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ASYMPTOTIC ESTIMATION OF GAUSSIAN QUADRATURE ERROR FOR A NONSINGULAR INTEGRAL IN POTENTIAL THEORY

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ABSTRACT. This paper considers the n-point Gauss-Jacobi approximation of nonsingular integrals of the form $\int_{-1}^{1} \mu(t)\phi(t)\log(z-t)\,\mathrm{d}t$, with Jacobi weight μ and polynomial ϕ, and derives an estimate for the quadrature error that is asymptotic as $n \to \infty$. The approach follows that previously described by Donaldson and Elliott. A numerical example illustrating the accuracy of the asymptotic estimate is presented. The extension of the theory to similar integrals defined on more general analytic arcs is outlined.

1. INTRODUCTION

Let $\mathcal{P}_\nu$ denote the linear space of polynomials of degree not exceeding ν, let $\mathcal{A}$ denote the space of functions which are analytic at all finite points of the cut plane $\mathbb{C} \setminus (-\infty, 1]$ and let $K : \mathcal{P}_\nu \mapsto \mathcal{A}$ be defined by

$$K\phi(z) := \int_{-1}^{1} \mu(t)\phi(t)\log(z-t)\,\mathrm{d}t\ ,\quad \phi \in \mathcal{P}_\nu\ ,\quad z \notin [-1,1]\ , \tag{1.1}$$

where $\mu(t) := (1-t)^\alpha(1+t)^\beta$ is the classical Jacobi weight function. We note that in general, apart from certain special choices of μ, $K\phi$ cannot be expressed as a finite combination of elementary functions. The n-point Gauss-Jacobi approximation K_n to K is defined by

$$K_n\phi(z) := \sum_{k=1}^{n} \mu_k \phi(t_k)\log(z-t_k)\quad ,\quad z \notin [-1,1]\ , \tag{1.2}$$

where $\{\mu_k\}$ and $\{t_k\}$ are the weights and abscissae of the n-point Gauss-Jacobi rule associated with μ. Let

$$E_n := K - K_n\ . \tag{1.3}$$

The central aim of the paper is to derive an estimate for $E_n\phi$ that is valid asymptotically as $n \to \infty$.

The function $K\phi$ arises directly in the numerical solution of harmonic boundary value problems on polygonal domains via the boundary integral representation using the single layer logarithmic potential. In this case, the weight μ models the solution singularity that arises at corner points on the physical boundary or at points where boundary conditions are discontinuous. The polynomial ϕ then corresponds to a typical basis function for the finite-dimensional space in which an approximation

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to the unknown boundary density function is to be constructed; a set of Jacobi polynomials is a natural choice for this basis; see, for example, Hough [Hou90]. Finally, in a collocation solution of the boundary integral equation, the field point z corresponds to a typical collocation point somewhere on the physical boundary but not on the integration arc $[-1, 1]$. Thus, an analytic estimate of the quadrature error $E_n\phi(z)$ is directly relevant in constructing the numerical solution to harmonic boundary value problems on polygonal domains.

In the case of domains with nonpolygonal boundaries, let Γ denote a typical boundary arc and let $\zeta : [-1, 1] \mapsto \Gamma$ be an analytic parametrisation of Γ. Also let $\mathcal{A}_\zeta$ denote the set of functions that is analytic at all finite points of the cut plane $\mathbb{C} \setminus \{B \cup \Gamma\}$, where B is any suitable branch cut extending from $\zeta(-1)$ to ∞. There then arises the more general operator $K_\zeta : \mathcal{P}_\nu \mapsto \mathcal{A}_\zeta$ defined by

$$K_\zeta\phi(z) := \int_{-1}^{1} \mu(t)\phi(t) \log(z - \zeta(t))\, \mathrm{d}t \, , z \notin \Gamma \, . \tag{1.4}$$

The corresponding Gauss-Jacobi estimate is

$$K_{\zeta,n}\phi(z) := \sum_{k=1}^{n} \mu_k \phi(t_k) \log(z - \zeta(t_k)) \quad , \tag{1.5}$$

with error defined by

$$E_{\zeta,n} := K_\zeta - K_{\zeta,n} \quad .$$

An additional reason for wishing to estimate the error function $E_n\phi$ is that, for many functions ζ of practical interest, the error $E_{\zeta,n}\phi$ can be expressed in terms of $E_n\phi$.

In Section 2, we derive our asymptotic estimate for the error $E_n\phi$ as $n \to \infty$; our central result is contained in Theorem 2.6 where a simple explicit formula is presented. A numerical example illustrates the accuracy of the aymptotic estimate at relatively small values of n. The analysis of Section 2 is based on the ideas described in Donaldson and Elliott [DE72]. In Section 3, we outline how $E_{\zeta,n}\phi$ is related to $E_n\phi$ for the case of an entire parametric function ζ.

2. Asymptotic analysis

The analysis requires that, for any given $z \notin [-1, 1]$, the domain of definition of the function $\log(z - \cdot)$ should be extended to the whole complex plane cut by a suitable branch cut linking z to the point at infinity. It turns out to be most convenient to use a hyperbolic branch cut H_z defined as follows. Let

$$S^+ := \{w \in \mathbb{C} : \Re(w) > 0 \, , \quad -\pi < \Im(w) \le \pi\} \quad .$$

Then it is readily established that if the domain of the hyperbolic cosine is restricted to S^+, then the resulting map $\cosh : S^+ \to \mathbb{C} \setminus [-1, 1]$ is bijective and hence has an inverse, say $\cosh_+^{-1} : \mathbb{C} \setminus [-1, 1] \to S^+$. Given any $z \notin [-1, 1]$ define

$$H_z := \{w \in \mathbf{C} : w = \cosh\left(\cosh_+^{-1}(z) + t\right) \ , \quad 0 \le t < \infty\} \quad . \tag{2.1}$$

It may be noted that $\cosh\left(\cosh_+^{-1}(z) + t\right) \equiv z \cosh t + \sqrt{z^2 - 1} \sinh t$, so that the latter formula represents a somewhat more obvious parametrization of H_z. The hyperbolic arc H_z starts at z and goes out to ∞ without ever crossing the segment $[-1, 1]$. For the special case when z is real, the above definition implies that if

$z > 1$, then H_z is the segment $[z, \infty)$ of the real axis whilst if $z < -1$, then H_z is the segment $(-\infty, z]$.

We remark that the expression for $E_n\phi(z)$ derived in Theorem 2.1 below is quite independent of the specific choice of the branch cut. One can readily appreciate that this should be the case since, from (1.1), we see that selecting an alternative branch for the logarithm function modifies the expression $K\phi(z)$ by integer multiples of $2\pi i \int_{-1}^{1} \mu(t)\phi(t)\,\mathrm{d}t$. This latter integral is evaluated exactly by the n-point Gauss-Jacobi rule provided $\nu < 2n$ and hence contributes nothing to $E_n\phi(z)$.

According to Donaldson and Elliott [DE72, Theorem 1] the n-point Gauss Jacobi error function can be expressed in the form

$$E_n\phi(z) = \frac{1}{2\pi i}\int_C \eta^{(\alpha,\beta)}(w)\,\mathrm{d}w \tag{2.2}$$

where

$$\eta^{(\alpha,\beta)}(w) := \frac{\Pi_n^{(\alpha,\beta)}(w)}{P_n^{(\alpha,\beta)}(w)}\phi(w)\log(z-w) \quad , \tag{2.3}$$

$P_n^{(\alpha,\beta)}(t)$ is the Jacobi polynomial of degree n associated with the weight $\mu(t)$ and $\Pi_n^{(\alpha,\beta)}$ denotes the Jacobi function of the second kind, namely

$$\Pi_n^{(\alpha,\beta)}(w) := \int_{-1}^{1} \frac{(1-t)^\alpha(1+t)^\beta P_n^{(\alpha,\beta)}(t)}{w-t}\,\mathrm{d}t \quad . \tag{2.4}$$

Note that $\Pi_n^{(\alpha,\beta)}$ is analytic in the cut plane $\mathbb{C}\setminus[-1,1]$. The integration path C in (2.2) may be any simple closed curve, traversed in the anticlockwise direction, which encircles the segment $[-1,1]$ but does not cross the branch cut H_z. One such path is formed from the circular paths C_0, C_1 and the two edges of the cross-cut L_1 as indicated in Figure 1. Here, C_0 and C_1 denote circular paths traversed in the anti-clockwise direction and defined by

$$\begin{aligned} C_0 &:= \{w \in \mathbb{C} : w = \rho\,\mathrm{e}^{it}\ ,\ \ 0 \le t \le 2\pi\}, \\ C_1 &:= \{w \in \mathbb{C} : w = z + \delta_1\,\mathrm{e}^{it}\ ,\ \ 0 \le t \le 2\pi\} \end{aligned} \tag{2.5}$$

where

$$\rho > \max(1, |z|) \tag{2.6}$$

and δ_1 is small enough to ensure that $C_1 \subset \operatorname{int} C_0$ and $(\operatorname{int} C_1 \cup C_1) \cap [-1,1] \equiv \emptyset$. Also,

$$L_1 := H_z \cap \operatorname{int} C_0 \cap \operatorname{ext} C_1$$

is that part of H_z which lies between C_0 and C_1, traversed in the direction from C_1 towards C_0. With the above definitions it follows that

$$\begin{aligned} \int_C \eta^{(\alpha,\beta)}(w)\,\mathrm{d}w &= \int_{C_0} \eta^{(\alpha,\beta)}(w)\,\mathrm{d}w - \int_{C_1} \eta^{(\alpha,\beta)}(w)\,\mathrm{d}w \\ &\quad - \int_{L_1} \eta_-^{(\alpha,\beta)}(w)\,\mathrm{d}w + \int_{L_1} \eta_+^{(\alpha,\beta)}(w)\,\mathrm{d}w \end{aligned} \tag{2.7}$$

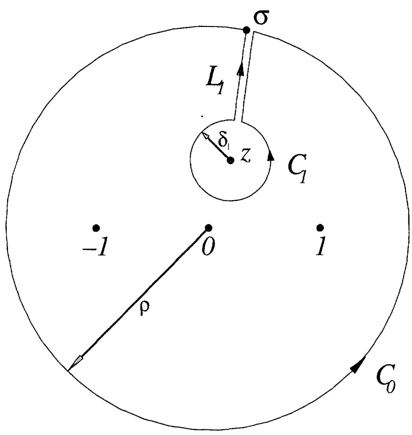

FIGURE 1. Components C_0, C_1 and L_1 of the contour C

where, since $\arg(z-w)$ decreases by 2π as w traverses C_1 in the anti-clockwise direction and recalling (2.3), we have written

$$\begin{aligned}\eta_-^{(\alpha,\beta)}(w) &= \frac{\Pi_n^{(\alpha,\beta)}(w)}{P_n^{(\alpha,\beta)}(w)}\phi(w)\log(z-w),\\ \eta_+^{(\alpha,\beta)}(w) &= \frac{\Pi_n^{(\alpha,\beta)}(w)}{P_n^{(\alpha,\beta)}(w)}\phi(w)\left(\log(z-w)-2\pi i\right).\end{aligned} \tag{2.8}$$

Hence it follows that

$$\int_{L_1}\left(\eta_+^{(\alpha,\beta)}(w)-\eta_-^{(\alpha,\beta)}(w)\right)\,\mathrm{d}w = -2\pi i\int_{L_1}\frac{\Pi_n^{(\alpha,\beta)}(w)}{P_n^{(\alpha,\beta)}(w)}\phi(w)\,\mathrm{d}w\ . \tag{2.9}$$

Substituting the previous result into (2.7) and allowing $\delta_1\to 0$ in conjunction with the fact that

$$\lim_{\delta_1\to 0}\int_{C_1}\eta^{(\alpha,\beta)}(w)\,\mathrm{d}w = 0\ ,$$

it follows from (2.2) that

$$E_n\phi(z) = -\int_z^\sigma \frac{\Pi_n^{(\alpha,\beta)}(w)}{P_n^{(\alpha,\beta)}(w)}\phi(w)\,\mathrm{d}w\Bigg|_{H_z} + \frac{1}{2\pi i}\int_{C_0}\eta^{(\alpha,\beta)}(w)\,\mathrm{d}w\ , \tag{2.10}$$

where

$$\sigma := H_z\cap C_0$$

and $\cdot|_{H_z}$ indicates that the integration path is along H_z.

The above expression can be further simplified by allowing $\rho\to\infty$; the result is summarised in the following theorem.

Theorem 2.1. *With notation as previously defined, if $\nu < 2n$, then*

$$E_n\phi(z) = -\int_{H_z}\frac{\Pi_n^{(\alpha,\beta)}(w)}{P_n^{(\alpha,\beta)}(w)}\phi(w)\,\mathrm{d}w\ . \tag{2.11}$$

Proof. We wish to prove that the second integral in (2.10) approaches zero as $\rho \to \infty$. Recalling the definitions (2.3), (2.4) and interchanging the order of integration gives

$$\int_{C_0} \eta^{(\alpha,\beta)}(w)\,\mathrm{d}w = \int_{-1}^{1} \mu(t) P_n^{(\alpha,\beta)}(t) \left(\int_{C_0} \frac{\phi(w)\log(z-w)}{P_n^{(\alpha,\beta)}(w)(w-t)}\,\mathrm{d}w \right) \mathrm{d}t \ . \tag{2.12}$$

In view of (2.6), $w \in C_0$ implies that the series

$$\frac{1}{w-t} = \sum_{k=0}^{\infty} \frac{t^k}{w^{k+1}} \tag{2.13}$$

is absolutely convergent. Substituting (2.13) into (2.12) and making use of the orthogonality of the Jacobi polynomials it follows that

$$\int_{C_0} \eta^{(\alpha,\beta)}(w)\,\mathrm{d}w =$$

$$\sum_{k=n}^{\infty} \left\{ \left(\int_{-1}^{1} \mu(t) P_n^{(\alpha,\beta)}(t) t^k\,\mathrm{d}t \right) \times \left(\int_{C_0} \frac{\phi(w)\log(z-w)}{P_n^{(\alpha,\beta)}(w) w^{k+1}}\,\mathrm{d}w \right) \right\} . \tag{2.14}$$

Now, since $\deg(\phi) \le \nu$, the rational function defined by the expression

$$\frac{\phi(w)}{P_n^{(\alpha,\beta)}(w) w^{\nu-n}}$$

is analytic on $\mathbb{C}\backslash[-1,1]$ and hence the maximum principle for such functions implies that

$$\max_{w\in C_0} \left| \frac{\phi(w)}{P_n^{(\alpha,\beta)}(w) w^{\nu-n}} \right| \le \max_{|w|=1} \left| \frac{\phi(w)}{P_n^{(\alpha,\beta)}(w)} \right| =: L_n^{(\alpha,\beta)} \ . \tag{2.15}$$

Also, in view of (2.6), $w \in C_0$ implies that $|z-w| < 2\rho$ and hence

$$\max_{w\in C_0} |\log(z-w)| < \log(2\rho) + 2\pi \ . \tag{2.16}$$

Therefore, using (2.15) and (2.16) it follows that

$$\begin{aligned} \left| \int_{C_0} \frac{\phi(w)\log(z-w)}{P_n^{(\alpha,\beta)}(w) w^{k+1}}\,\mathrm{d}w \right| &< L_n^{(\alpha,\beta)}(\log(2\rho)+2\pi) \int_{C_0} \frac{|\mathrm{d}w|}{|w|^{k+1-\nu+n}} \\ &= \frac{2\pi L_n^{(\alpha,\beta)}(\log(2\rho)+2\pi)}{\rho^{k-\nu+n}} \ . \end{aligned} \tag{2.17}$$

Furthermore, making use of the Rodrigues type formula

$$(1-t)^\alpha (1+t)^\beta P_n^{(\alpha,\beta)}(t) = -\left(\frac{(1-t)^{\alpha+1}(1+t)^{\beta+1} P_{n-1}^{(\alpha+1,\beta+1)}(t)}{2n} \right)' \ , \quad n \ge 1 \ ,$$

see Szegö [Sze75, §4.10], we see that

$$\begin{aligned} \left| \int_{-1}^{1} \mu(t) P_n^{(\alpha,\beta)}(t) t^k\,\mathrm{d}t \right| &= \frac{k}{2n} \left| \int_{-1}^{1} (1-t)^{\alpha+1}(1+t)^{\beta+1} P_{n-1}^{(\alpha+1,\beta+1)}(t) t^{k-1}\,\mathrm{d}t \right| \\ &\le \frac{k M_n^{(\alpha,\beta)}}{2} \int_{-1}^{1} |t|^{k-1}\,\mathrm{d}t \\ &= M_n^{(\alpha,\beta)} \ , \qquad k \ge n \ge 1 \ , \end{aligned} \tag{2.18}$$

where

$$M_n^{(\alpha,\beta)} := \max_{-1\le t\le 1}\left(\frac{(1-t)^{\alpha+1}(1+t)^{\beta+1}|P_{n-1}^{(\alpha+1,\beta+1)}(t)|}{n}\right) .$$

Thus, combining (2.14), (2.17) and (2.18) we conclude that

$$\left|\int_{C_0}\eta^{(\alpha,\beta)}(w)\,\mathrm{d}w\right| < \frac{2\pi M_n^{(\alpha,\beta)} L_n^{(\alpha,\beta)}(\log(2\rho)+2\pi)}{\rho^{2n-\nu}(1-\rho^{-1})} . \tag{2.19}$$

Hence, if $\nu < 2n$, then

$$\lim_{\rho\to\infty}\int_{C_0}\eta^{(\alpha,\beta)}(w)\,\mathrm{d}w = 0 . \tag{2.20}$$

Finally, it is clear that as $\rho\to\infty$, so $\sigma\to\infty$ and the integration path for the first integral in (2.10) becomes the full semi-infinite branch cut H_z. Hence, allowing $\rho\to\infty$ in (2.10) leads to the result (2.11). □

The approach to quadrature error estimation outlined by Donaldson and Elliott [DE72] is based on the use of asymptotic estimates for $\Pi_n^{(\alpha,\beta)}$ and $P_n^{(\alpha,\beta)}$ as $n\to\infty$. These are used in (2.11) to produce an asymptotic estimate for $E_n\phi$. The results are summarised in the following lemmas.

Lemma 2.2. *Given any $z\in\mathbb{C}\setminus[-1,1]$, let f be any function for which the associated function F defined by*

$$F(t) := \sinh(\xi+t) f\circ\cosh(\xi+t) \ , \ \xi := \cosh_+^{-1}(z) \ , \ 0\le t<\infty ,$$

has a well-defined Laplace transform

$$\mathcal{L}F(s) := \int_0^\infty \mathrm{e}^{-st}F(t)\,\mathrm{d}t .$$

Then

$$\int_{H_z}\frac{f(w)}{(w+\sqrt{w^2-1})^n}\mathrm{d}w = \frac{\mathcal{L}F(n)}{(z+\sqrt{z^2-1})^n} .$$

Proof. This is simply a matter of using the parametrisation for H_z given in the definition (2.1). Thus,

$$\begin{aligned}\int_{H_z}\frac{f(w)}{(w+\sqrt{w^2-1})^n}\mathrm{d}w &= \lim_{\tau\to\infty}\int_0^\tau \frac{\sinh(\xi+t)f\circ\cosh(\xi+t)}{(\cosh(\xi+t)+\sinh(\xi+t))^n}\,\mathrm{d}t\\ &= \frac{1}{(\mathrm{e}^{\xi})^n}\lim_{\tau\to\infty}\int_0^\tau \mathrm{e}^{-nt}F(t)\,\mathrm{d}t\\ &= \frac{\mathcal{L}F(n)}{(z+\sqrt{z^2-1})^n} ,\end{aligned}$$

where

$$\xi := \cosh_+^{-1}(z) . \tag{2.21}$$

□

Remark 2.3. In the previous lemma, and at various points below, we encounter the expression $z+\sqrt{z^2-1}$. Any ambiguity in its meaning is removed by always using the value whose magnitude exceeds 1. This is effectively achieved by defining $\sqrt{z^2-1}$ to be $(z-1)^{\frac{1}{2}}(z+1)^{\frac{1}{2}}$ with $|\arg(z\pm 1)<\pi|$.

The following lemma is proved in, for example, Henrici [Hen77, §11.5].

Lemma 2.4 (Watson-Doetsch). *If F has an asymptotic power series*

$$F(t) \sim t^\gamma \sum_{k=0}^{\infty} F_k t^{k\lambda} = t^\gamma F_0 + \dots$$

as $t \to 0$, $t > 0$, then $\mathcal{L}F$ has the asymptotic power series

$$\mathcal{L}F(n) \sim \frac{1}{n^{\gamma+1}} \sum_{k=0}^{\infty} \frac{F_k \Gamma(\gamma+1+k\lambda)}{n^{k\lambda}} = \frac{F_0\Gamma(\gamma+1)}{n^{\gamma+1}} + \dots$$

as $n \to \infty$.

The next lemma is proved in a more general form in Elliott [Ell71].

Lemma 2.5. *If w is bounded away from $[-1,1]$, then the leading term in the asymptotic expansion of $\Pi_n^{(\alpha,\beta)}(w)/P_n^{(\alpha,\beta)}(w)$ as $n \to \infty$ is given by*

$$\frac{\Pi_n^{(\alpha,\beta)}(w)}{P_n^{(\alpha,\beta)}(w)} \sim \frac{N_n^{(\alpha,\beta)}(w-1)^\alpha(1+w)^\beta}{(w+\sqrt{w^2-1})^{2n+\alpha+\beta+1}}$$

where

$$N_n^{(\alpha,\beta)} := 2^{4n+2\alpha+2\beta+2}\frac{\Gamma(n+1)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+2)\Gamma(2n+\alpha+\beta+1)} .$$

Theorem 2.6. *If $\nu < 2n$, then the operator $\tilde{E}_n : \mathcal{P}_\nu \mapsto \mathcal{A}$ defined by*

$$\tilde{E}_n\phi(z) := -\frac{N_n^{(\alpha,\beta)}(z-1)^\alpha(1+z)^\beta\phi(z)\sqrt{z^2-1}}{(2n+\alpha+\beta+1)(z+\sqrt{z^2-1})^{2n+\alpha+\beta+1}} ,$$

defines a computable estimate for the Gauss-Jacobi quadrature error operator E_n and is valid asymptotically as $n \to \infty$ with $z \notin [-1,1]$.

Proof. Substituting the leading expansion term given by Lemma 2.5 into (2.11) and utilizing Lemma 2.2 leads to the result

$$E_n\phi(z) \sim -\frac{N_n^{(\alpha,\beta)}\mathcal{L}F(2n+\alpha+\beta+1)}{(z+\sqrt{z^2-1})^{2n+\alpha+\beta+1}}$$

where F is defined by

$$F(t) := (\cosh(\xi+t)-1)^\alpha(1+\cosh(\xi+t))^\beta \sinh(\xi+t)\phi(\cosh(\xi+t)) .$$

Since $\cosh\xi = z \notin [-1,1]$, it follows that the above function F is analytic at $t=0$ and consequently Lemma 2.4 implies that, as $n \to \infty$,

$$E_n\phi(z) \sim \tilde{E}_n\phi(z) ,$$

where $\tilde{E}_n\phi(z)$ is defined in the statement of the theorem. □

We conclude this section by presenting an example to illustrate the accuracy of the estimate of Theorem 2.6. In this example, the Gauss-Jacobi estimate $K_n\phi$, which is needed for the calculation of $E_n\phi$, is computed via the IMSL Fortran algorithm DGQRUL; this is a double-precision implementation of the eigensystem formulation of Golub and Welsch [GW69] for the calculation of Gaussian quadrature points and weights.

TABLE 1. A comparison of exact and asymptotic quadrature errors

δ	n	$E_nT_4(z)$	$\lvert E_nT_4(z)\rvert$	$\tilde{E}_nT_4(z)$	$\lvert\tilde{E}_nT_4(z)\rvert$
0.1	3	$(-6.9+5.4i)\times 10^{-1}$	8.8×10^{-1}	$(-2.0+1.8i)\times 10^{-1}$	2.7×10^{-1}
	6	$(3.5+1.2i)\times 10^{-2}$	3.7×10^{-2}	$(2.2+0.76i)\times 10^{-2}$	2.3×10^{-2}
	12	$(-3.5+2.7i)\times 10^{-4}$	4.4×10^{-4}	$(-2.8+2.2i)\times 10^{-4}$	3.6×10^{-4}
	24	$(1.6+0.81i)\times 10^{-7}$	1.8×10^{-7}	$(1.5+0.71i)\times 10^{-7}$	1.7×10^{-7}
0.01	3	$-1.2+0.50i$	1.3	$(-3.0+4.7i)\times 10^{-1}$	5.6×10^{-1}
	6	$(0.60+3.3i)\times 10^{-1}$	3.4×10^{-1}	$(1.3+1.66i)\times 10^{-1}$	2.1×10^{-1}
	12	$(0.46-6.0i)\times 10^{-2}$	6.0×10^{-2}	$(0.32-5.7i)\times 10^{-2}$	5.7×10^{-2}
	24	$(9.4-0.26i)\times 10^{-3}$	9.4×10^{-3}	$(8.6-0.78i)\times 10^{-3}$	8.6×10^{-3}

Example 2.7. In [DE72], Donaldson and Elliott remark that asymptotic estimates for the quadrature error are frequently found to be good even for relatively small values of the number of quadrature points n. This example gives further support to this remark. We consider the Chebyshev weight function

$$\mu(x) := \frac{1}{\sqrt{1-x^2}} \quad .$$

Let T_r denote the Chebyshev polynomial of degree r associated with μ. Then we have the simple exact result

$$KT_r(z) = -\frac{\pi}{r}(z+\sqrt{z^2-1})^{-r} \ , \quad r>0 \ ;$$

see, for example, [Lev91, Appendix A]. In Table 1 we compare the exact quadrature error $E_nT_4(z)$ with its asymptotic estimate $\tilde{E}_nT_4(z)$ for the cases $n=3,6,12,24$ at the two points

$$z=\cos(\pi/16)+i\delta,\ \delta=0.1,0.01 \ .$$

The values $z=\cos(\pi/16)+i\delta$ are chosen because as $\delta\to 0$ the real and imaginary parts of $KT_4(z)$ tend to become equal in magnitude. Accurate values of $KT_4(z)$ correct to ten decimal places are :

$$\begin{aligned}\delta=0.1 \quad &: \quad -0.0455932530+0.2411950862i\\ \delta=0.01 \quad &: \quad -0.4439538326+0.4662079164i \ .\end{aligned}$$

A quadrature error estimate is perfectly acceptable for most practical purposes if it is correct to one significant figure. From this point of view, the magnitudes of the error estimates in Table 1 are very good for $n=12,24$ and are close to acceptable even at $n=6$.

3. ENTIRE PARAMETRIZATIONS

In this section we return to the comments made in the introduction and examine how the results of the previous section may be extended to the case of a general arc Γ with analytic parametrisation ζ.

In the first place we briefly note that the results of the previous section apply directly to the case where Γ is an arbitrary line segment. Such a line segment can be parametrised by $\zeta(t) = a + bt,\ b \neq 0,\ t \in [-1, 1]$. Thus, for any field point $z \notin \Gamma$ we may determine a unique complex parameter value, say w, such that $\zeta(w) = z$. Substituting this into (1.4) gives

$$K_\zeta \phi(z) = K\phi(w) + \log b \int_{-1}^{1} \mu(t)\phi(t)\,\mathrm{d}t \ .$$

Since the integral expression above is evaluated exactly by the n-point rule provided $\nu < 2n$, we conclude that for any line segment

$$E_{\zeta,n}\phi(z) = E_n\phi(\zeta^{-1}(z)) \ , \quad \nu < 2n \ . \tag{3.1}$$

The result (3.1) generalises to the case of a polynomial function ζ except that in this case the right-hand side will involve a summation over all the $\deg(\zeta)$ values of $\zeta^{-1}(z)$. However, perhaps the most natural generalisation of the result (3.1) is to the case where ζ is entire; see Henrici [Hen77, §8.3] for a summary of the properties of such functions. This class includes the practically important cases of polynomial and trigonometric polynomial parametrizations.

For the remainder of this section we assume that ζ is an entire function of finite genus γ. Then given any $z \in \mathbb{C}$, the function $\zeta_z : t \mapsto z - \zeta(t)$ is also of genus γ. Let

$$W_z := \{w \in \mathbb{C} : \zeta_z(w) = 0\}$$

be the set of zeros of ζ_z, otherwise known as the z-points of ζ. If these z-points satisfy

$$\sum_{w \in W_z} \frac{1}{|w|^{\gamma+1}} < \infty,$$

then ζ_z can be expressed as the canonical product

$$\zeta_z(t) = \exp \circ g_z(t) \prod_{w \in W_z} \sigma(tw^{-1}) \tag{3.2}$$

where g_z is a polynomial function of degree q, with coefficients depending on z,

$$\sigma(t) := (1 - t) \exp \circ \ell(t) \ , \tag{3.3}$$

ℓ is a polynomial of degree p defined by

$$\ell(t) := \begin{cases} 1 & , \ p = 0, \\ \displaystyle\sum_{k=1}^{p} \frac{t^k}{k} & , \ p > 0, \end{cases} \tag{3.4}$$

and $\gamma = \max(p, q)$.

Theorem 3.1. *If ζ is an entire function of finite genus γ and $\gamma + \nu < 2n$, then*

$$E_{\zeta,n}\phi(z) = \sum_{w \in W_z} E_n\phi(w) \ .$$

Proof. Using the form (3.2) in (1.4) and (1.5) gives

$$K_\zeta \phi(z) = \int_{-1}^{1} \mu(t)\phi(t) \left\{ g_z(t) + \sum_{w\in W_z} \log\sigma(tw^{-1}) \right\} \mathrm{d}t \tag{3.5}$$

$$K_{\zeta,n} \phi(z) = \sum_{k=1}^{n} \mu_k \phi(t_k) \left\{ g_z(t_k) + \sum_{w\in W_z} \log\sigma(t_k w^{-1}) \right\} \tag{3.6}$$

If the elements of W_z are indexed in order of increasing modulus, then the summation over W_z appearing in (3.5) and (3.6) is uniformly convergent for $t \in [-1, 1]$; see [Hen77, §8.3]. Therefore the integration and summation order in (3.5) can be interchanged, as can the order of the two summations in (3.6). Performing these interchanges and noting that, since $\deg(\phi g_z) \leq \nu + \gamma < 2n$, the Gaussian rule is exact for the integral involving ϕg_z, it follows that

$$E_{\zeta,n}\phi(z) = \sum_{w\in W_z} \left\{ \int_{-1}^{1} \mu(t)\phi(t) \log\sigma(tw^{-1})\,\mathrm{d}t - \sum_{k=1}^{n} \mu_k\phi(t_k)\log\sigma(t_k w^{-1}) \right\}. \tag{3.7}$$

Also, we note from (3.3) that

$$\log\sigma(tw^{-1}) = \log(w-t) + \ell(tw^{-1}) - \log w \ ,$$

which may be substituted termwise in the series of (3.7) and combined with the fact that $\deg(\phi\ell) \leq \nu + \gamma < 2n$ to deduce that

$$\begin{aligned} E_{\zeta,n}\phi(z) &= \sum_{w\in W_z} \left\{ \int_{-1}^{1} \mu(t)\phi(t)\log(w-t)\,\mathrm{d}t - \sum_{k=1}^{n} \mu_k\phi(t_k)\log(w-t_k) \right\} \\ &= \sum_{w\in W_z} E_n\phi(w) \ . \end{aligned}$$

□

An asymptotic estimate for $E_{\zeta,n}\phi$ can thus be defined as

$$\tilde{E}_{\zeta,n}\phi := \sum_{w\in W_z} \tilde{E}_n\phi(w) \tag{3.8}$$

where $\tilde{E}_n\phi(w)$ is defined by Theorem 2.1. However, in order to turn this definition into a genuinely computable estimate, a number of important practical issues would need to be addressed, including algorithms for truncating the series in (3.8) and for the efficient computation of the relevant points in W_z. It is hoped that these matters may be taken up in a future paper.

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