

MERGING THE BRAMBLE-PASCIAK-STEINBACH  
AND THE CROUZEIX-THOMÉE CRITERION  
FOR  $H^1$ -STABILITY OF THE  $L^2$ -PROJECTION  
ONTO FINITE ELEMENT SPACES

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ABSTRACT. Suppose  $\mathcal{S} \subset H^1(\Omega)$  is a finite-dimensional linear space based on a triangulation  $\mathcal{T}$  of a domain  $\Omega$ , and let  $\Pi : L^2(\Omega) \rightarrow L^2(\Omega)$  denote the  $L^2$ -projection onto  $\mathcal{S}$ . Provided the mass matrix of each element  $T \in \mathcal{T}$  and the surrounding mesh-sizes obey the inequalities due to Bramble, Pasciak, and Steinbach or that neighboring element-sizes obey the global growth-condition due to Crouzeix and Thomée,  $\Pi$  is  $H^1$ -stable: For all  $u \in H^1(\Omega)$  we have  $\|\Pi u\|_{H^1(\Omega)} \leq C \|u\|_{H^1(\Omega)}$  with a constant  $C$  that is independent of, e.g., the dimension of  $\mathcal{S}$ .

This paper provides a more flexible version of the Bramble-Pasciak-Steinbach criterion for  $H^1$ -stability on an abstract level. In its general version, (i) the criterion is applicable to *all* kind of finite element spaces and yields, in particular,  $H^1$ -stability for nonconforming schemes on arbitrary (shape-regular) meshes; (ii) it is *weaker than* (i.e., implied by) *either* the Bramble-Pasciak-Steinbach *or* the Crouzeix-Thomée criterion for regular triangulations into triangles; (iii) it guarantees  $H^1$ -stability of  $\Pi$  a priori for a class of *adaptively-refined* triangulations into right isosceles triangles.

1. THE  $L^2$ -PROJECTION IN A FINITE ELEMENT SPACE

Suppose the bounded Lipschitz domain  $\Omega$  in  $\mathbb{R}^d$  is partitioned into a triangulation  $\mathcal{T}$ , i.e.,  $\bar{\Omega} = \bigcup \mathcal{T}$  for a finite set  $\mathcal{T}$  of elements  $T$  which are closed and whose interiors are Lipschitz domains. The intersection of two distinct elements has zero  $d$ -dimensional Lebesgue measure. To describe nonconforming finite elements, let  $H$  be a closed subset of  $H^1(\mathcal{T})$ ,

$$(1) \quad H_0^1(\Omega) \subseteq H \subset H^1(\mathcal{T}) := \{u \in L^2(\Omega) : \forall T \in \mathcal{T}, u|_T \in H^1(T)\},$$

closed with respect to the semi-norm  $\|\nabla_{\mathcal{T}} \cdot\|$ , where  $\|\cdot\|$  denotes the  $L^2(\Omega)$ -norm and  $\nabla_{\mathcal{T}}$  is the  $\mathcal{T}$ -piecewise action of the gradient  $\nabla$  (different from the distributional gradient for discontinuous arguments). For instance, in the conforming setting, the choice of  $H = H_0^1(\Omega)$  or  $H = H^1(\Omega)$  is a typical example.

Suppose that  $\mathcal{S} \subset H$  is an  $n$ -dimensional subspace with a (not necessarily nodal) basis  $(\varphi_1, \varphi_2, \dots, \varphi_n)$ , and let  $\Pi$  denote the  $L^2(\Omega)$ -projection defined, for all  $u \in H$ ,

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by

$$(2) \quad \Pi u \in \mathcal{S} \quad \text{and} \quad \int_{\Omega} (u - \Pi u) \varphi_j dx = 0 \quad \text{for all } j = 1, \dots, n.$$

In this context, the  $L^2$ -projection  $\Pi$  is called  $H^1$ -stable if there exists a constant  $c_1 > 0$  with

$$(3) \quad \|\nabla_{\mathcal{T}} \Pi u\| \leq c_1 \|\nabla_{\mathcal{T}} u\| \quad \text{for all } u \in H.$$

Two sets of parameters, the  $n$  positive parameters  $(d_1, d_2, \dots, d_n)$  and the  $\mathcal{T}$ -piecewise constant weight  $h_{\mathcal{T}}$ , defined on  $T \in \mathcal{T}$  by  $h_{\mathcal{T}} > 0$ , will provide the link between the triangulation  $\mathcal{T}$  and the discrete space  $\mathcal{S}$ . Their choice is arbitrary up to the severe restriction of inequality (7) below.

To verify  $H^1$ -stability of the  $L^2$ -projection (3) we suppose that there exist a (possibly nonlinear) mapping  $P : H \rightarrow \mathcal{S}$  and a constant  $c_2 > 0$  that satisfy, for all  $u \in H$ ,

$$(4) \quad \|\nabla_{\mathcal{T}} P(u)\| + \|h_{\mathcal{T}}^{-1}(u - P(u))\| \leq c_2 \|\nabla_{\mathcal{T}} u\|.$$

*Remark 1.* In Sections 4, 5, and 6,  $h_{\mathcal{T}}$  will be the element-size and  $d_{\ell}$  a measure for the size of  $\text{supp } \varphi_{\ell}$ .

*Remark 2.* Approximation operators which satisfy (4) for  $h_{\mathcal{T}} = \text{diam}(T)$  can be found in [Ca, CF, Cl].

## 2. MASS MATRICES AND TWO INEQUALITIES

To define the mass matrix for a given  $T \in \mathcal{T}$ , let  $\ell(T, 1), \ell(T, 2), \dots, \ell(T, m(T))$  denote exactly those indices of basis functions whose restrictions  $\psi_{T,j} := \varphi_{\ell(T,j)}|_T \in H^1(T)$ ,  $1 \leq j \leq m(T)$ , on  $T$  are nonzero. Then the shape functions  $(\psi_{T,j} : j = 1, \dots, m(T))$  on  $T$  satisfy an inverse inequality (by equivalence of norms),

$$(5) \quad \left\| \sum_{j=1}^{m(T)} \xi_j \nabla \psi_{T,j} \right\|_{L^2(T)} \leq c_3 h_{\mathcal{T}}^{-1} \left\| \sum_{j=1}^{m(T)} \xi_j \psi_{T,j} \right\|_{L^2(T)}$$

for all  $(\xi_1, \dots, \xi_{m(T)}) \in \mathbb{R}^{m(T)}$ .

The (local)  $m(T) \times m(T)$ -dimensional mass matrix  $M(T)$  and the diagonal matrix  $\Lambda(T)$ ,

$$(6) \quad \Lambda(T)_{jk} = \frac{h_{\mathcal{T}}}{d_{\ell(T,j)}} \delta_{jk} \quad \text{and} \quad M(T)_{jk} = \int_T \psi_{T,j} \psi_{T,k} dx \quad \text{for all } j, k = 1, \dots, m(T),$$

( $\delta_{jk} \in \{0, 1\}$  denotes Kronecker's symbol) are supposed to satisfy, for constants  $c_4, c_5 > 0$ ,

$$(7) \quad c_4^2 x \cdot \Lambda(T)^2 M(T) \Lambda(T)^2 x \leq x \cdot M(T) x \leq c_5 x \cdot \Lambda(T)^2 M(T) x \quad \text{for all } x \in \mathbb{R}^{m(T)}.$$

*Remark 3.* Inverse estimates [BS, Ci] provide (5) for a size-independent constant  $c_3$  if  $h_{\mathcal{T}} = \text{diam}(T)$ .

*Remark 4.* The first inequality of (7) merely reflects a proper scaling of  $d_{\ell(T,j)}$  and  $h_{\mathcal{T}}$ .

*Remark 5.* The second inequality of (7) implies that  $\Lambda(T)^2 M(T)$  has positive definite symmetric part. This is the crucial condition and relates the mass-matrix  $M(T)$  to neighboring mesh-sizes.

*Remark 6.* We stress that (7) can *always* be satisfied even with  $c_4 = c_5 = 1$  if we let  $h_T = d_{\ell(T,j)}$  be equal to a global discretization parameter. For quasi-uniform meshes this implies (7).

*Remark 7.* In the original version [BPS, S],  $d_j$  is fixed as the arithmetic mean of all  $h_T$  with  $T \subset \text{supp } \varphi_j$ , where  $h_T^d$  is the  $d$ -dimensional volume of an element  $T \in \mathcal{T}$ . Then, the Bramble-Pasciak-Steinbach criterion [BPS, (4.2)] implies the crucial second inequality in (7) (and is, in particular situations, equivalent).

### 3. A MODIFIED BRAMBLE-PASCIAK-STEINBACH CRITERION FOR $H^1$ -STABILITY

Under the present assumptions (1)-(2) and (4)-(7) we have  $H^1$ -stability of  $\Pi$ .

**Theorem 1.** *We have (3) with  $c_1 = c_2 \max\{1, c_3 c_5/c_4\}$ .*

The proof is a review of arguments in [BPS] in an abstract setting, and is included here for completeness. Theorem 1 implies the Bramble-Pasciak-Steinbach criterion [BPS] for a special choice of  $h_T$  and  $d_j$  (of Remark 7).

*Proof.* Given  $u \in H$ , define  $q_h := P(u) - \Pi u = \sum_{\ell=1}^n q_\ell \varphi_\ell \in \mathcal{S}$  and  $p_h := \sum_{\ell=1}^n q_\ell d_\ell^{-2} \varphi_\ell \in \mathcal{S}$  so that

$$(8) \quad q_h|_T = \sum_{\ell=1}^n q_\ell \varphi_\ell|_T = \sum_{j=1}^{m(T)} \xi_{T,j} \psi_{T,j} \quad \text{on } T \in \mathcal{T}$$

for certain coefficient vectors  $x_T = (\xi_{T,1}, \dots, \xi_{T,m(T)}) = (q_{\ell(T,1)}, \dots, q_{\ell(T,m(T))})$ . The triangle inequality for  $\Pi u = P(u) - q_h$  and (4)-(5) show that

$$(9) \quad \|\nabla_T \Pi u\| \leq \|\nabla_T P(u)\| + \|\nabla_T q_h\| \leq c_2 \|\nabla_T u\| + c_3 \|h_T^{-1} q_h\|.$$

According to direct calculations with coefficients from (8), the second inequality in (7) yields

$$(10) \quad \begin{aligned} c_5^{-1} \|h_T^{-1} q_h\|^2 &= c_5^{-1} \sum_{T \in \mathcal{T}} h_T^{-2} x_T \cdot M(T) x_T \leq \sum_{T \in \mathcal{T}} h_T^{-2} x_T \cdot \Lambda(T)^2 M(T) x_T \\ &= \sum_{T \in \mathcal{T}} \sum_{j=1}^{m(T)} \frac{q_{\ell(T,j)}}{d_{\ell(T,j)}^2} \int_T \varphi_{\ell(T,j)} q_h \, dx = \int_\Omega p_h q_h \, dx \\ &= \int_\Omega p_h (P(u) - u) \, dx \leq c_2 \|h_T p_h\| \|\nabla_T u\| \end{aligned}$$

because of (2), Cauchy's inequality, and (4). Similar arguments and (7) lead to

$$\begin{aligned} c_4^2 \|h_T p_h\|^2 &= c_4^2 \sum_{T \in \mathcal{T}} h_T^{-2} x_T \cdot \Lambda(T)^2 M(T) \Lambda(T)^2 x_T \\ &\leq \sum_{T \in \mathcal{T}} h_T^{-2} x_T \cdot M(T) x_T = \|h_T^{-1} q_h\|^2. \end{aligned}$$

Utilizing this in (10), we obtain a bound of  $\|h_T^{-1} q_h\|$ , which we need in (9) to see (4).  $\square$

## 4. EXAMPLES FOR COURANT TRIANGLES

Suppose  $\mathcal{T}$  is a regular triangulation (in the sense of Ciarlet [BS, Ci]) of the bounded Lipschitz domain  $\Omega$  in the plane into triangles. Homogeneous Dirichlet conditions may apply on a (relatively closed and possibly empty) boundary part  $\Gamma_D$  (matched exactly by edges). Each node  $z \in \mathcal{N}$  with nodal basis function  $\varphi_z$  involves a positive real number  $d_z$  such that  $h_T/d_z + d_z/h_T \leq c_6$  for all triangles  $T \in \mathcal{T}$  of diameter  $h_T$  with vertex  $z$ . Let  $\mathcal{S} := \text{span}\{\varphi_z : z \in \mathcal{K}\}$ , where  $\mathcal{K} := \mathcal{N} \setminus \Gamma_D$  denotes the set of free nodes, and for the preceding notation identify  $(\varphi_z : z \in \mathcal{K})$  and the parameters  $(d_z : z \in \mathcal{K})$  with  $(\varphi_1, \varphi_2, \dots, \varphi_n)$  and  $(d_1, d_2, \dots, d_n)$ , respectively.

**Theorem 2.** *Suppose that  $d_z/d_\zeta \leq \kappa < \sqrt{2} + \sqrt{3} \approx 3.1462$  for all vertices  $z$  and  $\zeta$  of some triangle  $T \in \mathcal{T}$ . Then we have (3).*

*Proof.* The mass-matrix of a fixed  $T \in \mathcal{T}$  is a multiple of the  $3 \times 3$  matrix  $M$  with  $M_{jk} = 1 + \delta_{jk}$  and  $\Lambda(T)$  has diagonal entries  $\lambda_1, \lambda_2, \lambda_3 > 0$  with  $\lambda_j/\lambda_k \leq \kappa$ . The eigenvalues of  $\Lambda(T)^{-1}A\Lambda(T)^{-1}$  for  $A := (\Lambda(T)^2M + M\Lambda(T)^2)/2$  can be calculated [BPS, S], and their smallest value is  $(5 - \mu)$  for  $\mu^2 := \sum_{j,k=1}^3 \lambda_j^2/\lambda_k^2$ . A straightforward analysis reveals that  $\mu^2 \leq 3 + 2(1 + \kappa^2 + 1/\kappa^2) < 25$ , which shows that  $A$  is positive definite. Therefore,  $(x \cdot Ax)^{1/2}$  defines a norm which is equivalent to  $|x|$  in  $\mathbb{R}^3$ . This and  $h_T/d_z \leq c_6$  yield (7).  $\square$

*Remark 8.* The proof shows that  $\sum_{j,k=1}^3 \lambda_j^2/\lambda_k^2 \leq \nu < 22$  for some constant  $\nu$  suffices for (3). Given  $d_j$  as in Remark 7, this is the a posteriori criterion of [BPS, S] for two dimensions.

The technical assumption on the artificial, extended triangulation in the following theorem merely reduces the consideration to interior triangles for brevity.

**Theorem 3.** *Suppose  $\mathcal{T} \subset \hat{\mathcal{T}}$  for some regular triangulation  $\hat{\mathcal{T}}$  of a Lipschitz domain  $\hat{\Omega} \supset \Omega$  such that  $\hat{\mathcal{T}}$  consists of right isosceles triangles only, there are no hanging nodes, and each free node on the boundary is an interior node of  $\hat{\Omega}$ . Then we have (3).*

*Proof.* Theorem 2 yields the assertion if we take

$$d_z = \min\{|z - \zeta| : \zeta \in \mathcal{N}, \delta(z, \zeta) = 1\},$$

where  $\delta(z, \zeta) = 1$  characterizes neighboring vertices  $z$  and  $\zeta$ , i.e.,  $z, \zeta \in T \cap \mathcal{N}$  for at least one  $T \in \mathcal{T}$ . Since (up to scaling, transition, and rotation) there are only

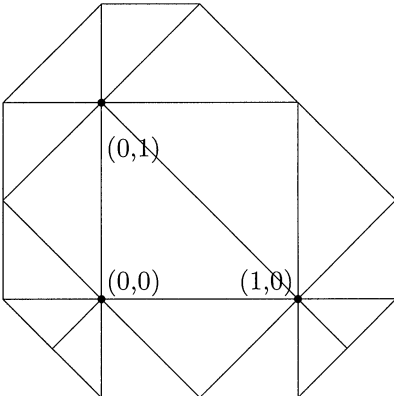


Figure 1: Part of a mesh as a smallest neighborhood of the reference triangle.

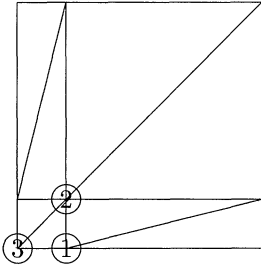


Figure 2: Reference mesh for comparisons in Example 1.

a finite number of possible configurations, it can be checked by a finite number of figures that  $d_z/d_\zeta \leq \sqrt{8}$ . Figure 1 illustrates a deduction: Suppose  $T$  has the vertices  $(0, 0)$ ,  $(1, 0)$ , and  $(0, 1)$ . Then, the patch  $\text{supp } \varphi_z$  of  $z = (0, 0)$  must include the polygonal domain with vertices  $(0, 1)$ ,  $(-.5, .5)$ ,  $(-.5, 0)$ ,  $(0, -.5)$ ,  $(.5, -.5)$ ,  $(1, 0)$ . This shows that  $1/\sqrt{8} \leq d_{(1,0)} \leq 1$ . Similarly, the patch  $\text{supp } \varphi_{(1,0)}$  must include the polygon  $(0, 0)$ ,  $(.5, -.5)$ ,  $(1, -.5)$ ,  $(1.5, 0)$ ,  $(1.5, 3)$ ,  $(1, 1)$ ,  $(0, 1)$ , whence  $1/\sqrt{8} \leq d_{(1,0)}, d_{(0,1)} \leq 1$ . Consequently,  $d_z/d_\zeta \leq \sqrt{8}$  for any choice of two vertices  $z$  and  $\zeta$  of  $T$ .  $\square$

**Example 1.** Let  $\mathcal{T}$  be the mesh of Figure 2 that consists of 8 triangles in a regular pattern that match the square  $\Omega := (0, H)^2$  for positive  $H = 1 + \lambda$ , where non-diagonals' lengths are either  $\lambda < 1$  or 1. For the nodes 1, 2, and 3 of Figure 2 the choice of  $(d_1, d_2, d_3)$  from [BPS], mentioned in Remark 7, is

$$((\lambda + 2\lambda^{1/2})/3, (\lambda^2 + \lambda^{1/2} + 1)/3, \lambda)/\sqrt{2}.$$

The conditions of the Bramble-Pasciak-Steinbach criterion (cf. Remark 8) and those of Theorem 2 are violated for  $\lambda < .1349$ , which corresponds to an aspect ratio larger than 7.4122. However, Theorem 1 with the parameters from Remark 6 guarantees (3) for any positive  $\lambda$  (with a  $\lambda$ -dependent constant  $c_1 = c_1(\lambda)$ ).

**Example 2.** Take a scaled copy of  $\Omega$  and the mesh from Example 1 and extend it by reflection about the  $x_1$ -axis, the  $x_2$ -axis, and about the anti-diagonal through the origin to  $h(-1, 1)^2$ ; and then extend it  $2h$ -periodically to the entire plane. The calculations of Example 1 remain valid and we conclude that, for a fixed  $\lambda < .1349$ , the Bramble-Pasciak-Steinbach criterion is not applicable, but Remark 6 (or the Crouzeix-Thomée criterion) guarantees (3) with an  $h$ -independent constant  $c_1 = c_1(\lambda)$ .

The nonconforming Crouzeix-Raviart finite element (cf., e.g., [BS, Ci]) concludes our first series of applications.

**Theorem 4.** *Suppose  $T$  is an arbitrary shape-regular triangulation into triangles and  $\mathcal{S}$  denotes the  $\mathcal{T}$ -piecewise affine functions which are continuous at midpoints of edges. Then we have (3).*

*Proof.* The mass-matrices are diagonal, so (7) is a consequence of shape-regularity. The operator  $P$  can be chosen exactly as in the conforming case.  $\square$

5. WEAKENING OF THE CROUZEIX-THOMÉE CRITERION FOR  $H^1$ -STABILITY

Part of the Crouzeix-Thomé criterion [CT] is the existence of  $c_7$  and  $1 \leq \kappa := \sqrt{\alpha} < \sqrt{2} + \sqrt{3}$  such that

$$(11) \quad |T_1|/|T_2| \leq c_7 \alpha^{l(T_1, T_2)} \quad \text{for all } T_1, T_2 \in \mathcal{T}.$$

Here,  $|T_j|$  is the area of  $T_j \in \mathcal{T}$  and the neighbor-index  $l(T_1, T_2)$  might be defined via a metric  $\delta$  on the nodes  $\mathcal{N}$ : For two distinct nodes  $z$  and  $\zeta$ ,  $\delta(z, \zeta)$  is the smallest integer  $j$  such that there exists a polygon  $(z_1, z_2, \dots, z_j)$  of nodes  $z_1, \dots, z_j \in \mathcal{N}$  which connects  $z = z_1$  with  $\zeta = z_j$  along edges, i.e.,  $\{z_i, z_{i+1}\} \subset \partial T_i$  for some  $T_i \in \mathcal{T}$  and all  $i = 1, \dots, j-1$ ;  $\delta(z, z) := 0$ . For any  $T, K \in \mathcal{T}$  and  $z \in \mathcal{N}$ , let  $\delta(z, T) := \min_{\zeta \in T \cap \mathcal{N}} \delta(z, \zeta)$  and  $\delta(K, T) = \min_{z \in K \cap \mathcal{N}} \delta(z, T)$ . Then,  $l(T_1, T_2) = \delta(T_1, T_2) + 1$  if  $T_1 \neq T_2$ , while  $l(T_1, T_2) = 0$  if and only if  $T_1 = T_2$ .

At first glance, the *local* Bramble-Pasciak-Steinbach and the *global* Crouzeix-Thomé criteria appear incomparable: a large constant  $c_7$  prohibits a direct application of (11) in the spirit of Theorems 2 and 3 (as  $d_z/d_\zeta \leq c_8 (|T_1|/|T_2|)^{1/2} \leq c_7^{1/2} c_8 \kappa \not\leq \sqrt{2} + \sqrt{3}$  for  $\delta(z, \zeta) = 1$ ). However, all necessities are provided by

$$(12) \quad d_z := \min_{T \in \mathcal{T}} h_T \kappa^{\delta(z, T)} \quad \text{for all } z \in \mathcal{N} \quad \text{and} \quad h_T := |T|^{1/2} \quad \text{for all } T \in \mathcal{T}.$$

**Theorem 5.** *Suppose (11)-(12) hold for a planar regular triangulation  $\mathcal{T}$ . Then, the conditions of Theorem 2 are satisfied and we have (3).*

*Proof.* Given  $z \in K \in \mathcal{T}$ , we have  $d_z \leq h_K$  ( $K$  is allowed in the minimization (12), and  $\delta(z, K) = 0$ ) and  $l(T, K) - \delta(z, T) \leq 1$ . With a minimizing  $T \in \mathcal{T}$  in (12), (11) shows that

$$(13) \quad h_K/d_z = \frac{h_K}{h_T \kappa^{\delta(z, T)}} = \frac{|K|^{1/2}}{|T|^{1/2}} \kappa^{-\delta(z, T)} \leq \sqrt{c_7} \kappa^{l(T, K) - \delta(z, T)} \leq \sqrt{c_7} \kappa.$$

To bound  $d_z/d_\zeta$  for  $z, \zeta \in \mathcal{N}$  with  $\delta(z, \zeta) = 1$ , let  $K \in \mathcal{T}$  satisfy  $d_\zeta = h_K \kappa^{\delta(z, K)}$ . The definition (12) and  $\delta(z, K) - \delta(\zeta, K) \leq 1$  show that

$$(14) \quad d_z/d_\zeta = \frac{d_z}{h_K \kappa^{\delta(\zeta, K)}} \leq \frac{h_K \kappa^{\delta(z, K)}}{h_K \kappa^{\delta(\zeta, K)}} = \kappa^{\delta(z, K) - \delta(\zeta, K)} \leq \kappa. \quad \square$$

**Example 3.** There exists an adaptively-refined mesh [CV, Figure 1] of right isosceles triangles where the modified Bramble-Pasciak-Steinbach criterion guarantees  $H^1$ -stability (cf. [S] or Theorem 3) while the Crouzeix-Thomé criterion is not applicable.

## REFERENCES

- [BPS] J. H. BRAMBLE, J. E. PASCIAK, O. STEINBACH: On the stability of the  $L_2$ -projection in  $H^1(\Omega)$ . *Math. Comp.* (this journal, previous article).
- [BS] S. C. BRENNER, L. R. SCOTT: *The Mathematical Theory of Finite Element Methods*. Texts Appl. Math. **15**, Springer, New-York (1994). MR **95f**:65001
- [Ca] C. CARSTENSEN: Quasi-interpolation and a posteriori error analysis in finite element method. *M2AN Math. Model. Numer. Anal.* **33** (1999) 1187-1202. MR **2001a**:65135
- [CF] C. CARSTENSEN, S. A. FUNKEN: Constants in Clément-interpolation error and residual based a posteriori estimates in finite element methods. *East-West Journal of Numerical Analysis* **8** (2000) 153-175.
- [CV] C. CARSTENSEN, R. VERFÜRTH: Edge residuals dominate a posteriori error estimates for low order finite element methods. *SIAM J. Numer. Anal.* **36** 5 (1999) 1571-1587. MR **2000g**:65115

- [Ci] P. G. CIARLET: *The finite element method for elliptic problems*. North-Holland, Amsterdam (1978). MR **58**:25001
- [Cl] P. CLÉMENT: Approximation by finite element functions using local regularization. *RAIRO Sér. Rouge Anal. Numér.* **9**, **R-2** (1975) 77–84. MR **53**:4569
- [CT] M. CROUZEIX AND V. THOMÉE: The stability in  $L^p$  and  $W^{1,p}$  of the  $L^2$ -projection onto finite element function spaces. *Math. Comput.* **48** (1987) 521–532. MR **88f**:41016
- [S] O. STEINBACH: On the stability of the  $L_2$  projection in fractional Sobolev spaces. *Numerische Mathematik* (2000) published online October 16, 2000.

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