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ON THE LEAST PRIME PRIMITIVE ROOT MODULO A PRIME

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ABSTRACT. We derive a conditional formula for the natural density $E(q)$ of prime numbers p having its least prime primitive root equal to q, and compare theoretical results with the numerical evidence.

1. THEORETICAL RESULT CONCERNING THE DENSITY OF PRIMES WITH A GIVEN LEAST PRIME PRIMITIVE ROOT

Let us denote, following Elliott and Murata [4], by $g(p)$ and $G(p)$ the least primitive and the least prime primitive root $\bmod p$, respectively. The first aim of this paper is to derive from the work of Matthews [5] a conditional (under the generalized Riemann hypothesis) formula for the density of primes p such that $G(p) = q$, where q is a given prime, and to compare this formula with the numerical evidence. Next we give for each prime $q \leq 349$ the least prime p such that $G(p) = q$, if such p exists below 2^{31}, and we compare $G(p)$ with $(\log p)(\log\log p)^2$, which, according to a conjecture of E. Bach [2], is the maximal order of $G(p)$ (i.e., $0 < \limsup \frac{G(p)}{(\log p)(\log\log p)^2} < \infty$). We also numerically investigate the average value of the least prime primitive root.

In order to formulate the theorem, we denote by p_n the nth prime and, for a given set M, by $|M|$ its cardinality. Now we can state

Theorem. *Assume that the Riemann hypothesis holds for each of the fields* $Q(\sqrt[k]{1}, \sqrt[l_1]{p_1}, \dots, \sqrt[l_n]{p_n})$, *where* $k = \text{l.c.m. } l_i$ *is squarefree. Then the set of primes* p *such that* $G(p) = p_n$ *has a natural density equal to*

$$E(p_n) = \sum_{m=1}^{n} (-1)^{m-1} \Delta_m \cdot c_{m,n}, \tag{1}$$

where

$$\Delta_m = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i - 1}\left(1 - \left(1 - \frac{1}{p_i}\right)^m\right)\right),$$

$$c_{m,n} = \frac{1}{2} \sum_{\substack{|M|=m \\ M\subset\{p_1,p_2,\dots,p_n\} \\ (M\ni p_n)}} \left\{ \prod_{p\in M-\{2\}} (1 + d_{m,p}) + \prod_{p\in M-\{2\}} (1 + (-1|p) d_{m,p}) \right\} \tag{2}$$

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and

$$d_{m,p} = \frac{1}{p-1}\left(1-\left(1-\frac{1}{p}\right)^m\right)\cdot\left(1-\frac{1}{p-1}\left(1-\left(1-\frac{1}{p}\right)^m\right)\right)^{-1}$$

The above formula for $c_{m,n}$ is due to the referee, which we gratefully acknowledge. The expression given in (2) can be efficiently evaluated using a generating function approach. Our original formula was less suitable for computation.

The proof is based on two lemmas, in which the letters p, q, r, are reserved for primes and $\log_2 x = \log\log x$.

Lemma 1. *Let $M = \{r_1, \dots, r_m\}$ be a set of primes,*

$$N_M(x) = \{p \le x : \text{ every } r \text{ in } M \text{ is a primitive root } \bmod p\}.$$

On the assumption of the Riemann hypothesis for each extension

$$Q(\sqrt[k]{1}, \sqrt[l_1]{r_1}, \dots, \sqrt[l_m]{r_m}),$$

where $k = \mathrm{l.c.m.} l_i$ is squarefree we have

$$|N_M(x)| = A_M \cdot \mathrm{Li}\, x + O_M(\mathrm{Li}\, x(\log x)^{-1}(\log_2 x)^{2^{|M|}-1}),$$

where A_M is defined as follows.

Let $c(p)$ be the natural density of the set

$$\{q : q \equiv 1(\bmod p), \text{ at least one of } r_1, \dots, r_m \text{ is a } p\text{th power residue } \bmod r\}$$

and let $\bar{c}(p) = 1 - c(p)$. Also let $G(r_1, \dots, r_m)$ denote the set of numbers of the form $a = r_1^{\varepsilon_1} \cdots r_m^{\varepsilon_m} \equiv 1(\bmod 4)$, $\varepsilon_i = 0$ or 1, and finally let

$$f(a) = \prod_{p|a} \frac{c(p)}{1-c(p)}.$$

Then

$$A_M = \prod_{i=1}^{\infty} \bar{c}(p_i) \sum_{a \in G(r_1, \dots, r_m)} f(a). \tag{3}$$

Proof. This is how Theorem 13.2 of Matthews [5] simplifies when his set M consists of primes.

Lemma 2. *In the notation of Lemma* 1

$$c(p) = \frac{1}{p-1}\left(1-\left(1-\frac{1}{p}\right)^m\right).$$

Proof. Let us put in Theorem 7.11* of [6] $K = Q(\zeta_p), L = Q(\zeta_p, \sqrt[p]{r_1}, \dots, \sqrt[p]{r_m})$, where ζ_p is a primitive root of unity of order p. The extension L/K is Abelian and its Galois group is isomorphic to F_p^m, a vector $[a_1, \dots, a_m] \in F_p^m$ acts on L by the formula

$$\sqrt[p]{r_i} \to \zeta_p^{a_i} \cdot \sqrt[p]{r_i} \ (1 \le i \le m).$$

Let, for a prime ideal $\mathfrak{q}$ of K not dividing $r_1, \dots, r_m$, $F_{L/K}(\mathfrak{q})$ be the Artin symbol, i.e., a vector $[a_1, \dots, a_m] \in F_p^m$ such that

$$r_i^{\frac{N\mathfrak{q}-1}{p}} \equiv \zeta_p^{a_i} (\bmod \mathfrak{q}) \quad (1 \le i \le m). \tag{4}$$

By Theorem 7.11* of [6] for each $[a_1, \dots, a_m] \in F_p^m$, the number of prime ideals $\mathfrak{q}$ of K with norm $\le x$ satisfying $F_{L/K}(\mathfrak{q}) = [a_1, \dots, a_m]$, hence (4), is $(\frac{1}{p^m} + o(1)) \cdot \frac{x}{\log x}$.

If at least one r_i is a pth power residue and $q \equiv 1 (\operatorname{mod} p)$, $q \nmid r_1, \dots, r_m$, then for each prime ideal $\mathfrak{q}$ of K dividing q we have in (1) at least one $a_i = 0$ $(1 \le i \le m)$. The number of vectors $[a_1, \dots, a_m]$ in F_p^m with this property is $p^m - (p-1)^m$. To each prime $q \equiv 1 (\operatorname{mod} p)$ correspond $p-1$ ideals $\mathfrak{q}$ of norm q. Since the number of prime ideals with norm $\le x$ not being a prime is $o(\frac{x}{\log x})$, the lemma follows.

Proof of the Theorem. By the sieve principle the number $N(x)$ of primes $\le x$ with $G(p) = p_n$ equals

$$\sum_{\substack{M \subset \{p_1, \dots, p_n\} \\ (M \ni p_n)}} (-1)^{|M|-1} N_M(x)$$

(cf. [4, Lemma 10], for a similar formula concerning $g(p)$); hence by Lemma 1,

$$N(x) = \sum_{\substack{M \subset \{p_1, \dots, p_n\} \\ (M \ni p_n)}} (-1)^{|M|-1} A_M \operatorname{Li} x + O_n(\operatorname{Li} x (\log x)^{-1} (\log_2 x)^{2^n - 1})$$

and

$$E(p_n) = \sum_{\substack{M \subset \{p_1, \dots, p_n\} \\ (M \ni p_n)}} (-1)^{|M|-1} A_M. \tag{5}$$

Now if $M = \{r_1, \dots, r_m\}$, we have by Lemma 2

$$\bar{c}(p_i) = 1 - \frac{1}{p_i - 1}\left(1 - \left(1 - \frac{1}{p_i}\right)^m\right),$$
$$\frac{c(p)}{1 - c(p)} = d_{m,p},$$

hence

$$\prod_{i=1}^{\infty} \bar{c}(p_i) = \Delta_m. \tag{6}$$

On the other hand if $M_\varepsilon = \{r : r \equiv \varepsilon \operatorname{mod} 4\}$, the condition $\prod_{\mu=1}^{m} r_\mu^{\varepsilon_\mu} \equiv 1 (\operatorname{mod} 4)$ is equivalent to

$$\varepsilon_\mu = 0 \text{ if } r_\mu = 2 \text{ and } \sum_{r_\mu \in M_{-1}} \varepsilon_\mu \equiv 0 (\operatorname{mod} 2).$$

Hence

$$\begin{aligned}
\sum_{a \in G(r_1, \dots, r_m)} f(a) &= \sum_{k=0}^{|M_1|} \sum_{\substack{N \subset M_1 \\ |N| = k}} \prod_{r \in N} d_{m,r} \cdot \sum_{k=0}^{[\frac{|M_{-1}|}{2}]} \sum_{\substack{N \subset M_{-1} \\ |N| = 2k}} \prod_{r \in N} d_{m,r} \\
&= \prod_{r \in M_1} (1 + d_{m,r}) \cdot \frac{\prod_{r \in M_{-1}} (1 + d_{m,r}) + \prod_{r \in M_{-1}} (1 - d_{m,r})}{2} \\
&= \frac{1}{2} \left\{ \prod_{p \in M - \{2\}} (1 + d_{m,p}) + \prod_{p \in M - \{2\}} (1 + (-1|p) d_{m,p}) \right\},
\end{aligned} \tag{7}$$

and (1) follows from (2), (3), (5), (6) and (7).

2. Results of numerical investigations

This section addresses two practical topics:

- It attempts to verify empirically the existence of positive densities $E(p_n)$ for all primes having their least prime primitive root equal to p_n. By formulas (1) and (2), values of $E(p_n)$ for $n \leq 25$ have been computed. These values were compared with the frequencies calculated empirically on computers.
- It attempts to answer the question of whether the average value of the least prime primitive root tends to a finite limit.

The computation of $E(p_n)$ was programmed for all $n \leq 25$ with the aid of an IBM PC (Pentium 100 Mhz) computer using Borland's PASCAL compiler. Table 1 shows the results of the computation of $E(p_n)$ according to formulas (1) and (2) for initial values of n. The constants Δ_n were computed with high accuracy and are as follows:

$\Delta_1 = 0.373955813619,$	$\Delta_2 = 0.147349400317,$	$\Delta_3 = 0.060821655315,$
$\Delta_4 = 0.026107446426,$	$\Delta_5 = 0.011565842109,$	$\Delta_6 = 0.005251758060,$
$\Delta_7 = 0.002430226781,$	$\Delta_8 = 0.001140851399,$	$\Delta_9 = 0.000541435518,$
$\Delta_{10} = 0.000259105371,$	$\Delta_{11} = 0.000124792269,$	$\Delta_{12} = 0.000060404308,$
$\Delta_{13} = 0.000029353746,$	$\Delta_{14} = 0.000014309885,$	$\Delta_{15} = 0.000006994080,$
$\Delta_{16} = 0.000003425724,$	$\Delta_{17} = 0.000001680934,$	$\Delta_{18} = 0.000000826053,$
$\Delta_{19} = 0.000000406471,$	$\Delta_{20} = 0.000000200235,$	$\Delta_{21} = 0.000000098737,$
$\Delta_{22} = 0.000000048730,$	$\Delta_{23} = 0.000000024068,$	$\Delta_{24} = 0.000000011896,$
$\Delta_{25} = 0.000000005883.$		

The calculation was similar to that of Wrench [9].

One can prove that $\lim_{n\to\infty} \frac{\Delta_n}{\Delta_{n+1}} = 2$.

Note that $E(2)$ is Artin's constant and that $E(3) = \Delta_1 - \Delta_2$. The referee has observed that $E(p_{n+1})/E(p_n)$ seems to tend to a limit, but we are unable to prove or disprove this.

Additionally the frequencies of least prime primitive roots for prime numbers from the interval $[3, 2147483647]$ were computed. The computations were done on several IBM PC Pentium computers. The program for the computations was optimized for 32-bit arithmetic. Results of computations are gathered in Table 2. The correctness of computations was monitored in several ways.

- The number of generated primes. To verify the number of generated primes that least prime primitive roots were searched for, the algorithm by D. C. Mapes from 1963, for finding isolated values of the $\pi(x)$ function (the number of primes $\leq x$) was used.
- Verification of the factorization of $p-1$, where p is a randomly selected prime, with the aid of procedures independently implemented by other people.
- Partial verification of computations by existing packages, e.g., GP/PARI, Maple.

Let us denote by $N(p_n, x)$ the number of least prime primitive roots equal to p_n for primes not exceeding x and respectively by $E(p_n, x)$ the natural density of primes not exceeding x, having their least primitive roots equal to p_n.

TABLE 1. Theoretical values of densities $E(p_n)$ of least prime primitive roots equal to p_n for $n \leq 25$

n	p_n	E_n
1	2	0.37395581
2	3	0.22660641
3	5	0.13906581
4	7	0.08639185
5	11	0.05640411
6	13	0.03669884
7	17	0.02468028
8	19	0.01691581
9	23	0.01159480
10	29	0.00799836
11	31	0.00561924
12	37	0.00394799
13	41	0.00280419
14	43	0.00200731
15	47	0.00144059
16	53	0.00103755
17	59	0.00075313
18	61	0.00054722
19	67	0.00040018
20	71	0.00029321
21	73	0.00021534
22	79	0.00015895
23	83	0.00011751
24	89	0.00008706
25	97	0.00006471

Graphs of the functions $E(p_n, x)$ for primes $p_n < 32$ and $x < 21 \cdot 10^8$ are given below. Figures 1–11 show us that the behavior of natural densities of primes with a given least primitive root equal to a small prime number is extremely regular. The functions $E(p_n, x)$ for primes $p_n < 32$ stabilize very early and at least four decimal digits after the dot are constant.

Let us denote by $E^*(x)$ the average value of the least prime primitive root of primes not exceeding x, that is

$$E^*(x) = \frac{1}{\pi(x)} \sum_{p \leq x} G(p),$$

and the above sum is extended for all primes p less than or equal to x.

TABLE 2. Frequencies of least prime primitive roots of prime numbers less than or equal to $x = 2,000,000,000$. $N(p_n, x)$ denotes the number of least prime primitive roots equal to p_n for primes not exceeding x.

p_n	$N(p_n, x)$	$E(p_n, x)$	p_n	$N(p_n, x)$	$E(p_n, x)$
2	36730667	0.3739545079	131	787	0.0000080124
3	22258719	0.2266157680	137	632	0.0000064344
5	3659479	0.1390670022	139	471	0.0000047952
7	8486600	0.0864019792	149	362	0.0000036855
11	5539490	0.0563974854	151	248	0.0000025249
13	3603666	0.0366888830	157	183	0.0000018631
17	2424059	0.0246793174	163	139	0.0000014152
19	1662660	0.0169275228	167	98	0.0000009977
23	1139840	0.0116046982	173	75	0.0000007636
29	786125	0.0080035298	179	71	0.0000007229
31	551842	0.0056182972	181	53	0.0000005396
37	387927	0.0039494804	191	40	0.0000004072
41	275476	0.0028046181	193	39	0.0000003971
43	197240	0.0020080982	197	21	0.0000002138
47	140579	0.0014312332	199	22	0.0000002240
53	101667	0.0010350706	211	20	0.0000002036
59	73978	0.0007531692	223	8	0.0000000814
61	53542	0.0005451105	227	3	0.0000000305
67	39135	0.0003984330	229	2	0.0000000204
71	28765	0.0002928561	233	6	0.0000000611
73	20912	0.0002129048	239	4	0.0000000407
79	15548	0.0001582940	241	3	0.0000000305
83	11486	0.0001169388	251	3	0.0000000305
89	8462	0.0000861515	257	2	0.0000000204
97	6217	0.0000632952	263	2	0.0000000204
101	4721	0.0000480644	277	1	0.0000000102
103	3470	0.0000353280	283	1	0.0000000102
107	2498	0.0000254321	307	1	0.0000000102
109	1818	0.0000185090	347	1	0.0000000102
113	1419	0.0000144468	349	1	0.0000000102
127	980	0.0000099774			

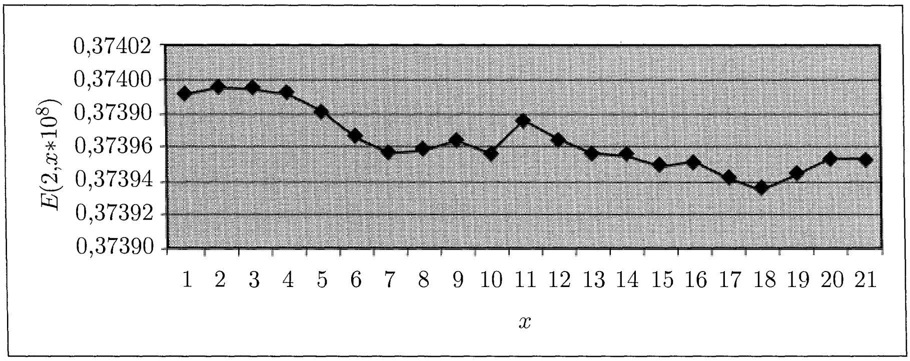

FIGURE 1. The natural density of primes with the least prime primitive root equal to 2

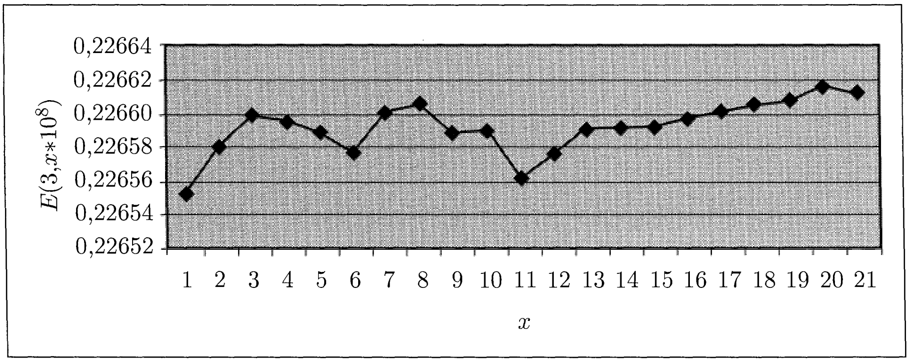

FIGURE 2. The natural density of primes with the least prime primitive root equal to 3

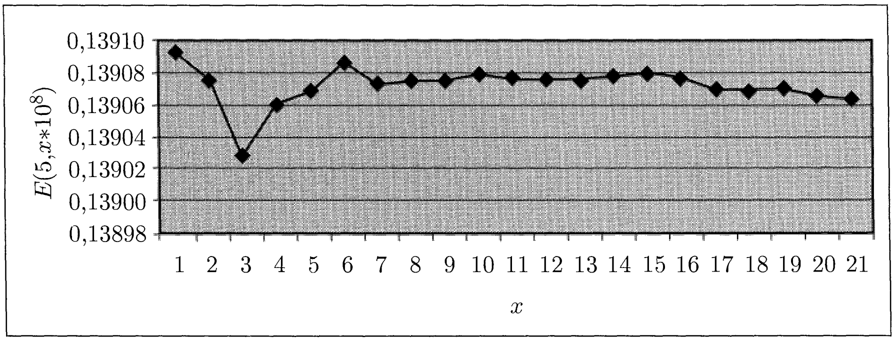

FIGURE 3. The natural density of primes with the least prime primitive root equal to 5

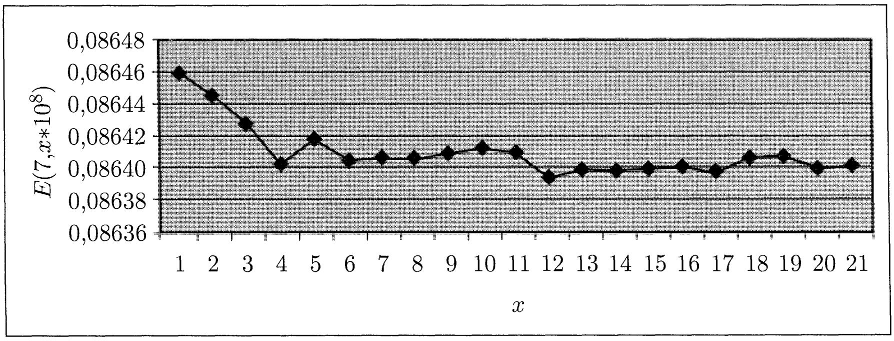

FIGURE 4. The natural density of primes with the least prime primitive root equal to 7

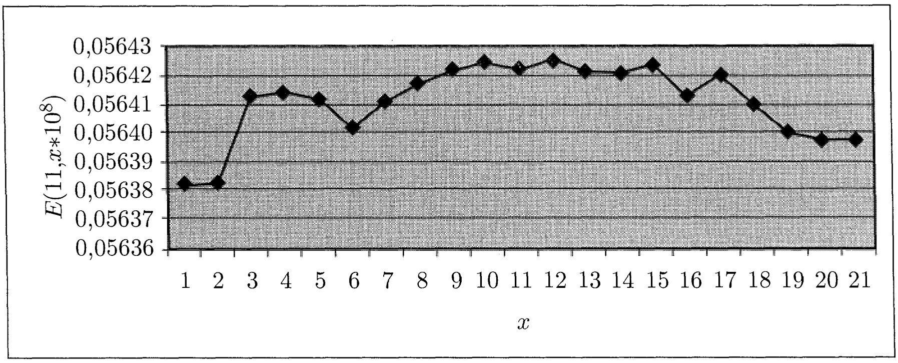

FIGURE 5. The natural density of primes with the least prime primitive root equal to 11

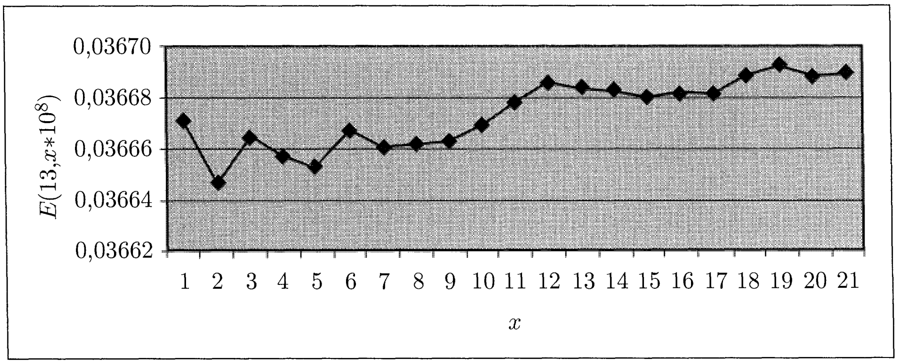

FIGURE 6. The natural density of primes with the least prime primitive root equal to 13

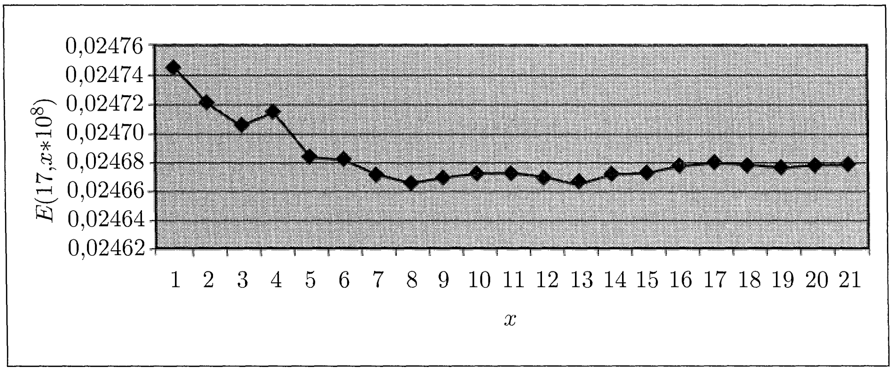

FIGURE 7. The natural density of primes with the least prime primitive root equal to 17

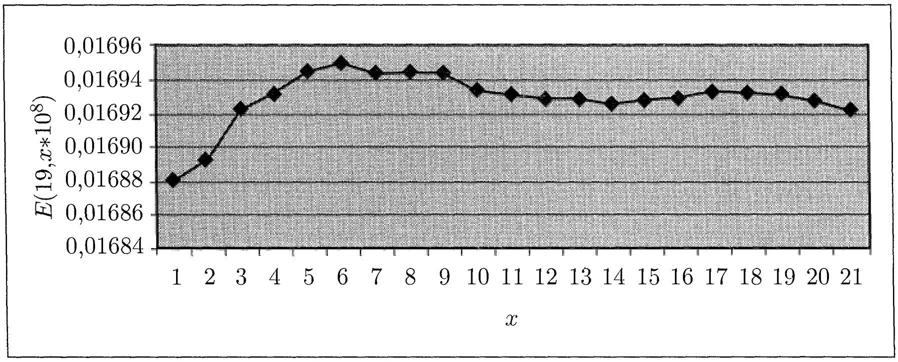

FIGURE 8. The natural density of primes with the least prime primitive root equal to 19

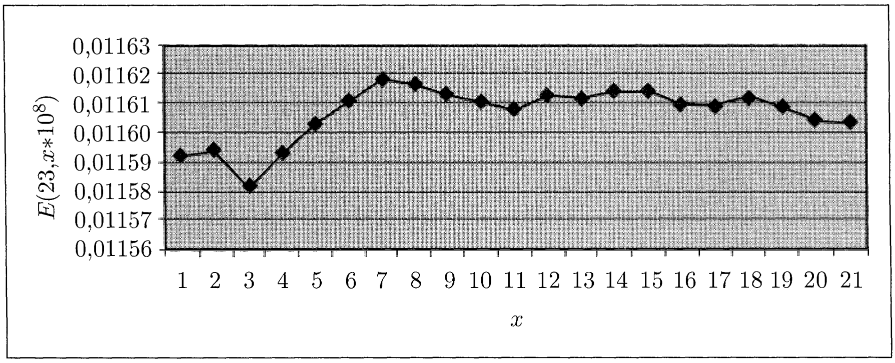

FIGURE 9. The natural density of primes with the least prime primitive root equal to 23

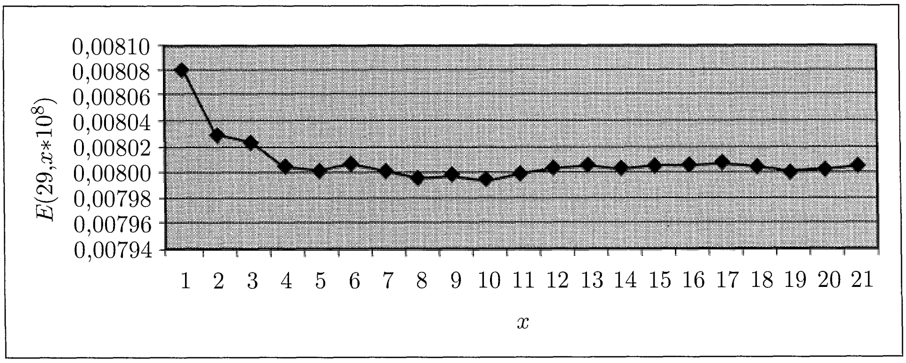

FIGURE 10. The natural density of primes with the least prime primitive root equal to 29

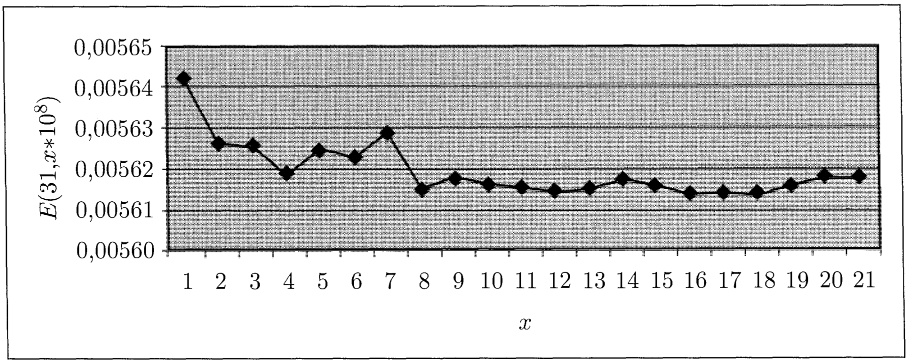

FIGURE 11. The natural density of primes with the least prime primitive root equal to 31

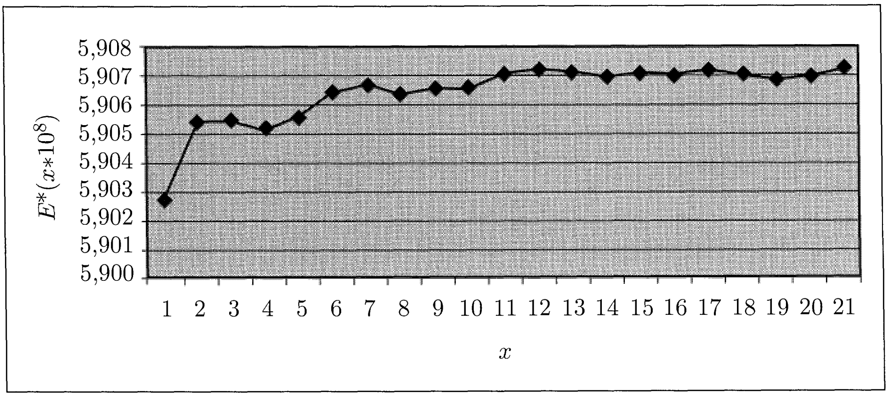

FIGURE 12. The average value $E^*(x)$ of the least prime primitive root of primes $\leq x$

TABLE 3. Average values E^* of least prime primitive roots of prime numbers (primitive roots of prime numbers) not exceeding $x \cdot 10^8$

x	$E^*(x \cdot 10^8)$	x	$E^*(x \cdot 10^8)$	x	$E^*(x \cdot 10^8)$
1	5.9027080833	8	5.9063536374	15	5.9070799864
2	5.9054200778	9	5.9065722812	16	5.9070094456
3	5.9054950599	10	5.9066169463	17	5.9072094835
4	5.9052252014	11	5.9070787552	18	5.9071083838
5	5.9056411614	12	5.9072307772	19	5.9068949876
6	5.9064682273	13	5.9071263097	20	5.9070018498
7	5.9066463619	14	5.9069476033	21	5.9072779365

TABLE 4. The growth rate of least prime primitive roots

$G(p)$	p	$\frac{G(p)}{\log p}$	$\frac{G(p)}{\log^2 p}$	$\frac{G(p)}{e^\gamma \log p(\log\log p)^2}$
2	3	1.820478	1.657070	115.559706
3	7	1.541695	0.792274	1.953084
5	23	1.594644	0.508578	0.685570
7	41	1.884977	0.507591	0.614838
11	109	2.344741	0.499801	0.551000
19	191	3.617481	0.688745	0.738260
43	271	7.675667	1.370136	1.451413
53	2791	6.679980	0.841927	0.874297
79	11971	8.412988	0.895928	0.941673
107	31771	10.321899	0.995715	1.059694
149	190321	12.256850	1.008257	1.102962
151	2080597	10.379315	0.713444	0.812905
163	3545281	10.808210	0.716671	0.824196
211	4022911	13.874717	0.912359	1.051558
223	73189117	12.314619	0.680044	0.824189
263	137568061	14.034429	0.748917	0.917462
277	443571241	13.912348	0.698748	0.873004
307	565822531	15.232866	0.755831	0.948147
347	1160260711	16.625214	0.796535	1.011101
349	1622723341	16.456541	0.775982	0.990421

TABLE 5. The least prime numbers $p < 2^{31}$ and their least prime primitive roots

$G(p)$	$g(p)$	p	Factorization of $p-1$
2	2	3	2
3	3	7	2,3
5	5	23	2,11
7	6	41	2(3), 5
11	6	109	2(2), 3(3)
13	13	457	2(3), 3, 19
17	17	311	2, 5, 31
19	19	191	2, 5, 19
23	10	2137	2(3), 3, 89
29	21	409	2(3), 3, 17
31	10	1021	2(2), 3, 5, 17
37	14	1031	2, 5, 103
41	6	1811	2, 5, 181
43	6	271	2, 3(3), 5
47	6	14293	2(2), 3(2), 397
53	6	2791	2, 3(2), 5, 31
59	38	55441	2(4), 3(2), 5, 7, 11
61	12	35911	2, 3(3), 5, 7, 19
67	6	57991	2, 3, 5, 1933
71	22	221101	2(2), 3, 5(2), 11, 67
73	6	23911	2, 3, 5, 797
79	10	11971	2, 3(2), 5, 7, 19
83	69	110881	2(5), 3(2), 5, 7, 11
89	6	103091	2, 5, 13(2), 61
97	44	71761	2(4), 3, 5, 13, 23
101	6	513991	2, 3(2), 5, 5711
103	35	290041	2(3), 3, 5, 2417
107	10	31771	2, 3(2), 5, 353
109	14	448141	2(2), 3, 5, 7, 11, 97
113	33	2447761	2(4), 3, 5, 7, 31, 47
127	6	674701	2(2), 3, 5(2), 13, 173
131	10	3248701	2(2), 3, 5(2), 7(2), 13, 17
137	10	2831011	2, 3, 5, 7, 13, 17, 61
139	18	690541	2(2), 3, 5, 17, 677

Table 5. (continued)

149	14	190321	$2(4), 3, 5, 13, 61$
151	6	2080597	$2(2), 3, 7, 17, 31, 47$
157	33	4076641	$2(5), 3(2), 5, 19, 149$
163	14	3545281	$2(6), 3(2), 5, 1231$
167	33	11643607	$2, 3(2), 13, 17, 2927$
173	18	16135981	$2(2), 3, 5, 7, 103, 373$
179	94	5109721	$2(3), 3, 5, 7(2), 11, 79$
181	15	9633751	$2, 3, 5(4), 7, 367$
191	38	25400761	$2(3), 3, 5, 7, 11, 2749$
193	15	25738831	$2, 3(3), 5, 13, 7333$
197	22	399263281	$2(4), 3, 5, 13, 73, 1753$
199	6	37565431	$2, 3, 5, 7, 41, 4363$
211	6	4022911	$2, 3(2), 5, 44699$
223	6	73189117	$2(2), 3(3), 7, 11, 13, 677$
227	6	298155271	$2, 3, 5, 7, 71, 19997$
229	6	741488749	$2(2), 3, 7, 11, 13, 61729$
223	6	453507991	$2, 3, 5, 13, 31, 37511$
239	12	187155691	$2, 3, 5, 1223, 5101$
241	14	449032321	$2(7), 3(2), 5, 11, 19, 373$
251	22	672618871	$2, 3(3), 5, 7, 11, 32353$
257	10	794932741	$2(2), 3(2), 5, 7, 630899$
263	14	137568061	$2(2), 3(2), 5, 7, 23, 47, 101$
277	57	443571241	$2(3), 3, 5, 7, 29, 131, 139$
283	22	1095701881	$2(3), 3, 5, 7, 13, 19, 5281$
307	12	565822531	$2, 3(3), 5, 7, 13, 23029$
347	15	1160260711	$2, 3, 5, 7(2), 17, 29, 1601$
349	6	1622723341	$2(2), 3, 5, 7, 1151, 2557$

It is still an open problem whether $E^*(x)$ tends to a constant value when x tends to infinity. Table 3 and the graph of the function $E^*(x)$ for $x < 2.1 \cdot 10^9$ (Figure 12) allow us to believe that $E^*(x)$ will really tend to a constant.

With the aid of computer programs, the average values of least prime primitive roots were computed. Table 3 collects these values.

Table 4 registers the very first occurrence of a prime number as a least prime primitive root greater than the previous one. With the aid of the table one can approximate the growth rate of prime primitive roots. It can easily be seen that the growth rate of the least prime primitive root of primes is well approximated by small powers of logarithms of these primes.

E. Bach [2] surmises, giving probabilistic arguments, that

$$\limsup_{p\to\infty} \frac{G(p)}{\log p(\log\log p)^2} = e^\gamma,$$

where γ in the above formula is equal to the Euler constant $0.5772\ldots$.

The validity of the above limit may be of great importance for practical purposes, e.g., for primality testing. The existence a small primitive root of a prime number is the basic assumption in many primality testing strategies.

Table 4 supports the correctness of Bach's computations. In the Table 5 we present minimal prime numbers p having prescribed least prime primitive roots $q \leq 349$, corresponding to $G(p) = q$, the least primitive root $g(p)$, and the factorization of $p - 1$. We see that all primes below 350 with the exception of $311, 313, 317, 331$ and 337 occur as the least prime primitive root of a prime less than 2^{31}.

3. Conclusions and proposals for future investigations

We derived a conditional formula for the natural density $E(p_n)$ of prime numbers p having its least prime primitive root equal to p_n. For every prime number from the interval $[3, 2147483647 (= 2^{31} - 1)]$ the least prime primitive root has been found. Under the generalized Riemann hypothesis, densities $E(p_n)$ of prime numbers having their least prime primitive root equal to the prime p_n, where $p_n < 100$, were computed (Table 1). These values were compared with empirical values (Table 2). The agreement of both: theoretical and practical results are surprisingly good.

Relying on the computed material, the average value of the least prime primitive root has been found (Table 3).

It seems reasonable (Table 4) to majorize the value of the least prime primitive root of a prime by a constant multiple of the square of the natural logarithm of that prime. It would be useful to find a stronger theoretical estimate than that found by Ankeny [1] on the extended Riemann hypothesis, namely $G(p) = O(Y^2 \log^2 Y)$, where $Y = 2^{\omega(p-1)} \log p$ and $\omega(n)$ is the number of distinct prime factors of n. It is highly probable that the estimate can be improved to the form $G(p) < \log^{1+\varepsilon} p$, where ε can be any positive number as suggested by E. Bach.

We extended the investigations of least (unrestricted) primitive roots to the bound $3 \cdot 10^{10}$, but they were stopped because of highly time-consuming computations. The results will be submitted for publication in the near future. It would be very useful to extend the computations of least (prime) primitive roots for all primes $p < 10^{11}$ or higher, but for this project much more powerful machines should be applied.

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Note added in proof

The computation described in the paper has been carried further, up to the limit 10^{12} by A. Paszkiewicz and 10^{14} by Tomas Oliveira e Silva, University of Aveiro, Portugal.

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