method is to construct families $(J_{a,b}[Q])$, $0 \leq a, b \leq m - 1$, $Q \in \mathcal{I}$, of sets of principal ideals generated by Jacobi sums of the type studied in Section 1, where \mathcal{I} is a set of prime ideals of $\mathbb{Z}[\zeta_m]$ above rational primes in S .

Let ν be a positive integer and, for $1 \leq i \leq \nu$, let r_i be prime numbers (not necessarily distinct) not dividing m . Let f_i be the smallest positive integer such that $r_i^{f_i} \equiv 1 \pmod m$, R_i a prime ideal of $\mathbb{Z}[\zeta_m]$ above r_i , $s_i \in \mathbb{Z}[\zeta_m]$ a generator of $\mathbb{Z}[\zeta_m]/R_i \cong \mathbb{F}_{r_i^{f_i}}$ (the field with $r_i^{f_i}$ elements) such that $s_i^{(r_i^{f_i}-1)/m} \equiv \zeta_m \pmod{R_i}$. For $1 \leq i \leq \nu$ and $0 \leq a, b \leq m - 1$, let $\mathfrak{J}_{i,a,b}$ be the Jacobi sum

$$(18) \quad \mathfrak{J}_{i,a,b} = - \sum_{\substack{\gamma \in \mathbb{Z}[\zeta_m]/R_i \\ \gamma \neq 0,1}} \zeta_m^{a \operatorname{ind}_{s_i}(\gamma) + b \operatorname{ind}_{s_i}(1-\gamma)},$$

where $\operatorname{ind}_{s_i}(\gamma)$ is the least nonnegative integer such that $s_i^{\operatorname{ind}_{s_i}(\gamma)} \equiv \gamma \pmod{R_i}$. We assume that the numbers $\mathfrak{J}_{i,a,b}$ are known (i.e. that they have been calculated).

If c is an integer relatively prime with m , denote by σ_c the automorphism of $\mathbb{Q}(\zeta_m)$ such that $\sigma_c(\zeta_m) = \zeta_m^c$. If $a + b \not\equiv 0 \pmod m$, the prime ideal factorization of the ideal $(\mathfrak{J}_{i,a,b})$ of $\mathbb{Z}[\zeta_m]$ is given by

$$(19) \quad (\mathfrak{J}_{i,a,b}) = \prod_{\substack{1 \leq c \leq m-1 \\ \text{g.c.d.}(c,m)=1}} \sigma_c^{-1}(\bar{R}_i)^{\lfloor \frac{(a+b)c}{m} \rfloor - \lfloor \frac{ac}{m} \rfloor - \lfloor \frac{bc}{m} \rfloor},$$

where the bar denotes complex conjugation, and $\lfloor \rho \rfloor$ denotes the integral part of a real number ρ (see [4], page 13, Fac 3).

Define $r = \prod_{i=1}^{\nu} r_i$ and $r' = \prod_{i=1}^{\nu} r_i^{f_i}$. Let

$$\mathcal{C} = \{ \alpha \in \mathbb{Z}[\zeta_m] : (\alpha) = R_1 \dots R_{\nu} Q, \text{ with } N_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(Q) = q \in S \},$$

\mathcal{A} a nonempty subset of \mathcal{C} , and $\mathcal{I} = \{ Q = (\alpha)(R_1 \dots R_{\nu})^{-1} : \alpha \in \mathcal{A} \}$ (a set of prime ideals of $\mathbb{Z}[\zeta_m]$ above primes in S). For $0 \leq a, b \leq m - 1$ such that $m \nmid a + b$, set $\mathfrak{J}_{a,b} = \prod_{i=1}^{\nu} \mathfrak{J}_{i,a,b}$, and for $\alpha \in \mathcal{A}$, set

$$(20) \quad \mathfrak{K}_{a,b}[\alpha] = \prod_{\substack{1 \leq c \leq m-1 \\ \text{g.c.d.}(c,m)=1}} \sigma_c^{-1}(\bar{\alpha})^{\lfloor \frac{(a+b)c}{m} \rfloor - \lfloor \frac{ac}{m} \rfloor - \lfloor \frac{bc}{m} \rfloor}.$$

Then, for $\alpha \in \mathcal{A}$, we have $(\mathfrak{K}_{a,b}[\alpha]/\mathfrak{J}_{a,b}) = (J_{a,b}[Q])$ (equality of ideals of $\mathbb{Z}[\zeta_m]$), with $J_{a,b} = J_{a,b}[Q]$ as in (5), where $Q \in \mathcal{I}$ is the prime ideal $(\alpha)(R_1 \dots R_{\nu})^{-1}$. To prove this equality just check, using (19), that both sides have the same prime ideal factorization.

The choice of the set \mathcal{A} will determine whether our family of polynomials has a nice description. One way to make this choice is the following. Take $\alpha_0, \alpha_1 \in \mathbb{Z}[\zeta_m]$

such that $(\alpha_0, \alpha_1) = R_1 \dots R_\nu$ and define $\mathcal{A} = \mathcal{A}_1$, where

$$\mathcal{A}_1 = \left\{ \alpha = \alpha_0 + \alpha_1 \beta : \beta = \sum_{i=0}^{\varphi(m)-1} b_i \zeta_m^i \in \mathbb{Z}[\zeta_m] \right. \\ \left. \text{and } N_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(\alpha) = r'q, \text{ with } q \in S \right\}.$$

The parameters of the family we construct will then be the coefficients b_i of β . In the examples we work with the simpler sets

$$\mathcal{A}_2 = \{ \alpha = \alpha_0 + \alpha_1 n : n \in \mathbb{Z} \text{ and } N_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(\alpha) = r'q, \text{ with } q \in S \}.$$

The second step is to identify the Jacobi sums $J_{a,b}[Q]$, $Q \in \mathcal{I}$, among the generators of the principal ideals $(J_{a,b}[Q])$. One way to do that is to start with a subset \mathcal{A} of \mathcal{C} such that if $\alpha \in \mathcal{A}$ the numbers $\mathfrak{K}_{a,b}[\alpha]$ are products of Jacobi sums (as the ones defined in (18)). Then we know after Weil [14] that, using the notation above, for $\alpha \in \mathcal{A}$ and $Q = (\alpha)(R_1 \dots R_\nu)^{-1}$, $J_{a,b}[Q] = \mathfrak{K}_{a,b}[\alpha]/\mathfrak{J}_{a,b}$. Also, by [14], we know that there is a divisor \mathfrak{f} of m^2 such that any nonempty subset \mathcal{A} of the set $\mathcal{C}_{\mathfrak{f}} = \{ \alpha \in \mathcal{C} : \alpha \equiv 1 \pmod{\mathfrak{f}} \}$ has the desired property. Another way to identify the $J_{a,b}[Q]$ among the generators of the ideals $(J_{a,b}[Q])$, which works at least when $m = p$ is a prime and was used in [12], relies on the fact that only one of the numbers $\delta \zeta_m^k \mathfrak{K}_{a,b}[\alpha]/\mathfrak{J}_{a,b}$, $\delta \in \{1, -1\}$, $0 \leq k \leq m - 1$, satisfies congruence (13), and that number is $J_{a,b}[Q]$.

From the family $J_{a,b}[Q]$, $Q \in \mathcal{I}$, of sets of Jacobi sums, we construct, using (4) and (15), a family $C[Q]$, $Q \in \mathcal{I}$, of matrices with entries $c_{i,j} = c_{i,j}[Q]$, whose characteristic polynomials form, by (3), the desired family $P_q(x)$, $q \in \mathcal{P}$, of irreducible polynomials of Gaussian periods of degree m . Here $\mathcal{P} = \{ q = N_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(Q) : Q \in \mathcal{I} \} \subseteq S$. Note that ideals $Q \in \mathcal{I}$ are in the inverse ideal class of the ideal $\mathcal{R} = \prod_{i=1}^{\nu} R_i$.

In what follows we give examples of this construction and a MAPLE program to search for more examples.

Example 1. For $m = 7$, and primes of the form

$$q = 49n^6 - 49n^5 + 49n^4 + 35n^3 + 21n^2 + 7n + 1,$$

the irreducible polynomials of the Gaussian periods of degree m in $\mathbb{Q}(\zeta_q)$ are

$$P_q(x) = x^7 + x^6 + (-21n^6 + 21n^5 - 21n^4 - 15n^3 - 9n^2 - 3n)x^5 \\ + (-21n^9 + 28n^8 + 7n^7 - 48n^6 + 36n^5 + 20n^4 + 12n^3 + 3n^2)x^4 \\ + (91n^{12} - 147n^{11} + 252n^{10} - 85n^9 + 73n^8 \\ + 100n^7 + 21n^6 + 10n^5 - 2n^4 - n^3)x^3 \\ + (112n^{15} - 203n^{14} + 175n^{13} + 113n^{12} - 227n^{11} \\ + 127n^{10} - 23n^9 - 45n^8 - 25n^7 - 14n^6 - 2n^5)x^2 \\ + (-84n^{18} + 238n^{17} - 518n^{16} + 629n^{15} - 442n^{14} \\ + 196n^{12} - 8n^{11} - 22n^{10} - 26n^9 - 11n^8 - n^7)x \\ - 97n^{21} + 357n^{20} - 609n^{19} + 434n^{18} + 52n^{17} - 282n^{16} + 94n^{15} \\ + 56n^{14} + 7n^{13} - 3n^{12} - 8n^{11} - 2n^{10}.$$

To obtain this result we start with the elements $1 + n(\zeta_m - 1)^2$ in $\mathbb{Z}[\zeta_m]$, which have norms $q = q(n)$ and generate prime ideals $Q = (1 + n(\zeta_m - 1)^2)$. We calculate the Jacobi sums $J_{a,b}[Q]$ using Stickelberger's theorem and the fact that if $m = p$ is a prime then $J_{a,b}[Q] \equiv 1 \pmod{(\zeta_m - 1)^2}$. We use the values of the Jacobi sums found to calculate the matrices $C = C[Q]$. Finally we calculate the characteristic polynomials of the $C[Q]$, which are the irreducible polynomials we wanted to find. All these calculations are performed by the program at the end of the article, where we must enter only the values $m:=7$; and $F:=z->1+n*(z-1)^2$;

In general the "smallest" examples I found for m prime start with the elements $\alpha = 1 + n(\zeta_m - \zeta_m^{-1})^3$ which have norms $q = q(n)$ that are polynomials in n^2 . The coefficients of the resulting polynomials $P_q(x)$ are also polynomials in n^2 . Something similar works for arbitrary m , where the right expression for α can be found by trial and error ($q(n)$ must be an irreducible polynomial in $\mathbb{Z}[n]$ and the matrix $C[(\alpha)]$ must have its entries in $\mathbb{Z}[n]$). This is illustrated in Examples 2, 3 and 4.

Example 2. For $m = 7$, and primes of the form

$$q = 343n^6 + 833n^4 + 70n^2 + 1,$$

the irreducible polynomials of the Gaussian periods of degree m in $\mathbb{Q}(\zeta_q)$ are

$$\begin{aligned} P_q(x) = & x^7 + x^6 + (-147n^6 - 357n^4 - 30n^2)x^5 \\ & + (-294n^8 - 749n^6 - 145n^4 - 8n^2)x^4 \\ & + (7203n^{12} + 30086n^{10} + 32403n^8 + 3436n^6 + 96n^4)x^3 \\ & + (28812n^{14} + 128723n^{12} + 152306n^{10} \\ & \quad + 21199n^8 + 1008n^6 + 16n^4)x^2 \\ & + (-117649n^{18} - 617057n^{16} - 787577n^{14} \\ & \quad + 47481n^{12} + 45234n^{10} + 3104n^8 + 32n^6)x \\ & - 705894n^{20} - 3186127n^{18} - 3505999n^{16} + 213835n^{14} \\ & + 39841n^{12} + 904n^{10} + 16n^8. \end{aligned}$$

To obtain this result we proceed in a similar way as in Example 1. Enter the values $m:=7$; and $F:=z->1+n*(z-z^-(m-1))^3$; in the program at the end of the article.

Example 3. For $m = 9$, and primes of the form

$$q = 2187n^6 + 729n^4 + 54n^2 + 1,$$

the irreducible polynomials of the Gaussian periods of degree m in $\mathbb{Q}(\zeta_q)$ are

$$\begin{aligned}
 P_q(x) = & x^9 + x^8 + (-972n^6 - 324n^4 - 24n^2)x^7 \\
 & + (-3888n^8 - 1548n^6 - 180n^4 - 8n^2)x^6 \\
 & + (196830n^{12} + 148716n^{10} + 34830n^8 + 2856n^6 + 80n^4)x^5 \\
 & + (629856n^{14} + 535086n^{12} + 148716n^{10} + 16830n^8 + 840n^6 + 16n^4)x^4 \\
 & + (-14880348n^{18} - 10786284n^{16} - 2259900n^{14} \\
 & \quad - 106164n^{12} + 7128n^{10} + 480n^8)x^3 \\
 & + (-25509168n^{20} - 18659484n^{18} - 6167340n^{16} \\
 & \quad - 1097388n^{14} - 95652n^{12} - 3480n^{10} - 32n^8)x^2 \\
 & + (387420489n^{24} + 70150212n^{22} - 29878794n^{20} \\
 & \quad - 7934436n^{18} - 489159n^{16} + 3672n^{14} + 720n^{12})x \\
 & - 29229255n^{24} - 1653372n^{22} + 2523798n^{20} \\
 & + 384156n^{18} + 22761n^{16} + 792n^{14} + 16n^{12}.
 \end{aligned}$$

To obtain this result we proceed in a similar way as in Example 1. This time enter the values $m:=9$; and $F:=z \rightarrow 1+3*n*(z-z^{m-1})$; in the program at the end of the article. Observe that the resulting matrix C has entries in $\mathbb{Z}[n]$.

Example 4. For $m = 12$, and primes of the form

$$q = 1296n^4 + 72n^2 + 1,$$

the irreducible polynomials of the Gaussian periods of degree m in $\mathbb{Q}(\zeta_q)$ are

$$\begin{aligned}
 P_q(x) = & x^{12} + x^{11} + (-594n^4 - 33n^2)x^{10} + (216n^6 - 153n^4 - 9n^2)x^9 \\
 & + (120771n^8 + 8937n^6 + 186n^4)x^8 \\
 & + (-116640n^{10} + 8586n^8 + 1044n^6 + 24n^4)x^7 \\
 & + (-9713196n^{12} - 858762n^{10} - 26784n^8 - 304n^6)x^6 \\
 & + (19840464n^{14} + 581742n^{12} - 28998n^{10} - 1368n^8 - 16n^6)x^5 \\
 & + (278337303n^{16} + 30561138n^{14} + 1165428n^{12} + 18144n^{10} + 96n^8)x^4 \\
 & + (-1055008800n^{18} - 84367899n^{16} \\
 & \quad - 1851660n^{14} + 1512n^{12} + 288n^{10})x^3 \\
 & + (-806018850n^{20} - 210194757n^{18} \\
 & \quad - 14311728n^{16} - 377136n^{14} - 3456n^{12})x^2 \\
 & + (7971615000n^{22} + 1069672635n^{20} \\
 & \quad + 52743879n^{18} + 1137240n^{16} + 9072n^{14})x \\
 & - 8968066875n^{24} - 1102740075n^{22} \\
 & \quad - 50585310n^{20} - 1026432n^{18} - 7776n^{16}.
 \end{aligned}$$

To obtain this result we proceed in a similar way as in Example 1. This time enter the values $m:=12$; and $F:=z \rightarrow 1+6*n*(z-z^{m-1})$; in the program at the end of the article. Observe that the resulting matrix C has entries in $\mathbb{Z}[n]$.

Example 5. Let $m = 7$ and $w = \zeta_7$ a 7-th primitive root of 1. Take $r_1 = 2$. Set $R_1 = (w^5 - 2w^4 + 3w^3 - w^2 + 2, 2(w - 1)^2) = (1 + w + w^3)$. We have $(2) = R_1 \overline{R_1}$. The element $s_1 = 1 + w^3$ is a generator of $\mathbb{Z}[w]/R_1 \cong \mathbb{F}_8$ (the field with 8 elements), such that $s_1 = s_1^{(8-1)/7} \equiv w \pmod{R_1}$. Let

$$\mathcal{A}_3 = \{ \alpha = w^5 - 2w^4 + 3w^3 - w^2 + 2 + 2(w - 1)^2 n \mid n \in \mathbb{Z} \text{ and } N_{\mathbb{Q}(w)/\mathbb{Q}}(\alpha) = 8q, \text{ with } q \in S \},$$

and

$$\mathcal{I} = \{ Q = (\alpha)R_1^{-1} : \alpha \in \mathcal{A}_3 \}.$$

Observation. Since $\mathbb{Z}[w]$ is a principal ideal domain, we could simplify our example by dividing the elements of \mathcal{A}_3 by a generator of R_1 . That, however, would not illustrate how the method works in the general situation. The first cases in which we really need to work with auxiliary Jacobi sums $\mathfrak{J}_{i,a,b}$ occur when $m = 23$, which is too large for a complete example, in paper, of a family of irreducible polynomials of Gaussian periods (but see Example 6).

If $\alpha = w^5 - 2w^4 + 3w^3 - w^2 + 2 + 2(w - 1)^2 n \in \mathcal{A}_3$, then

$$N_{\mathbb{Q}(w)/\mathbb{Q}}(\alpha) = 8(392n^6 + 98n^4 + 161n^3 + 14n^2 - 35n + 113).$$

So we are searching for the irreducible polynomials of the Gaussian periods of degree 7 corresponding to the primes q of the form

$$q = 392n^6 + 98n^4 + 161n^3 + 14n^2 - 35n + 113.$$

Set $\mathfrak{J}_{a,b} = \mathfrak{J}_{1,a,b}$, the Jacobi sums corresponding to s_1 and R_1 . By (18) we have

$$\begin{aligned} \mathfrak{J}_{1,1} &= \mathfrak{J}_{1,5} = -2(w + w^2 + w^4), \\ \mathfrak{J}_{1,2} &= \mathfrak{J}_{1,4} = -(3 + w^3 + w^5 + w^6), \\ \mathfrak{J}_{1,3} &= \overline{\mathfrak{J}}_{1,1} = -2(w^3 + w^5 + w^6). \end{aligned}$$

For $Q \in \mathcal{I}$ and $\alpha \in \mathcal{A}_3$ such that $(\alpha) = R_1 Q$, define $\mathfrak{K}_{a,b}[\alpha]$ as in (20). We have

$$\begin{aligned} \mathfrak{K}_{1,1}[\alpha] &= (24n^2 - 12n - 6)w^5 + (-36n^2 - 18n - 6)w^4 \\ &\quad + (-56n^3 - 12n^2 + 32n - 4)w^3 + (12n^2 + 6n - 24)w^2 \\ &\quad + (-48n^2 - 18n + 12)w - 24n^2 + 24n + 6, \end{aligned}$$

$$\mathfrak{K}_{1,2}[\alpha] = (6n - 21)w^4 + (6n - 21)w^2 + (6n - 21)w - 56n^3 - 4n - 22,$$

$$\begin{aligned} \mathfrak{K}_{1,3}[\alpha] &= (-48n^2 - 24n + 18)w^5 + (-12n^2 - 6n + 24)w^4 \\ &\quad + (-60n^2 - 24n + 36)w^3 + (-56n^3 - 24n^2 + 26n + 20)w^2 \\ &\quad + (12n^2 - 18n + 18)w - 36n^2 + 18n + 30, \end{aligned}$$

$$\mathfrak{K}_{1,4}[\alpha] = (6n - 21)w^4 + (6n - 21)w^2 + (6n - 21)w - 56n^3 - 4n - 22,$$

$$\begin{aligned} \mathfrak{K}_{1,5}[\alpha] &= (24n^2 - 12n - 6)w^5 + (-36n^2 - 18n - 6)w^4 \\ &\quad + (-56n^3 - 12n^2 + 32n - 4)w^3 + (12n^2 + 6n - 24)w^2 \\ &\quad + (-48n^2 - 18n + 12)w - 24n^2 + 24n + 6. \end{aligned}$$

Using the formula $(J_{a,b}[Q]) = (\mathfrak{K}_{a,b}[\alpha]/\mathfrak{J}_{a,b})$, and the fact that $J_{a,b}[Q] \equiv 1 \pmod{(w-1)^2}$, we get

$$\begin{aligned} J_{1,1}[Q] &= -w^3 \mathfrak{K}_{1,1}[\alpha]/\mathfrak{J}_{1,1} \\ &= (-14n^3 - 3n^2 + 5n - 1)w^5 + (-14n^3 + 15n^2 + 8n - 7)w^4 \\ &\quad + (-9n^2 - 6n - 3)w^3 + (-14n^3 - 12n^2 + 11n + 5)w^2 \\ &\quad + (6n^2 + 9n - 3)w + 3n^2 - 6n + 3, \end{aligned}$$

$$\begin{aligned} J_{1,2}[Q] &= -w^3 \mathfrak{K}_{1,2}[\alpha]/\mathfrak{J}_{1,2} \\ &= (-7n^3 + n - 8)w^4 + (-7n^3 + n - 8)w^2 \\ &\quad + (-7n^3 + n - 8)w - 21n^3 - 3n - 3, \end{aligned}$$

$$\begin{aligned} J_{1,3}[Q] &= -w^2 \mathfrak{K}_{1,3}[\alpha]/\mathfrak{J}_{1,3} \\ &= (27n^2 - 3n - 12)w^5 + (14n^3 + 12n^2 - 11n - 5)w^4 \\ &\quad + (14n^3 + 18n^2 - 2n - 8)w^3 + (14n^3 + 3n^2 - 17n - 8)w^2 \\ &\quad + (9n^2 - 6n - 6)w + 14n^3 + 15n^2 - 17n - 2, \end{aligned}$$

$$\begin{aligned} J_{1,4}[Q] &= -w^3 \mathfrak{K}_{1,4}[\alpha]/\mathfrak{J}_{1,4} \\ &= (-7n^3 + n - 8)w^4 + (-7n^3 + n - 8)w^2 \\ &\quad + (-7n^3 + n - 8)w - 21n^3 - 3n - 3, \end{aligned}$$

$$\begin{aligned} J_{1,5}[Q] &= -w^3 \mathfrak{K}_{1,5}[\alpha]/\mathfrak{J}_{1,5} \\ &= (-14n^3 - 3n^2 + 5n - 1)w^5 + (-14n^3 + 15n^2 + 8n - 7)w^4 \\ &\quad + (-9n^2 - 6n - 3)w^3 + (-14n^3 - 12n^2 + 11n + 5)w^2 \\ &\quad - 6n + (6n^2 + 9n - 3)w + 3n^2 + 3. \end{aligned}$$

For $1 \leq i \leq 5$ write

$$J_u = J_{1,u}[Q] = \sum_{k=0}^6 d_{u,k} \zeta_p^k, \quad \text{with } d_{u,k} \in \mathbb{Z}[n] \quad \text{such that } \sum_{k=0}^6 d_{u,k} = 1.$$

Denote by A the matrix $[d_{u,k}]_{\substack{1 \leq u \leq 5, \\ 0 \leq k \leq 6}}$. From the results above we obtain

$$A^t = \begin{bmatrix} 4-9n+3n^2+6n^3 & 1-3n-15n^3 & 4-9n+3n^2+6n^3 & 1-3n-15n^3 & 4-9n+3n^2+6n^3 \\ -2+6n+6n^2+6n^3 & -4+n-n^3 & 2n-3n^2-8n^3 & -4+n-n^3 & -2+6n+6n^2+6n^3 \\ 6+8n-12n^2-8n^3 & -4+n-n^3 & -2-9n-9n^2+6n^3 & -4+n-n^3 & 6+8n-12n^2-8n^3 \\ -2-9n-9n^2+6n^3 & 4+6n^3 & -2+6n+6n^2+6n^3 & 4+6n^3 & -2-9n-9n^2+6n^3 \\ -6+5n+15n^2-8n^3 & -4+n-n^3 & 1-3n+6n^3 & -4+n-n^3 & -6+5n+15n^2-8n^3 \\ 2n-3n^2-8n^3 & 4+6n^3 & -6+5n+15n^2-8n^3 & 4+6n^3 & 2n-3n^2-8n^3 \\ 1-3n+6n^3 & 4+6n^3 & 6+8n-12n^2-8n^3 & 4+6n^3 & 1-3n+6n^3 \end{bmatrix}.$$

Formula (15) is, in the case $m = p$ prime, equivalent to the following:

$$(i, j) = -\frac{1}{p} \left(\delta_{0,i} + \delta_{0,j} + \delta_{i,j} - f - 1 + \sum_{u=1}^{p-2} d_{u,i+ju} \right),$$

where $f = (q - 1)/p$ (see, for example, [11], formula 7). Using this and (4), we calculate the matrix $C = [c_{i,j}]$. We have

$$C = \begin{bmatrix} X_0 - f & X_1 - f & X_2 - f & X_3 - f & X_4 - f & X_5 - f & X_6 - f \\ X_1 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_7 \\ X_2 & X_7 & X_5 & X_{10} & X_{11} & X_{11} & X_8 \\ X_3 & X_8 & X_{10} & X_4 & X_9 & X_{11} & X_9 \\ X_4 & X_9 & X_{11} & X_9 & X_3 & X_8 & X_{10} \\ X_5 & X_{10} & X_{11} & X_{11} & X_8 & X_2 & X_7 \\ X_6 & X_7 & X_8 & X_9 & X_{10} & X_7 & X_1 \end{bmatrix},$$

where $f = 56n^6 + 14n^4 + 23n^3 + 2n^2 - 5n + 16$ and

$$X_0 = 8n^6 + 2n^4 + 5n^3 - n^2 + 4n,$$

$$X_1 = 4 + 8n^6 + 2n^4 + 3n^3 - n^2 - 3n,$$

$$X_2 = 8n^6 + 2n^4 + 5n^3 + 5n^2 - 2n + 2,$$

$$X_3 = 8n^6 + 2n^4 - n^3 + 2n^2 + n + 2,$$

$$X_4 = 8n^6 + 2n^4 + 5n^3 - 4n^2 - 2n + 5,$$

$$X_5 = 8n^6 + 2n^4 + 5n^3 - n^2 - 2n + 2,$$

$$X_6 = 8n^6 + 2n^4 + n^3 + 2n^2 - n,$$

$$X_7 = 8n^6 + 2n^4 + 2n^3 - n^2 + n + 3,$$

$$X_8 = 8n^6 + 2n^4 + 6n^3 - n^2 - 3n + 2,$$

$$X_9 = 8n^6 + 2n^4 + 4n^3 + 2n^2 - n + 1,$$

$$X_{10} = 8n^6 + 2n^4 + 5n^3 + 2n^2 + n + 3,$$

$$X_{11} = 8n^6 + 2n^4 - n^2 + 2.$$

Therefore, by (3), for all primes of the form

$$q = 392n^6 + 98n^4 + 161n^3 + 14n^2 - 35n + 113,$$

the irreducible polynomials of the Gaussian periods of degree 7 in $\mathbb{Q}(\zeta_q)$ are

$$\begin{aligned}
 P_q(x) = \det(xI - C) = & x^7 + x^6 + (-168n^6 - 42n^4 - 69n^3 - 6n^2 + 15n - 48)x^5 \\
 & + (-224n^9 + 168n^8 - 672n^7 + 78n^6 - 93n^5 \\
 & \quad - 195n^4 - 49n^3 + 108n^2 - 189n + 37)x^4 \\
 & + (6608n^{12} + 2856n^{11} + 28n^{10} + 6140n^9 \\
 & \quad + 1251n^8 + 1395n^7 + 3850n^6 + 1635n^5 \\
 & \quad + 338n^4 + 1271n^3 - 57n^2 + 443n + 312)x^3 \\
 & + (14784n^{15} + 12768n^{14} + 23856n^{13} + 8184n^{12} \\
 & \quad + 8100n^{11} + 26226n^{10} + 4935n^9 + 4377n^8 \\
 & \quad + 16176n^7 + 1200n^6 - 2373n^5 \\
 & \quad + 6063n^4 + 792n^3 - 501n^2 + 573n - 12)x^2 \\
 & + (-36736n^{18} + 41664n^{17} + 64176n^{16} - 122352n^{15} \\
 & \quad - 30492n^{14} + 16518n^{13} - 146848n^{12} \\
 & \quad - 50097n^{11} + 22722n^{10} - 82665n^9 - 46842n^8 \\
 & \quad + 3279n^7 - 29398n^6 - 16158n^5 + 1698n^4 \\
 & \quad - 4317n^3 - 4050n^2 - 894n - 49)x \\
 & - 33664n^{21} + 146496n^{20} + 24640n^{19} - 276528n^{18} \\
 & - 158904n^{17} - 275688n^{16} - 447508n^{15} - 216771n^{14} \\
 & - 185387n^{13} - 290411n^{12} - 179430n^{11} - 127792n^{10} \\
 & - 130448n^9 - 65166n^8 - 28901n^7 - 26116n^6 \\
 & - 18399n^5 - 9110n^4 - 2993n^3 - 519n^2 - 39n - 1.
 \end{aligned}$$

Example 6. Let $m = 23$, $r_1 = 47$ and $R_1 = (1 + \zeta_{23}^2 - \zeta_{23}^3, 47)$ (a nonprincipal prime ideal of $\mathbb{Z}[\zeta_{23}]$; see, for example, [3], page 104). Set

$$\mathcal{A}_4 = \{ \alpha = 1 + \zeta_{23}^2 - \zeta_{23}^3 + 47n : n \in \mathbb{Z} \text{ and } N_{\mathbb{Q}(\zeta_{23})/\mathbb{Q}}(\alpha) = 47q, \text{ with } q \in S \},$$

and

$$\mathcal{I} = \{ Q = (\alpha)R_1^{-1} : \alpha \in \mathcal{A}_4 \}.$$

With notation as in (18), put $\tilde{\mathfrak{J}}_{a,b} = \tilde{\mathfrak{J}}_{1,a,b}$, and $s_1 = -2$, which is a primitive root modulo 47 such that $s_1^{(47-1)/23} = (-2)^2 \equiv \zeta_{23} \pmod{R_1}$. Using the MAPLE program at the end of Section 1, with $m = 23$, $q = 47$ and $s = -2$, we find that

$$\begin{aligned}
 \tilde{\mathfrak{J}}_{1,1} = & 2 - 2\zeta_{23}^2 + 2\zeta_{23}^8 - 2\zeta_{23}^9 + 2\zeta_{23}^{12} + 2\zeta_{23}^{13} \\
 & + 2\zeta_{23}^{14} - 2\zeta_{23}^{15} + 2\zeta_{23}^{16} - 2\zeta_{23}^{18} - 2\zeta_{23}^{20} - \zeta_{23}^{21}.
 \end{aligned}$$

For $\alpha \in \mathcal{A}_4$, let

$$\mathfrak{K}_{1,1}[\alpha] = \prod_{c=1}^{22} \sigma_c^{-1}(\bar{\alpha})^{\lfloor \frac{2c}{23} \rfloor}.$$

We can obtain the family of Jacobi sums $J_{1,1}[Q]$, $Q \in \mathcal{I}$, using the formula

$$J_{1,1}[Q] = \left(\frac{n+1}{23}\right) \zeta_{23}^{-k} \mathfrak{R}_{1,1}[\alpha] / \mathfrak{J}_{1,1} = \left(\frac{n+1}{23}\right) \zeta_{23}^{-k} \mathfrak{R}_{1,1}[\alpha] \bar{\mathfrak{J}}_{1,1} / 47,$$

where $(\alpha) = R_1 Q$, $\alpha \in \mathcal{A}_4$, $\left(\frac{\cdot}{23}\right)$ is the Legendre symbol, and

$$k \equiv 11 \left(\frac{n+1}{23}\right) (n+1)^{10} \pmod{23}.$$

To prove this equality, check that the numbers on both sides generate the same ideals in $\mathbb{Z}[\zeta_{23}]$, and that the right hand side is $\equiv 1 \pmod{(\zeta_{23} - 1)^2}$. We do not write the expanded expression of $J_{1,1}[Q]$ in $\mathbb{Z}[\zeta_{23}, n]$, since it occupies more than one page.

Proceeding in a similar way we can find all the families of Jacobi sums $J_{1,1}[Q], \dots, J_{1,21}[Q]$, $Q \in \mathcal{I}$. With these families we can construct, using (3), (4), and (15) (or better [11], formulas (6) and (7), as in Example 4), the family of irreducible polynomials $P_q(x) \in \mathbb{Z}[n, x]$, of Gaussian periods of degree 23, corresponding to the primes of the form

$$\begin{aligned} q = q(n) = & 130033429462229783044185156533092847n^{22} \\ & + 60866711663171387807916456249532822n^{21} \\ & + 13597882392836161106023889162129673n^{20} \\ & + 1928777644373923561138140306685060n^{19} \\ & + 194929655548428445008641839505405n^{18} \\ & + 14930782127113668128321502600414n^{17} \\ & + 900082610499760135395267887259n^{16} \\ & + 43773014492389550657520626736n^{15} \\ & + 1746389479019419656026933311n^{14} \\ & + 57795967528053201788638220n^{13} + 1594119954503408569331187n^{12} \\ & + 36397389727152969816873n^{11} + 666486961951621859180n^{10} \\ & + 8874252237258368851n^9 + 54335329669656750n^8 \\ & - 992442355341030n^7 - 37699732250660n^6 - 646801716550n^5 \\ & - 6475959625n^4 - 5641786n^3 + 1224820n^2 + 22033n + 139. \end{aligned}$$

These primes are norms of the prime ideals in \mathcal{I} . Note that the prime ideals in $\mathbb{Q}[\zeta_{23}]$ above primes of the form $q(n)$ are not principal.

In the following program enter the values of m , an integer > 2 , and F , a polynomial function in z , with coefficients depending on one or more parameters n_1, \dots, n_k , which, when z is replaced by ζ_m and the n_i by integers, gives elements of $\mathbb{Z}[\zeta_m]$ that are either $\equiv 1 \pmod{m^2}$, or $\equiv 1$ modulo a smaller divisor of m^2 , provided that the resulting matrix C still has its entries in $\mathbb{Z}[n_1, \dots, n_k]$ (these entries are always in $\mathbb{Q}[n_1, \dots, n_k]$). The smallest such divisor of m^2 for which the program works is, likely, the conductor of the Hecke character defined in Weil’s article [14], which we called f in the discussion above. The resulting value of q must be irreducible in $\mathbb{Z}[n_1, \dots, n_k]$. (With the help of a computer it is easy to check that in fact

the matrix $C = [c_{i,j}]$ satisfies the conditions of [12], Proposition 2, or, equivalently, that the matrix H , whose entry in row a and column b is equal to the Jacobi sum $J_{a,b}$ when $m \nmid a + b$, satisfies the conditions of Propositions 2 and 3.) The resulting polynomial P gives, for all values of the parameters such that $q = q(n_1, \dots, n_k)$ is a prime, the irreducible polynomials of the Gaussian periods of degree m in $\mathbb{Q}(\zeta_q)$.

A MAPLE program to find families of irreducible polynomials of Gaussian periods of degree m for arbitrary $m > 2$

```
with(numtheory): with(linalg):
m:=10; F:=z->1+n*10*(z-z^(m-1));
R:=cyclotomic(m,z);
for i0 from 0 to m-1 do;
T[i0]:=modp(i0^(phi(m)-1),m); od;
for i1 from 0 to m-1 do;
if igcd(i1,m)=1 then t[i1]:=1;
else t[i1]:=0; fi; od;
q:=rem(expand(product(F(z^c)^t[c],c=0..m-1)),R,z);
factor(q);
f:=(q-1)/m; A:=array(1..m-1,1..m-1,1..m):
for i2 from 1 to m-1 do;
for j2 from 1 to m-1 do;
for k2 from 1 to m do;
A[i2,j2,k2]:=((floor((i2+j2)*(k2-1)/m)-floor(i2*(k2-1)/m)- floor(j2*(k2-1)/m))*t[k2-1];
od: od: B:=array(1..m-1,1..m-1):
for i3 from 1 to m-1 do;
for j3 from 1 to m-1 do;
B[i3,j3]:=expand(product(F(z^(m-T[k3-1]))^A[i3,j3,k3], k3=1..m),z); od: od:
H:=array(1..m-1,1..m-1):
for i4 from 1 to m-1 do;
for j4 from 1 to m-1 do;
H[i4,j4]:=sort(collect(rem(B[i4,j4],R,z),z)); od: od:
evalm(H);
Id:=array(identity,1..m,1..m):
C:=array(1..m,1..m):
for i5 from 1 to m do;
for j5 from 1 to m do;
C[i5,j5]:=rem(-f*Id[1,i5]+(-1/m^2)*(m*Id[1,i5]+m*Id[1,j5]+m*Id[i5,j5]-q-1+
sum(sum(z^((m-i5+1)*a+(m-j5+1)*b)*H[a,b],a=1..m-1), b=1..m-1)-
sum(z^((m-i5+j5)*l)*H[1,m-l],l=1..m-1)),R,z); od: od:
evalm(C);
P:=sort(collect(charpoly(C,x),x));
```

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