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NEW CONVERGENCE RESULTS ON THE GENERALIZED RICHARDSON EXTRAPOLATION PROCESS GREP$^{(1)}$ FOR LOGARITHMIC SEQUENCES

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ABSTRACT. Let $a(t) \sim A + \varphi(t)\sum_{i=0}^{\infty}\beta_i t^i$ as $t \to 0+$, where $a(t)$ and $\varphi(t)$ are known for $0 < t \le c$ for some $c > 0$, but A and the β_i are not known. The generalized Richardson extrapolation process GREP$^{(1)}$ is used in obtaining good approximations to A, the limit or antilimit of $a(t)$ as $t \to 0+$. The approximations $A_n^{(j)}$ to A obtained via GREP$^{(1)}$ are defined by the linear systems $a(t_l) = A_n^{(j)} + \varphi(t_l)\sum_{i=0}^{n-1}\bar{\beta}_i t_l^i$, $l = j, j+1, \ldots, j+n$, where $\{t_l\}_{l=0}^{\infty}$ is a decreasing positive sequence with limit zero. The study of GREP$^{(1)}$ for slowly varying functions $a(t)$ was begun in two recent papers by the author. For such $a(t)$ we have $\varphi(t) \sim \alpha t^{\delta}$ as $t \to 0+$ with δ possibly complex and $\delta \ne 0, -1, -2, \ldots$. In the present work we continue to study the convergence and stability of GREP$^{(1)}$ as it is applied to such $a(t)$ with different sets of collocation points t_l that have been used in practical situations. In particular, we consider the cases in which (i) t_l are arbitrary, (ii) $\lim_{l\to\infty} t_{l+1}/t_l = 1$, (iii) $t_l \sim cl^{-q}$ as $l \to \infty$ for some $c, q > 0$, (iv) $t_{l+1}/t_l \le \omega \in (0,1)$ for all l, (v) $\lim_{l\to\infty} t_{l+1}/t_l = \omega \in (0,1)$, and (vi) $t_{l+1}/t_l = \omega \in (0,1)$ for all l.

1. INTRODUCTION AND GENERAL BACKGROUND

In two recent papers Sidi [Si6] and Sidi [Si7] we began a theoretical investigation of the convergence and stability of GREP$^{(1)}$, the simplest case and prototype of the generalized Richardson extrapolation process GREP$^{(m)}$ due to the author, see Sidi [Si1]. Here m is a positive integer. GREP$^{(m)}$ is a very effective extrapolation method that is used in accelerating the convergence of a very large family of infinite sequences that arise from and/or can be identified with functions $A(y)$ that belong to a certain class of functions denoted $\mathbf{F}^{(m)}$.

In the present work we continue the investigation of [Si6] and [Si7] by adding various theoretical results pertaining to the application of GREP$^{(1)}$ to functions $A(y) \in \mathbf{F}^{(1)}$ that vary slowly. What is meant by slowly varying $A(y)$ will become clear shortly.

We recall that $A(y) \in \mathbf{F}^{(1)}$ if there exist a constant A and a function $\beta(y)$ such that

$$A(y) = A + \phi(y)\beta(y), \quad y \in (0, b] \text{ for some } b > 0, \tag{1.1}$$

where y can be a continuous or discrete variable, and $\beta(\xi)$, as a function of the continuous variable ξ, is continuous in $[0, \hat{\xi}]$ for some $\hat{\xi} > 0$ and has a Poincaré-type

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TABLE 1. The Romberg table.

$A_0^{(0)}$				
$A_0^{(1)}$	$A_1^{(0)}$			
$A_0^{(2)}$	$A_1^{(1)}$	$A_2^{(0)}$		
$A_0^{(3)}$	$A_1^{(2)}$	$A_2^{(1)}$	$A_3^{(0)}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

asymptotic expansion of the form

$$\beta(\xi) \sim \sum_{i=0}^{\infty} \beta_i \xi^{ir} \quad \text{as } \xi \to 0+, \quad \text{for some fixed } r > 0. \tag{1.2}$$

We also recall that $A(y) \in \mathbf{F}_{\infty}^{(1)}$ if the function $B(t) \equiv \beta(t^{1/r})$, as a function of the continuous variable t, is infinitely differentiable in $[0, \hat{\xi}^r]$. (Therefore, in the variable t, $B(t) \in C[0, \hat{t}]$ for some $\hat{t} > 0$ and (1.2) reads $B(t) \sim \sum_{i=0}^{\infty} \beta_i t^i$ as $t \to 0+$.)

We assume that the functions $A(y)$ and $\phi(y)$ are computable hence known for $y \in (0, b]$ (keeping in mind that y may be discrete or continuous depending on the situation) and that the constant r is known. The aim is to find (or approximate) A that is $\lim_{y \to 0+} A(y)$ when this limit exists and the antilimit of $A(y)$ otherwise.

Approximations to A can be obtained by GREP$^{(1)}$ that is defined via the linear systems

$$A(y_l) = A_n^{(j)} + \phi(y_l) \sum_{i=0}^{n-1} \bar{\beta}_i y_l^{ir}, \quad l = j, j+1, \ldots, j+n, \tag{1.3}$$

where $\{y_l\} \subset (0, b]$ such that $y_0 > y_1 > y_2 > \cdots$, and $\lim_{l \to \infty} y_l = 0$. Here $A_n^{(j)}$ is the approximation to A and $\bar{\beta}_i$ are additional auxiliary unknowns.

The approximations $A_n^{(j)}$ can be arranged in a two-dimensional array called the Romberg table (see Table 1).

Two limiting processes pertaining to the $A_n^{(j)}$ are of importance: (i) Process I, in which n is held fixed and $j \to \infty$, and (ii) Process II, in which j is held fixed and $n \to \infty$. Thus, Process I concerns the convergence of the columns in the Romberg table, while Process II concerns the convergence of the diagonals.

If we set $t = y^r$, $a(t) = A(y)$, $\varphi(t) = \phi(y)$, and $t_l = y_l^r$, $l = 0, 1, \ldots$, then, provided $\varphi(t_l) \neq 0$, $l = 0, 1, \ldots$, which we assume throughout, we can express $A_n^{(j)}$ as in

$$A_n^{(j)} = \frac{D_n^{(j)}\{a(t)/\varphi(t)\}}{D_n^{(j)}\{1/\varphi(t)\}}, \tag{1.4}$$

where $D_n^{(j)}\{g(t)\}$ denotes the divided difference of the function $g(t)$ over the set of points $\{t_j, t_{j+1}, \ldots, t_{j+n}\}$ and is thus given by

$$D_n^{(j)}\{g(t)\} = \sum_{i=0}^{n} c_{ni}^{(j)} g(t_{j+i}); \quad c_{ni}^{(j)} = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{1}{t_{j+i} - t_{j+k}}, \quad i = 0, 1, \ldots, n. \tag{1.5}$$

From (1.4) and (1.5) it is clear that $A_n^{(j)}$ can also be expressed as in

$$(1.6)\qquad A_n^{(j)} = \sum_{i=0}^{n} \gamma_{ni}^{(j)} a(t_{j+i}); \quad \gamma_{ni}^{(j)} = \frac{c_{ni}^{(j)}/\varphi(t_{j+i})}{D_n^{(j)}\{1/\varphi(t)\}}, \quad i = 0, 1, \dots, n.$$

Obviously, $\sum_{i=0}^{n} \gamma_{ni}^{(j)} = 1$.

A quantity that is of special importance in the stability analysis of GREP$^{(1)}$ is

$$(1.7)\qquad \Gamma_n^{(j)} = \sum_{i=0}^{n} |\gamma_{ni}^{(j)}| = \frac{1}{|D_n^{(j)}\{1/\varphi(t)\}|} \sum_{i=0}^{n} \frac{|c_{ni}^{(j)}|}{|\varphi(t_{j+i})|}.$$

Of course, $\Gamma_n^{(j)} \geq 1$. We recall that $\Gamma_n^{(j)}$ determines the rate at which errors (round-off or other) in the $a(t_l)$ propagate into $A_n^{(j)}$. We also recall that if $\sup_j \Gamma_n^{(j)} < \infty$ with n fixed, then the sequence $\{A_n^{(j)}\}_{j=0}^{\infty}$ is stable, i.e., Process I is stable. Similarly, if $\sup_n \Gamma_n^{(j)} < \infty$ with j fixed, the sequence $\{A_n^{(j)}\}_{n=0}^{\infty}$ is stable, i.e., Process II is stable.

The $A_n^{(j)}$ and $\Gamma_n^{(j)}$ can be computed recursively in a very economical fashion by the W-algorithm of Sidi [Si4] and [Si6] as follows:

1. Set $M_0^{(j)} = a(t_j)/\varphi(t_j)$, $N_0^{(j)} = 1/\varphi(t_j)$, and $H_0^{(j)} = (-1)^j |N_0^{(j)}|$, $j = 0, 1, \dots$.
2. Compute $M_n^{(j)}$, $N_n^{(j)}$, and $H_n^{(j)}$ recursively from

$$(1.8)\qquad M_n^{(j)} = \frac{M_{n-1}^{(j+1)} - M_{n-1}^{(j)}}{t_{j+n} - t_j}, \quad N_n^{(j)} = \frac{N_{n-1}^{(j+1)} - N_{n-1}^{(j)}}{t_{j+n} - t_j}, \quad \text{and}$$

$$H_n^{(j)} = \frac{H_{n-1}^{(j+1)} - H_{n-1}^{(j)}}{t_{j+n} - t_j}, \quad j = 0, 1, \dots, \quad n = 1, 2, \dots.$$

3. Set

$$(1.9)\qquad A_n^{(j)} = \frac{M_n^{(j)}}{N_n^{(j)}} \quad \text{and} \quad \Gamma_n^{(j)} = \left| \frac{H_n^{(j)}}{N_n^{(j)}} \right|.$$

Surprisingly, GREP$^{(1)}$ is quite amenable to rigorous and refined analysis, and the conclusions that we draw from the study of GREP$^{(1)}$ are relevant to GREP$^{(m)}$ with arbitrary m, in general. It is important to note that the analytic study of GREP$^{(1)}$ is made possible by the divided difference representations of $A_n^{(j)}$ and $\Gamma_n^{(j)}$ that are given in (1.4) and (1.7) respectively. With the help of these we are able to produce results that are optimal or nearly optimal in many cases. We must also add that not all problems associated with GREP$^{(1)}$ have been solved, however. In particular, various problems concerning Process II are still open.

The slowly varying functions $A(y)$ we alluded to above are those for which

$$(1.10)\quad \varphi(t) = t^{\delta} H(t); \quad \delta \neq 0, -1, -2, \dots, \quad H(t) \sim \sum_{i=0}^{\infty} h_i t^i \text{ as } t \to 0+, \; h_0 \neq 0.$$

Here δ can be complex in general. Thus, $A(y)$ has the asymptotic behavior $A(y) \sim A + h_0 y^{\gamma}$ as $y \to 0+$, for some γ that is related to δ through $\gamma = r\delta$. We note that a sufficient condition for (1.10) to hold is that $H(t) \in C^{\infty}[0, \hat{t}]$ for some $\hat{t} > 0$, although this condition is not necessary in general. In the next section we present examples of such $A(y)$ that arise from some classes of infinite integrals and series. In particular, they arise very naturally from convergent or divergent infinite sequences that behave logarithmically, as has been shown in [Si6]. We recall that it is very difficult to accelerate the convergence of such sequences, the source of the difficulty being the instability of the extrapolation processes.

In this work we aim at presenting a detailed convergence and stability analysis of GREP$^{(1)}$ for slowly varying $A(y) = a(t)$ and different choices of the t_l. As will be seen later, some of these choices give rise to instability, while others do not. In particular, in Section 4 we deal with the case in which the t_l are arbitrary. In Section 5 we consider the case in which $\lim_{l\to\infty}(t_{l+1}/t_l) = 1$ in general and $t_l \sim cl^{-q}$ as $l \to \infty$ for some $c > 0$ and $q > 0$. In Section 6 we look at the cases in which $\lim_{l\to\infty}(t_{l+1}/t_l) = \omega$ for some $\omega \in (0,1)$. Finally, in Section 7 we analyze the case in which $t_{l+1} \leq \omega t_l$ for all l, again for some $\omega \in (0,1)$. These choices of the t_l are the ones most commonly used in applications. Thus, by drawing the proper analogies, the results of this chapter and the next apply very naturally to the $D^{(1)}$-, $d^{(1)}$-transformations of Levin and Sidi [LS], and to a new sequence transformation that we denote the $\tilde{d}^{(m)}$-transformation.

We shall come back to these transformations in Section 8, where we shall actually show how the conclusions drawn from the study of GREP$^{(1)}$ given here can be used to enhance their performance in finite-precision arithmetic.

In the next section we give some technical preliminaries that will be of use in the remainder of this work.

2. Examples of slowly varying $a(t)$

We now present practical examples of functions $a(t)$ that vary slowly.

Example 2.1. If $f(x) \sim \sum_{i=0}^{\infty} \nu_i x^{\gamma-i}$ as $x \to \infty$, for $\nu_0 \neq 0$ and for some possibly complex $\gamma \neq -1, 0, 1, 2, \dots$, then we know from Theorem 4.1 in [Si7] that

$$F(x) = \int_a^x f(t)\,dt = I[f] + xf(x)g(x); \quad g(x) \sim \sum_{i=0}^{\infty} g_i x^{-i} \text{ as } x \to \infty,\ g_0 \neq 0.$$

This means that $F(x) \leftrightarrow a(t)$, $I[f] \leftrightarrow A$, $x^{-1} \leftrightarrow t$, $xf(x) \leftrightarrow \varphi(t)$ with $\varphi(t)$ as above and with $\delta = -\gamma - 1$. This is valid both when $\int_a^\infty f(t)\,dt$ converges and when it diverges. In the former case $I[f] = \int_a^\infty f(t)\,dt$, while in the latter $I[f]$ is the Hadamard finite part of $\int_a^\infty f(t)\,dt$. The use of GREP$^{(1)}$ as in this example (with $t_l = 1/x_l$ for an increasing unbounded sequence $\{x_l\}$) results in the Levin-Sidi $D^{(1)}$-transformation for infinite integrals.

Example 2.2. If $a_n \sim \sum_{i=0}^{\infty} \nu_i n^{\gamma-i}$ as $n \to \infty$, for $\nu_0 \neq 0$ and for some possibly complex $\gamma \neq -1, 0, 1, 2, \dots$, then we know from Theorem 4.1 of [Si6] that

$$A_n = \sum_{k=1}^{n} a_k = S(\{a_k\}) + na_n g(n); \quad g(n) \sim \sum_{i=0}^{\infty} g_i n^{-i} \text{ as } n \to \infty,\ g_0 \neq 0.$$

This means that $A_n \leftrightarrow a(t)$, $S(\{a_k\}) \leftrightarrow A$, $n^{-1} \leftrightarrow t$, $na_n \leftrightarrow \varphi(t)$ with $\varphi(t)$ as above and with $\delta = -\gamma - 1$. This is valid both when $\sum_{n=1}^{\infty} a_n$ converges and when it diverges. In the former case $S(\{a_k\}) = \sum_{n=1}^{\infty} a_n$ while in the latter $S(\{a_k\})$ is the antilimit of $\sum_{n=1}^{\infty} a_n$. The use of GREP$^{(1)}$ as in this example (with $t_l = 1/R_l$ for an increasing sequence of integers $\{R_l\}$) results in the Levin-Sidi $d^{(1)}$-transformation for infinite series. With $t_l = 1/(l+1)$, $l = 0, 1, \dots$, the $d^{(1)}$-transformation reduces to the u-transformation of Levin [Le] that is one of the best convergence acceleration methods known. The convergence and stability of the Levin transformations were analyzed in the papers Sidi [Si2] and [Si3], whose results were summarized in Sidi [Si5].

Example 2.3. If $a_n \sim \sum_{i=0}^{\infty} \nu_i n^{\gamma-i/m}$ as $n \to \infty$, for $\nu_0 \neq 0$ and for some possibly complex $\gamma \neq -1, 0, 1, 2, \dots$, and a positive integer $m > 1$, then we can proceed as in the proof of Theorem 4.1 in [Si6] to show that

$$A_n = \sum_{k=1}^{n} a_k = S(\{a_k\}) + n a_n g(n); \quad g(n) \sim \sum_{i=0}^{\infty} g_i n^{-i/m} \text{ as } n \to \infty, \ g_0 \neq 0.$$

This means that $A_n \leftrightarrow a(t)$, $S(\{a_k\}) \leftrightarrow A$, $n^{-1/m} \leftrightarrow t$, $na_n \leftrightarrow \varphi(t)$ with $\varphi(t)$ as above and with $\delta = -\gamma - 1$. This is valid when $\sum_{n=1}^{\infty} a_n$ converges and when it diverges. In the former case $S(\{a_k\}) = \sum_{n=1}^{\infty} a_n$ while in the latter $S(\{a_k\})$ is the antilimit of $\sum_{n=1}^{\infty} a_n$. The use of GREP$^{(1)}$ as in this example (with $t_l = 1/R_l$ for an increasing sequence of integers $\{R_l\}$) obviously results in a new bona fide sequence transformation that we now denote the $\tilde{d}^{(m)}$-transformation. Numerical experiments show that this is a very effective convergence acceleration method for the type of infinite series considered here. Of course, the $\tilde{d}^{(1)}$-transformation is nothing but the $d^{(1)}$-transformation.

3. Technical preliminaries

We start by deriving an error formula for $A_n^{(j)}$ that has been stated and proved as Lemma 3.1 in [Si6].

Lemma 3.1. *The error in* $A_n^{(j)}$ *is given by*

$$A_n^{(j)} - A = \frac{D_n^{(j)}\{B(t)\}}{D_n^{(j)}\{1/\varphi(t)\}}; \quad B(t) \equiv \beta(t^{1/r}). \tag{3.1}$$

In some of our analysis we assume the functions $\varphi(t)$ and $B(t)$ to be differentiable, while in others no such requirement is imposed. Obviously, the assumption in the former case is quite strong, and this makes some of the proofs easier.

The following simple result on $A_n^{(j)}$ will become useful shortly.

Lemma 3.2. *If* $B(t) \in C^\infty[0, t_j]$ *and* $\psi(t) \equiv 1/\varphi(t) \in C^\infty(0, t_j]$, *then for any nonzero complex number* c,

$$A_n^{(j)} - A = \frac{\Re[cB^{(n)}(t'_{jn,1})] + \mathrm{i}\Im[cB^{(n)}(t'_{jn,2})]}{\Re[c\psi^{(n)}(t''_{jn,1})] + \mathrm{i}\Im[c\psi^{(n)}(t''_{jn,2})]} \tag{3.2}$$
$$\text{for some } t'_{jn,1},\ t'_{jn,2},\ t''_{jn,1},\ t''_{jn,2} \in (t_{j+n}, t_j).$$

Proof. It is known that if $f \in C^n[a,b]$ is real and $a \le x_0 \le x_1 \le \cdots \le x_n \le b$, then the divided difference $f[x_0, x_1, \dots, x_n]$ satisfies

$$f[x_0, x_1, \dots, x_n] = \frac{f^{(n)}(\xi)}{n!} \text{ for some } \xi \in (x_0, x_n).$$

Applying this to the real and imaginary parts of the complex-valued function $u(t) \in C^n(0, t_j)$, we have

(3.3)
$$D_n^{(j)}\{u(t)\} = \frac{1}{n!}\left[\Re u^{(n)}(t_{jn,1}) + \mathrm{i}\Im u^{(n)}(t_{jn,2})\right], \quad \text{for some } t_{jn,1},\ t_{jn,2} \in (t_{j+n}, t_j).$$

The result now follows. □

The introduction of the constant c in (3.2) will serve us in the proof of Theorem 4.1 in the next section.

Note that in many of our problems it is known that $B(t) \in C^\infty[0, \hat{t}]$, while $\psi(t) \in C^\infty(0, \hat{t}]$ only, for some $\hat{t} > 0$. That is to say, $B(t)$ has an infinite number of derivatives at $t = 0$ while $\psi(t)$ does not. This is an important observation.

Useful simplifications take place for the case $\varphi(t) = t$ that has been studied most extensively in Bulirsch and Stoer [BS] and Laurent [Laure]. The recursion relation for the $A_n^{(j)}$ in Lemma 3.3 was first given in [BS], for example.

Lemma 3.3. *If $\varphi(t) = t$, then the $A_n^{(j)}$ and the $\Gamma_n^{(j)}$ can be computed recursively from*

$$A_0^{(j)} = a(t_j) \quad \textit{and} \quad \Gamma_0^{(j)} = 1, \quad j = 0, 1, \dots, \tag{3.4}$$
$$A_n^{(j)} = \frac{t_j A_{n-1}^{(j+1)} - t_{j+n} A_{n-1}^{(j)}}{t_j - t_{j+n}} \quad \textit{and} \quad \Gamma_n^{(j)} = \frac{t_j \Gamma_{n-1}^{(j+1)} + t_{j+n}\Gamma_{n-1}^{(j)}}{t_j - t_{j+n}},$$
$$j = 0, 1, \dots, \quad n = 1, 2, \dots .$$

Lemma 3.4. *If $\varphi(t) = t$, then*

$$A_n^{(j)} - A = (-1)^n D_n^{(j)}\{B(t)\}\left(\prod_{i=0}^{n} t_{j+i}\right). \tag{3.5}$$

Thus, for some $t'_{jn,1},\ t'_{jn,2} \in (t_{j+n}, t_j)$, we have

$$A_n^{(j)} - A = (-1)^n \frac{\Re[B^{(n)}(t'_{jn,1})] + \mathrm{i}\Im[B^{(n)}(t'_{jn,2})]}{n!}\left(\prod_{i=0}^{n} t_{j+i}\right). \tag{3.6}$$

The proofs are achieved by invoking the fact that

$$D_n^{(j)}\{t^{-1}\} = (-1)^n \left(\prod_{i=0}^{n} t_{j+i}\right)^{-1}. \tag{3.7}$$

We leave the details to the reader.

Obviously, by imposing suitable growth conditions on $B^{(n)}(t)$, Lemma 3.4 can be turned into powerful convergence theorems.

The last result of this section is a slight refinement of a result of [Si1] concerning Process I as it applies to GREP$^{(1)}$.

Theorem 3.5. *Let $\sup_j \Gamma_n^{(j)} = \Lambda_n < \infty$ and $\phi(y_{j+1}) = O(\phi(y_j))$ as $j \to \infty$. Then, with n fixed,*

$$A_n^{(j)} - A = O(\varphi(t_j) t_j^{n+\mu}) \quad \textit{as } j \to \infty, \tag{3.8}$$

where $\beta_{n+\mu}$ is the first nonzero β_i with $i \geq n$.

Proof. Using in (3.1) the fact that $D_n^{(j)}\{\sum_{i=0}^{n-1} c_i t^i\} = 0$ for arbitrary constants c_i we first have

$$(3.9) \qquad A_n^{(j)} - A = \sum_{i=0}^{n} \gamma_{ni}^{(j)} \varphi(t_{j+i})[B(t_{j+i}) - u(t_{j+i})]; \quad u(t) = \sum_{k=0}^{n-1} \beta_k t^k.$$

The result now follows by taking moduli on both sides and realizing that $B(t) - u(t) \sim \beta_{n+\mu} t^{n+\mu}$ as $t \to 0+$ and recalling that $t_j > t_{j+1} > t_{j+2} > \cdots$. We leave the details to the reader. □

Note that Theorem 3.5 does not assume that $B(t)$ and $\varphi(t)$ are differentiable. It does, however, assume that Process I is stable.

As we shall see in the sequel, (3.8) holds under appropriate conditions on the t_l even when Process I is clearly unstable.

In connection with Process I we would like to remark that, under suitable conditions we can obtain a full asymptotic expansion for $A_n^{(j)} - A$ as $j \to \infty$. If we define

$$(3.10) \qquad \Phi_{n,k}^{(j)} = \frac{D_n^{(j)}\{t^k\}}{D_n^{(j)}\{1/\varphi(t)\}}, \quad k = 0, 1, \ldots,$$

and recall that $D_n^{(j)}\{t^k\} = 0$, for $k = 0, 1, \ldots, n-1$, then this expansion assumes the simple and elegant form

$$(3.11) \qquad A_n^{(j)} - A \sim \sum_{k=n}^{\infty} \beta_k \Phi_{n,k}^{(j)} \quad \text{as } j \to \infty.$$

If $\beta_{n+\mu}$ is the first nonzero β_i with $i \geq n$, then $A_n^{(j)} - A$ also satisfies the asymptotic equality

$$(3.12) \qquad A_n^{(j)} - A \sim \beta_{n+\mu} \Phi_{n,n+\mu}^{(j)} \quad \text{as } j \to \infty.$$

Of course, all this will be true provided that (i) $\{\Phi_{n,k}^{(j)}\}_{k=n}^{\infty}$ is an asymptotic sequence as $j \to \infty$, i.e., $\lim_{j\to\infty} \Phi_{n,k+1}^{(j)}/\Phi_{n,k}^{(j)} = 0$ for all $k \geq n$, and (ii) $A_n^{(j)} - A - \sum_{k=n}^{s-1} \beta_k \Phi_{n,k}^{(j)} = O(\Phi_{n,s}^{(j)})$ as $j \to \infty$, for each $s \geq n$. In the next sections we will aim at such results whenever possible. We will actually show that they are possible in most of the cases we treat.

We close this section with the well-known Hermite-Gennochi formula for divided differences that will be used later.

Lemma 3.6 (Hermite-Gennochi). *Let $f(x)$ be in $C^n[a, b]$, and let $x_0, x_1, \ldots, x_n$ be all in $[a, b]$. Then*

$$f[x_0, x_1, \ldots, x_n] = \int_{T_n} f^{(n)}\left(\sum_{i=0}^{n} \xi_i x_i\right) d\xi_1 \cdots d\xi_n,$$

where

$$T_n = \{(\xi_1, \ldots, \xi_n) : 0 \leq \xi_i \leq 1, \ i = 1, \ldots, n, \ \sum_{i=1}^{n} \xi_i \leq 1\}; \quad \xi_0 = 1 - \sum_{i=1}^{n} \xi_i.$$

For a proof of this lemma see, e.g., Atkinson [A]. Note that the argument $z = \sum_{i=0}^{n} \xi_i x_i$ of $f^{(n)}$ above is actually a convex combination of $x_0, x_1, \dots, x_n$ since $0 \le \xi_i \le 1$, $i = 0, 1, \dots, n$, and $\sum_{i=0}^{n} \xi_i = 1$. If we order the x_i such that $x_0 \le x_1 \le \cdots \le x_n$, then $z \in [x_0, x_n] \subseteq [a, b]$.

4. Analysis with arbitrary t_l

We start with the following surprising result that holds for arbitrary $\{t_l\}$. (Recall that so far the t_l satisfy only $t_0 > t_1 > \cdots > 0$ and $\lim_{l\to\infty} t_l = 0$.)

Theorem 4.1. *Let $\varphi(t) = t^{\delta} H(t)$, where δ is in general complex and $\delta \ne 0, -1, -2, \dots$, and $H(t) \in C^{\infty}[0, \hat{t}]$ for some $\hat{t} > 0$ with $h_0 \equiv H(0) \ne 0$. Let also $B(t) \in C^{\infty}[0, \hat{t}]$, and let $\beta_{n+\mu}$ be the first nonzero β_i with $i \ge n$ in (1.2). Then, provided $n \ge -\Re\delta$, we have*

$$A_n^{(j)} - A = O(\varphi(t_j) t_j^{n+\mu}) \quad \text{as } j \to \infty. \tag{4.1}$$

Consequently, if $n > -\Re\delta$, we have $\lim_{j\to\infty} A_n^{(j)} = A$. All this is valid for arbitrary $\{t_l\}$.

Proof. By the assumptions on $B(t)$ we have

$$B^{(n)}(t) \sim \sum_{i=n+\mu}^{\infty} i(i-1)\cdots(i-n+1)\beta_i t^{i-n} \quad \text{as } t \to 0+,$$

from which

$$B^{(n)}(t) \sim (\mu+1)_n \beta_{n+\mu} t^{\mu} \quad \text{as } t \to 0+, \tag{4.2}$$

and by the assumptions on $\varphi(t)$ we have for $\psi(t) \equiv 1/\varphi(t)$

$$\psi^{(n)}(t) = \sum_{k=0}^{n} \binom{n}{k} (t^{-\delta})^{(k)} [1/H(t)]^{(n-k)} \sim (t^{-\delta})^{(n)}/H(t) \sim (t^{-\delta})^{(n)}/h_0 \quad \text{as } t \to 0+,$$

from which

$$\psi^{(n)}(t) \sim (-1)^n h_0^{-1} (\delta)_n t^{-\delta-n} \sim (-1)^n (\delta)_n \psi(t) t^{-n} \quad \text{as } t \to 0+. \tag{4.3}$$

Also, Lemma 3.2 is valid for all sufficiently large j under the present assumptions on $B(t)$ and $\varphi(t)$, since $t_j < \hat{t}$ for all sufficiently large j. Substituting (4.2) and (4.3) in (3.2) with $|c| = 1$ there, we obtain

(4.4)
$$A_n^{(j)} - A = (-1)^n (\mu+1)_n \times \frac{[\Re(c\beta_{n+\mu}) + o(1)](t'_{jn,1})^{\mu} + \mathrm{i}\,[\Im(c\beta_{n+\mu}) + o(1)](t'_{jn,2})^{\mu}}{[\Re\alpha_{jn,1} + o(1)](t''_{jn,1})^{-\Re\delta-n} + \mathrm{i}\,[\Im\alpha_{jn,2} + o(1)](t''_{jn,2})^{-\Re\delta-n}} \quad \text{as } j \to \infty,$$

with $\alpha_{jn,s} \equiv c{h_0}^{-1}(\delta)_n (t''_{jn,s})^{-\mathrm{i}\Im\delta}$ and the $o(1)$ terms uniform in c, $|c| = 1$. Here we have also used the fact that $\lim_{j\to\infty} t'_{jn,s} = \lim_{j\to\infty} t''_{jn,s} = 0$. Next, by $0 < t'_{jn,s} < t_j$ and $\mu \ge 0$, it follows that $(t'_{jn,s})^{\mu} \le t_j^{\mu}$. This implies that the numerator of the quotient in (4.4) is $O(t_j^{\mu})$ as $j \to \infty$, *uniformly* in c, $|c| = 1$. As for the denominator, we start by observing that $a = h_0^{-1}(\delta)_n \ne 0$. Therefore, either $\Re a \ne 0$ or $\Im a \ne 0$, and we assume without loss of generality that $\Re a \ne 0$. If we now choose $c = (t''_{jn,1})^{\mathrm{i}\Im\delta}$, we obtain $\alpha_{jn,1} = a$ and hence $\Re\alpha_{jn,1} = \Re a \ne 0$, as a result of which the modulus of the denominator can be bounded from below by $|\Re a+o(1)|(t''_{jn,1})^{-\Re\delta-n}$, which in turn is bounded below by $|\Re a+o(1)|\,t_j^{-\Re\delta-n}$ since $0 < t''_{jn,s} < t_j$ and $\Re\delta + n \geq 0$. The result now follows by combining everything in (4.4) and by invoking $t^\delta = O(\varphi(t))$ as $t \to 0+$ that follows from $\varphi(t) \sim h_0 t^\delta$ as $t \to 0+$. □

Theorem 4.1 implies that the column sequence $\{A_n^{(j)}\}_{j=0}^\infty$ converges to A if $n > -\Re\delta$, and it also gives an upper bound on the rate of convergence through (4.1). The fact that convergence takes place for *arbitrary* $\{t_l\}$ and that we are actually able to prove that it does is quite unexpected.

By restricting $\{t_l\}$ only slightly in Theorem 4.1, we can show that $A_n^{(j)} - A$ has the full asymptotic expansion given in (3.11) and, as a result, satisfies the asymptotic equality of (3.12) as well. We start with the following lemma that turns out to be very useful in the sequel.

Lemma 4.2. *Let $g(t) = t^\theta u(t)$, where θ is in general complex and $u(t) \in C^\infty[0,\hat{t}]$ for some $\hat{t} > 0$. Pick the t_l to satisfy, in addition to $t_{l+1} < t_l$, $l = 0,1,\dots$, also $t_{l+1} \geq \nu t_l$ for all sufficiently large l with some $\nu \in (0,1)$. Then the following are true:*

(i) *The nonzero members of $\{D_n^{(j)}\{t^{\theta+i}\}\}_{i=0}^\infty$ form an asymptotic sequence as $j \to \infty$.*

(ii) *$D_n^{(j)}\{g(t)\}$ has the bona fide asymptotic expansion*

$$D_n^{(j)}\{g(t)\} \sim \sum_{i=0}^{\infty}{}^{*} g_i D_n^{(j)}\{t^{\theta+i}\} \quad \text{as } j \to \infty; \quad g_i = u^{(i)}(0)/i!,\ i = 0,1,\dots, \tag{4.5}$$

where the star on the summation means that only those terms for which $D_n^{(j)}\{t^{\theta+i}\} \neq 0$, i.e., for which $\theta + i \neq 0,1,\dots,n-1$, are taken into account.

Remark. The extra condition $t_{l+1} \geq \nu t_l$ for all large l that we have imposed on the t_l is satisfied, e.g., when $\lim_{l\to\infty}(t_{l+1}/t_l) = \lambda$ for some $\lambda \in (0,1]$, and such cases are considered further in the next sections.

Proof. Let α be in general complex and $\alpha \neq 0,1,\dots,n-1$. Denote $\binom{\alpha}{n} = M$ for simplicity of notation. Then, by (3.3), for any complex number c such that $|c| = 1$, we have

$$cD_n^{(j)}\{t^\alpha\} = \Re\left[cM(t_{jn,1})^{\alpha-n}\right] + \mathrm{i}\Im\left[cM(t_{jn,2})^{\alpha-n}\right] \quad \text{for some } t_{jn,1},\ t_{jn,2} \in (t_{j+n}, t_j), \tag{4.6}$$

from which we also have

$$|D_n^{(j)}\{t^\alpha\}| \geq \max\{|\Re\left[cM(t_{jn,1})^{\alpha-n}\right]|, |\Im\left[cM(t_{jn,2})^{\alpha-n}\right]|\}. \tag{4.7}$$

Since $M \neq 0$, we have either $\Re M \neq 0$ or $\Im M \neq 0$. Assume without loss of generality that $\Re M \neq 0$ and choose $c = (t_{jn,1})^{-\mathrm{i}\Im\alpha}$. Then $\Re\left[cM(t_{jn,1})^{\alpha-n}\right] = (\Re M)(t_{jn,1})^{\Re\alpha-n}$ and hence

$$|D_n^{(j)}\{t^\alpha\}| \geq |\Re M|(t_{jn,1})^{\Re\alpha-n} \geq |\Re M| \min_{t\in[t_{j+n},t_j]}(t^{\Re\alpha-n}). \tag{4.8}$$

Invoking in (4.8), if necessary, also the fact that $t_{j+n} \geq \nu^n t_j$ that is implied by the conditions on the t_l, we obtain

$$(4.9) \qquad |D_n^{(j)}\{t^\alpha\}| \geq C_{n1}^{(\alpha)} t_j^{\Re\alpha - n} \text{ for all large } j, \text{ with some constant } C_{n1}^{(\alpha)} > 0.$$

We can similarly show from (4.6) that

$$(4.10) \qquad |D_n^{(j)}\{t^\alpha\}| \leq C_{n2}^{(\alpha)} t_j^{\Re\alpha - n} \text{ for all large } j, \text{ with some constant } C_{n2}^{(\alpha)} > 0.$$

The proof of part (i) can now be achieved by using (4.9) and (4.10).

To prove part (ii) we need to show (in addition to part (i)) that, for any integer s for which $D_n^{(j)}\{t^{\theta+s}\} \neq 0$, i.e., for which $\theta + s \neq 0, 1, \ldots, n-1$, there holds

$$(4.11) \qquad D_n^{(j)}\{g(t)\} - \sum_{i=0}^{s-1} g_i D_n^{(j)}\{t^{\theta+i}\} = O\left(D_n^{(j)}\{t^{\theta+s}\}\right) \text{ as } j \to \infty.$$

Now $g(t) = \sum_{i=0}^{s-1} g_i t^{\theta+i} + v_s(t) t^{\theta+s}$, where $v_s(t) \in C^\infty[0, \hat{t}]$. As a result,

$$(4.12) \qquad D_n^{(j)}\{g(t)\} - \sum_{i=0}^{s-1} g_i D_n^{(j)}\{t^{\theta+i}\} = D_n^{(j)}\{v_s(t) t^{\theta+s}\}.$$

Next, by (3.3) and by the fact that

$$[v_s(t)t^{\theta+s}]^{(n)} = \sum_{i=0}^{n} \binom{n}{i} [v_s(t)]^{(n-i)} (t^{\theta+s})^{(i)} \sim v_s(t)(t^{\theta+s})^{(n)} \sim g_s(t^{\theta+s})^{(n)} \quad \text{as } t \to 0+,$$

and by the additional condition on the t_l again, we obtain

$$(4.13) \qquad D_n^{(j)}\{v_s(t)t^{\theta+s}\} = O(t_j^{\Re\theta+s-n}) = O(D_n^{(j)}\{t^{\theta+s}\}) \text{ as } j \to \infty,$$

the last equality being a consequence of (4.9). Here we assume that $g_s \neq 0$ without loss of generality. By substituting (4.13) in (4.12), the result in (4.11) follows. This completes the proof. □

Theorem 4.3. *Let $\varphi(t)$ and $B(t)$ be exactly as in Theorem 4.1, and choose the t_l as in Lemma 4.2. Then $A_n^{(j)} - A$ has the complete asymptotic expansion given in* (3.11) *and hence satisfies the asymptotic equality in* (3.12) *as well. Furthermore, if $\beta_{n+\mu}$ is the first nonzero β_i with $i \geq n$, then for all large j there holds*

(4.14)

$$\Omega_1 |\varphi(t_j)|\, t_j^{n+\mu} \leq |A_n^{(j)} - A| \leq \Omega_2 |\varphi(t_j)|\, t_j^{n+\mu}, \quad \text{for some } \Omega_1 > 0 \text{ and } \Omega_2 > 0,$$

whether $n \geq -\Re\delta$ or not.

Proof. The proof of the first part can be achieved by applying Lemma 4.2 to $B(t)$ and to $\psi(t) \equiv 1/\varphi(t)$. The proof of the second part can be achieved by employing (4.9) as well. We leave the details to the reader. □

Remark. It is important to make the following observations concerning the behavior of $A_n^{(j)} - A$ as $j \to \infty$ in Theorem 4.3. First, any column sequence $\{A_n^{(j)}\}_{j=0}^{\infty}$ converges at least as quickly as (or diverges at most as quickly as) the column sequence $\{A_{n-1}^{(j)}\}_{j=0}^{\infty}$ that precedes it. In other words, each column sequence is at least as good as the one preceding it. In particular, when $\beta_m \neq 0$ but $\beta_{m+1} = \cdots = \beta_{s-1} = 0$ and $\beta_s \neq 0$, we have

$$(4.15)\qquad \begin{aligned} A_n^{(j)} - A &= o(A_m^{(j)} - A) \ \text{ as } j \to \infty, \quad m+1 \leq n \leq s, \\ A_{s+1}^{(j)} - A &= o(A_s^{(j)} - A) \ \text{ as } j \to \infty. \end{aligned}$$

In addition, for all large j we have

$$(4.16)\qquad \begin{aligned} &\theta_{n1}|A_s^{(j)} - A| \leq |A_n^{(j)} - A| \leq \theta_{n2}|A_s^{(j)} - A|, \\ &\qquad\qquad m+1 \leq n \leq s-1, \quad \text{for some } \theta_{n1}, \ \theta_{n2} > 0, \end{aligned}$$

which implies that the column sequences $\{A_n^{(j)}\}_{j=0}^{\infty}$, $m+1 \leq n \leq s$, behave the same way for all large j.

In the next sections we continue the treatment of Process I by restricting the t_l further, and treat the issue of stability for Process I as well. In addition, we treat the convergence and stability of Process II.

5. Analysis with $\lim_{l\to\infty}(t_{l+1}/t_l) = 1$

5.1. Process I with $\varphi(t) = t^\delta H(t)$ and complex δ.

Theorem 5.1. *Assume that $\varphi(t)$ and $B(t)$ are exactly as in Theorem 4.1. In addition, choose the t_l such that $\lim_{l\to\infty}(t_{l+1}/t_l) = 1$. Then $A_n^{(j)} - A$ has the complete asymptotic expansion given in* (3.11) *and satisfies* (3.12) *and hence satisfies also the asymptotic equality*

$$(5.1)\qquad A_n^{(j)} - A \sim (-1)^n \frac{(\mu+1)_n}{(\delta)_n} \beta_{n+\mu} \varphi(t_j) t_j^{n+\mu} \quad \textit{as } j \to \infty,$$

where, again, $\beta_{n+\mu}$ is the first nonzero β_i with $i \geq n$ in (1.2). *This result is valid whether $n \geq -\Re\delta$ or not. In addition, Process I is unstable, i.e.,* $\sup_j \Gamma_n^{(j)} = \infty$.

Proof. First, Theorem 4.3 applies and thus (3.11) and (3.12) are valid.

Let us apply the Hermite-Gennochi formula of Lemma 3.6 to the function t^α, where α may be complex in general. By the assumption that $\lim_{l\to\infty}(t_{l+1}/t_l) = 1$, we have that the argument $z = \sum_{i=0}^{n} \xi_i t_{j+i}$ of the integrand in Lemma 3.6 satisfies $z \sim t_j$ as $j \to \infty$. As a result, we obtain

$$(5.2)\qquad D_n^{(j)}\{t^\alpha\} \sim \binom{\alpha}{n} t_j^{\alpha-n} \quad \text{as } j \to \infty, \quad \text{provided } \alpha \neq 0, 1, \ldots, n-1.$$

Next, applying Lemma 4.2 to $B(t)$ and to $\psi(t) \equiv 1/\varphi(t)$, and realizing that $\psi(t) \sim \sum_{i=0}^{\infty} \psi_i t^{-\delta+i}$ as $t \to 0+$ for some constants ψ_i with $\psi_0 = h_0^{-1}$, and using (5.2) as well, we have

(5.3)

$$D_n^{(j)}\{B(t)\} \sim \sum_{i=n+\mu}^{\infty} \beta_i D_n^{(j)}\{t^i\} \sim \beta_{n+\mu} D_n^{(j)}\{t^{n+\mu}\} \sim \binom{n+\mu}{n} \beta_{n+\mu} t_j^{\mu} \quad \text{as } j \to \infty,$$

and

(5.4)

$$D_n^{(j)}\{\psi(t)\} \sim \sum_{i=0}^{\infty}{}^{*} \psi_i D_n^{(j)}\{t^{-\delta+i}\} \sim h_0^{-1} D_n^{(j)}\{t^{-\delta}\} \sim \binom{-\delta}{n} \psi(t_j) t_j^{-n} \quad \text{as } j \to \infty.$$

The result in (5.1) is obtained by dividing (5.3) by (5.4).

For the proof of the second part we start by observing that when $\lim_{l\to\infty}(t_{l+1}/t_l) = 1$ we also have $\lim_{j\to\infty}(t_{j+k}/t_{j+i}) = 1$ for arbitrary fixed i and k. Therefore, for every $\epsilon > 0$, there exists a positive integer J, such that

$$(5.5) \qquad |t_{j+i} - t_{j+k}| = \left|1 - \frac{t_{j+k}}{t_{j+i}}\right| t_{j+i} < \epsilon t_{j+i} \le \epsilon t_j \quad \text{for } 0 \le i, k \le n \text{ and } j > J.$$

As a result of this,

$$(5.6) \qquad |c_{ni}^{(j)}| = \prod_{\substack{k=0 \\ k\neq i}}^{n} \frac{1}{|t_{j+i} - t_{j+k}|} > (\epsilon t_j)^{-n} \quad \text{for } i = 0, 1, \ldots, n, \text{ and } j > J.$$

Next, by the assumption that $H(t) \sim h_0$ as $t \to 0+$ and by $\lim_{j\to\infty}(t_{j+i}/t_j) = 1$, we have that $\psi(t_{j+i}) \sim \psi(t_j)$ as $j \to \infty$, from which $|\psi(t_{j+i})| \ge K_1|\psi(t_j)|$ for $0 \le i \le n$ and all j, where $K_1 > 0$ is a constant independent of j. Combining this with (5.6), we have

$$(5.7) \qquad \sum_{i=0}^{n} |c_{ni}^{(j)}|\,|\psi(t_{j+i})| \ge K_1(n+1)(\epsilon t_j)^{-n}|\psi(t_j)| \quad \text{for all } j > J.$$

Similarly, $|D_n^{(j)}\{\psi(t)\}| \le K_2|\psi(t_j)|\, t_j^{-n}$ for all j, where $K_2 > 0$ is another constant independent of j. (K_2 depends only on n.) Substituting this and (5.7) in (1.7), we obtain

(5.8)

$$\Gamma_n^{(j)} \ge M_n \epsilon^{-n} \quad \text{for all } j > J, \text{ with } M_n = (K_1/K_2)(n+1) \text{ independent of } \epsilon \text{ and } j.$$

Since ϵ can be chosen arbitrarily close to 0, (5.8) implies that $\sup_j \Gamma_n^{(j)} = \infty$. □

Obviously, the remarks following Theorem 4.3 are valid under the conditions of Theorem 5.1 too. In particular, (4.15) and (4.16) hold. Furthermore, (4.16) can now be refined to read

$$A_n^{(j)} - A \sim \theta_n (A_s^{(j)} - A) \quad \text{as } j \to \infty, \quad m+1 \le n \le s-1, \text{ for some } \theta_n \neq 0.$$

Finally, the column sequences $\{A_n^{(j)}\}_{j=0}^{\infty}$ with $n > -\Re\delta$ converge even though they are unstable.

In Theorem 5.3 below we show that the results of Theorem 5.1 remain unchanged if we restrict the t_l somewhat while we still require that $\lim_{l\to\infty}(t_{l+1}/t_l) = 1$, but relax the conditions on $\varphi(t)$ and $B(t)$ considerably. In fact, we do not put any differentiability requirements either on $\varphi(t)$ or on $B(t)$ this time, and obtain an asymptotic equality for $\Gamma_n^{(j)}$ in addition.

The following lemma that is analogous to Lemma 4.2 will be of use in the proof of Theorem 5.3.

Lemma 5.2. *Let $g(t) \sim \sum_{i=0}^{\infty} g_i t^{\theta+i}$ as $t \to 0+$, where $g_0 \neq 0$ and θ is in general complex, and let the t_l satisfy*

$$t_l \sim cl^{-q} \quad \text{and} \quad t_l - t_{l+1} \sim cpl^{-q-1} \quad \text{as } l \to \infty, \quad \text{for some } c > 0,\ p > 0,\ \text{and } q > 0. \tag{5.9}$$

Then the following are true:

(i) *The nonzero members of $\{D_n^{(j)}\{t^{\theta+i}\}\}_{i=0}^{\infty}$ form an asymptotic sequence as $j \to \infty$.*

(ii) *$D_n^{(j)}\{g(t)\}$ has the bona fide asymptotic expansion*

$$D_n^{(j)}\{g(t)\} \sim \sum_{i=0}^{\infty}{}^{*} g_i D_n^{(j)}\{t^{\theta+i}\} \quad \text{as } j \to \infty, \tag{5.10}$$

where the star on the summation means that only those terms for which $D_n^{(j)}\{t^{\theta+i}\} \neq 0$, i.e., for which $\theta + i \neq 0, 1, \ldots, n-1$, are taken into account.

Remark. Note that $\lim_{l\to\infty}(t_{l+1}/t_l) = 1$ under (5.9), and that (5.9) is satisfied by $t_l = c(l+\eta)^{-q}$, for example. Also, the first part of (5.9) does not necessarily imply the second part.

Proof. Part (i) is true by Lemma 4.2 since $\lim_{l\to\infty}(t_{l+1}/t_l) = 1$. In particular, (5.2) holds. To prove part (ii) we need to show in addition that, for any integer s for which $D_n^{(j)}\{t^{\theta+s}\} \neq 0$, there holds

$$D_n^{(j)}\{g(t)\} - \sum_{i=0}^{s-1} g_i D_n^{(j)}\{t^{\theta+i}\} = O\left(D_n^{(j)}\{t^{\theta+s}\}\right) \quad \text{as } j \to \infty. \tag{5.11}$$

Now $g(t) = \sum_{i=0}^{m-1} g_i t^{\theta+i} + v_m(t) t^{\theta+m}$, where $|v_m(t)| \leq C_m$ for some constant $C_m > 0$ and for all t sufficiently close to 0, and this holds for every m. Let us fix s and take $m > \max\{s + n/q, -\Re\theta\}$. We can write

$$D_n^{(j)}\{g(t)\} = \sum_{i=0}^{s-1} g_i D_n^{(j)}\{t^{\theta+i}\} + \sum_{i=s}^{m-1} g_i D_n^{(j)}\{t^{\theta+i}\} + D_n^{(j)}\{v_m(t) t^{\theta+m}\}. \tag{5.12}$$

Let us assume without loss of generality that $g_s \neq 0$. Then by part (i) of the lemma

$$\sum_{i=s}^{m-1} g_i D_n^{(j)}\{t^{\theta+i}\} \sim g_s D_n^{(j)}\{t^{\theta+s}\} \sim g_s \binom{\theta+s}{n} t_j^{\theta+s-n} \quad \text{as } j \to \infty\ .$$

Therefore, the proof will be complete if we show that $D_n^{(j)}\{v_m(t) t^{\theta+m}\} = O\left(t_j^{\theta+s-n}\right)$ as $j \to \infty$. Using also the fact that $t_{j+i} \sim t_j$ as $j \to \infty$, we first have that

$$|D_n^{(j)}\{v_m(t) t^{\theta+m}\}| \leq C_m \sum_{i=0}^{n} |c_{ni}^{(j)}|\, t_{j+i}^{\Re\theta+m} \leq C_m t_j^{\Re\theta+m} \left(\sum_{i=0}^{n} |c_{ni}^{(j)}| \right). \tag{5.13}$$

Next, from (5.9)

$$t_{j+i} - t_{j+k} \sim cp(k-i) j^{-q-1} \sim p(k-i) j^{-1} t_j \quad \text{as } j \to \infty, \tag{5.14}$$

as a result of which,

(5.15)

$$c_{ni}^{(j)} = \prod_{\substack{k=0\\k\neq i}}^{n} \frac{1}{t_{j+i}-t_{j+k}} \sim (-1)^i \frac{1}{n!}\binom{n}{i}\left(\frac{j}{pt_j}\right)^n \quad \text{and} \quad \sum_{i=0}^{n} |c_{ni}^{(j)}| \sim \frac{1}{n!}\left(\frac{2j}{pt_j}\right)^n$$

$$\text{as } j \to \infty.$$

Substituting (5.15) in (5.13) and noting that (5.9) implies $j \sim (t_j/c)^{-1/q}$ as $j \to \infty$, we obtain

$$D_n^{(j)}\{v_m(t)t^{\theta+m}\} = O(t_j^{\Re\theta+m-n-n/q}) = O(t_j^{\Re\theta+s-n}) \quad \text{as } j \to \infty, \tag{5.16}$$

by the fact that $\Re\theta + m - n - n/q > \Re\theta + s - n$. The result now follows. □

Theorem 5.3. *Assume that* $\varphi(t) = t^\delta H(t)$, *with* δ *in general complex and* $\delta \neq 0, -1, -2, \dots$, $H(t) \sim \sum_{i=0}^\infty h_i t^i$ *as* $t \to 0+$ *and* $B(t) \sim \sum_{i=0}^\infty \beta_i t^i$ *as* $t \to 0+$. *Let us pick the* t_l *to satisfy* (5.9). *Then* $A_n^{(j)} - A$ *has the complete asymptotic expansion given in* (3.11) *and satisfies* (3.12), *and hence satisfies also the asymptotic equality in* (5.1). *In addition,* $\Gamma_n^{(j)}$ *satisfies the asymptotic equality*

$$\Gamma_n^{(j)} \sim \frac{1}{|(\delta)_n|}\left(\frac{2j}{p}\right)^n \quad \text{as } j \to \infty. \tag{5.17}$$

That is to say, Process I is unstable.

Proof. The assertion concerning $A_n^{(j)}$ can be proved by applying Lemma 5.2 to $B(t)$ and to $\psi(t) \equiv 1/\varphi(t)$, and proceeding as in the proof of Theorem 5.1.

We now turn to the analysis of $\Gamma_n^{(j)}$. To prove the asymptotic equality in (5.17), we need the precise asymptotic behaviors of $\sum_{i=0}^n |c_{ni}^{(j)}||\psi(t_{j+i})|$ and $D_n^{(j)}\{\psi(t)\}$ as $j \to \infty$. By (5.15) and by the fact that $\psi(t_{j+i}) \sim \psi(t_j)$ as $j \to \infty$ for all fixed i, we obtain

$$\sum_{i=0}^{n} |c_{ni}^{(j)}|\,|\psi(t_{j+i})| \sim \left(\sum_{i=0}^{n} |c_{ni}^{(j)}|\right)|\psi(t_j)| \sim \frac{1}{n!}\left(\frac{2j}{pt_j}\right)^n |\psi(t_j)| \quad \text{as } j \to \infty. \tag{5.18}$$

Combining now (5.18) and (5.4) in (1.7), the result in (5.17) follows. □

So far all our results have been on Process I. What characterizes these results is that they are all obtained by considering only the *local* behavior of $B(t)$ and $\varphi(t)$ as $t \to 0+$. The reason for this is that $A_n^{(j)}$ is determined only by $a(t_l)$, $j \le l \le j+n$, and that in Process I we are letting $j \to \infty$ or, equivalently, $t_l \to 0$, $j \le l \le j+n$. In Process II, on the other hand, we are holding j fixed and letting $n \to \infty$. This means, of course, that $A_n^{(j)}$ is being influenced by the behavior of $a(t)$ on the *fixed* interval $(0, t_j]$. Therefore, we need to employ *global* information on $a(t)$ in order to analyze Process II. It is precisely this point that makes Process II much more difficult to study than Process I.

An additional source of difficulty when analyzing Process II with $\varphi(t) = t^\delta H(t)$ is complex values of δ. Indeed, except for Theorem 6.2 in the next section, we do not have any results on Process II under the assumption that δ is complex. Our analysis in the remainder of this section assumes real δ.

5.2. **Process II with $\varphi(t) = t$.** We now would like to present results pertaining to Process II. We start with the case $\varphi(t) = t$. Our first result concerns convergence and follows trivially from Lemma 3.4 as follows: Assuming that $B(t) \in C^\infty[0, t_j]$ and letting

$$\|B^{(n)}\| = \max_{0 \le t \le t_j} |B^{(n)}(t)| \tag{5.19}$$

we have

$$\left| \frac{D_n^{(j)}\{B(t)\}}{D_n^{(j)}\{t^{-1}\}} \right| \le \frac{\|B^{(n)}\|}{n!} \left(\prod_{i=0}^{n} t_{j+i} \right), \tag{5.20}$$

from which we have that $\lim_{n\to\infty} A_n^{(j)} = A$ when $\varphi(t) = t$ provided that

$$\frac{\|B^{(n)}\|}{n!} = o\left(\prod_{i=0}^{n} t_{j+i}^{-1} \right) \quad \text{as } n \to \infty. \tag{5.21}$$

In the special case $t_l = c/(l + \eta)^q$ for some positive $c, \eta,$ and q, this condition reads

$$\|B^{(n)}\| = o((n!)^{q+1} c^{-n} n^{(j+\eta)q}) \quad \text{as} \quad n \to \infty. \tag{5.22}$$

We are thus assured of convergence in this case under a very generous growth condition on $\|B^{(n)}\|$, especially when $q \ge 1$.

Our next result in the theorem below pertains to stability of Process II.

Theorem 5.4. *Consider $\varphi(t) = t$ and pick the t_l such that $\lim_{l\to\infty}(t_{l+1}/t_l) = 1$. Then Process II is unstable, i.e., $\sup_n \Gamma_n^{(j)} = \infty$. If the t_l are as in* (5.9), *then $\Gamma_n^{(j)} \to \infty$ as $n \to \infty$ faster than n^σ for every $\sigma > 0$. If, in particular, $t_l = c/(l + \eta)^q$ for some positive $c, \eta,$ and q, then*

$$\Gamma_n^{(j)} > E_q^{(j)} n^{-1/2} \left(\frac{e}{q} \right)^{qn} \quad \textit{for some} \quad E_q^{(j)} > 0, \quad q = 1, 2. \tag{5.23}$$

Proof. We already know that when $\varphi(t) = t$ we can compute the $\Gamma_n^{(j)}$ by the recursion relation in (3.4), which can also be written in the form

$$\Gamma_n^{(j)} = \Gamma_{n-1}^{(j+1)} + \frac{w_n^{(j)}}{1 - w_n^{(j)}} \left(\Gamma_{n-1}^{(j)} + \Gamma_{n-1}^{(j+1)} \right); \quad w_n^{(j)} = \frac{t_{j+n}}{t_j} < 1. \tag{5.24}$$

Hence,

$$\Gamma_n^{(j)} \ge \Gamma_{n-1}^{(j+1)} \ge \Gamma_{n-2}^{(j+2)} \ge \cdots, \tag{5.25}$$

from which $\Gamma_n^{(j)} \ge \Gamma_s^{(j+n-s)}$ for *arbitrary fixed s*. Applying now Theorem 5.1, we have $\lim_{n\to\infty} \Gamma_s^{(j+n-s)} = \infty$ for $s \ge 1$ from which $\lim_{n\to\infty} \Gamma_n^{(j)} = \infty$ follows. When the t_l are as in (5.9), we have from (5.17) that $\Gamma_s^{(j+n-s)} \sim \frac{1}{s!}\left(\frac{2}{p}\right)^s n^s$ as $n \to \infty$. From this and from the fact that s is arbitrary, we now deduce that $\Gamma_n^{(j)} \to \infty$ as $n \to \infty$ faster than n^σ for every $\sigma > 0$. To prove the last part we start with

$$\gamma_{ni}^{(j)} = \prod_{\substack{k=0 \\ k \ne i}}^{n} \frac{t_{j+k}}{t_{j+k} - t_{j+i}}, \quad i = 0, 1, \ldots, n, \tag{5.26}$$

that follows from (1.6) and (3.7). The result in (5.23) follows from $\Gamma_n^{(j)} > |\gamma_{nn}^{(j)}|$. $\square$

We can expand on the last part of Theorem 5.4 by deriving *upper* bounds on $\Gamma_n^{(j)}$ when $t_l = c/(l+\eta)^q$ for some positive c, η, and q. In this case, we first show that $w_n^{(j+1)} \geq w_n^{(j)}$ from which we can prove by induction and with the help of (5.24) that $\Gamma_n^{(j)} \leq \Gamma_n^{(j+1)}$. Using this in (5.24), we obtain the inequality $\Gamma_n^{(j)} \leq \Gamma_{n-1}^{(j+1)}[(1+w_n^{(j)})/(1-w_n^{(j)})]$, and, by induction,

$$\Gamma_n^{(j)} \leq \Gamma_0^{(j+n)} \left(\prod_{i=0}^{n-1} \frac{1+w_{n-i}^{(j+i)}}{1-w_{n-i}^{(j+i)}} \right). \tag{5.27}$$

Finally, we set $\Gamma_0^{(j+n)} = 1$ and bound the product in (5.27). It can be shown that when $q = 2$, $\Gamma_n^{(j)} = O(n^{-1/2}(e^2/3)^n)$ as $n \to \infty$. This result is due to Laurie [Lauri]. On the basis of this result Laurie concludes that the error propagation in Romberg integration with the harmonic sequence of stepsizes is relatively mild.

5.3. **Process II with $\varphi(t) = t^\delta H(t)$ and real δ.** We now want to extend Theorem 5.4 to the general case in which $\varphi(t) = t^\delta H(t)$, where δ is real and $\delta \neq 0, -1, -2, \dots$, and $H(t) \in C^\infty[0, t_j]$ with $H(t) \neq 0$ on $[0, t_j]$. In order to do this we need additional analytical tools. We shall make use of these tools in the next sections as well. The results we have obtained for the case $\varphi(t) = t$ will also prove to be very useful in the sequel.

Lemma 5.5. *Let δ_1 and δ_2 be two real numbers and $\delta_1 \neq \delta_2$. Define $\Delta_i(t) = t^{-\delta_i}$, $i = 1, 2$. Then, provided $\delta_1 \neq 0, -1, -2, \dots$,*

$$D_n^{(j)}\{\Delta_2(t)\} = \frac{(\delta_2)_n}{(\delta_1)_n} \tilde{t}_{jn}^{\delta_1-\delta_2} D_n^{(j)}\{\Delta_1(t)\} \quad \textit{for some } \tilde{t}_{jn} \in (t_{j+n}, t_j).$$

Corollary 5.6. *Let $\delta_1 > \delta_2$ in Lemma 5.5. Then for arbitrary $\{t_l\}$*

$$\left| \frac{(\delta_2)_n}{(\delta_1)_n} \right| t_{j+n}^{\delta_1-\delta_2} \leq \frac{|D_n^{(j)}\{\Delta_2(t)\}|}{|D_n^{(j)}\{\Delta_1(t)\}|} \leq \left| \frac{(\delta_2)_n}{(\delta_1)_n} \right| t_j^{\delta_1-\delta_2},$$

from which we also have

$$\frac{|D_n^{(j)}\{\Delta_2(t)\}|}{|D_n^{(j)}\{\Delta_1(t)\}|} \leq K n^{\delta_2-\delta_1} t_j^{\delta_1-\delta_2} = o(1) \quad \textit{as } j \to \infty \quad \textit{and/or} \quad \textit{as } n \to \infty,$$

for some constant $K > 0$ independent of j and n. Consequently, for arbitrary real θ and arbitrary $\{t_l\}$, the nonzero members of $\{D_n^{(j)}\{t^{\theta+i}\}\}_{i=0}^\infty$ form an asymptotic sequence as $j \to \infty$.

For the proofs of these results see [Si6].

The next lemma expresses $\Gamma_n^{(j)}$ and $A_n^{(j)} - A$ in factored forms. By analyzing each of the factors it becomes easier to obtain good bounds from which powerful results on Process II can be obtained.

Lemma 5.7. *Consider $\varphi(t) = t^\delta H(t)$ with δ real and $\delta \neq 0, -1, -2, \dots$. Define*

$$X_n^{(j)} = \frac{D_n^{(j)}\{t^{-1}\}}{D_n^{(j)}\{t^{-\delta}\}} \quad \textit{and} \quad Y_n^{(j)} = \frac{D_n^{(j)}\{t^{-\delta}\}}{D_n^{(j)}\{t^{-\delta}/H(t)\}}. \tag{5.28}$$

Define also

$$\check{\Gamma}_n^{(j)}(\delta) = \frac{1}{|D_n^{(j)}\{t^{-\delta}\}|} \sum_{i=0}^n |c_{ni}^{(j)}| \, t_{j+i}^{-\delta}. \tag{5.29}$$

Then

$$\check{\Gamma}_n^{(j)}(\delta) = |X_n^{(j)}| (\check{t}_{jn})^{1-\delta} \check{\Gamma}_n^{(j)}(1) \quad \text{for some } \check{t}_{jn} \in (t_{j+n}, t_j), \tag{5.30}$$

$$\Gamma_n^{(j)} = |Y_n^{(j)}|\, |H(\hat{t}_{jn})|^{-1} \check{\Gamma}_n^{(j)}(\delta) \quad \text{for some } \hat{t}_{jn} \in (t_{j+n}, t_j), \tag{5.31}$$

and

$$A_n^{(j)} - A = X_n^{(j)} Y_n^{(j)} \frac{D_n^{(j)}\{B(t)\}}{D_n^{(j)}\{t^{-1}\}}. \tag{5.32}$$

In addition,

$$X_n^{(j)} = \frac{n!}{(\delta)_n} (\tilde{t}_{jn})^{\delta-1} \quad \text{for some } \tilde{t}_{jn} \in (t_{j+n}, t_j), \tag{5.33}$$

These results are valid for all choices of $\{t_l\}$.

Proof. To prove (5.30) we start by writing (5.29) in the form

$$\check{\Gamma}_n^{(j)}(\delta) = |X_n^{(j)}| \frac{1}{|D_n^{(j)}\{t^{-1}\}|} \sum_{i=0}^{n} \left(|c_{ni}^{(j)}|\, t_{j+i}^{-1} \right) t_{j+i}^{1-\delta}.$$

The result follows by observing that, by continuity of $t^{1-\delta}$ for $t > 0$,

$$\sum_{i=0}^{n} \left(|c_{ni}^{(j)}|\, t_{j+i}^{-1} \right) t_{j+i}^{1-\delta} = \left(\sum_{i=0}^{n} |c_{ni}^{(j)}|\, t_{j+i}^{-1} \right) (\check{t}_{jn})^{1-\delta} \quad \text{for some } \check{t}_{jn} \in (t_{j+n}, t_j),$$

and by invoking (5.29) with $\delta = 1$. The proof of (5.31) proceeds along the same lines, while (5.32) is a trivial identity. Finally, (5.33) follows from Lemma 5.5. □

In the next two theorems we adopt the notation and definitions of Lemma 5.7. The first of these theorems concerns the stability of Process II, while the second concerns its convergence.

Theorem 5.8. *Let δ be real and $\delta \neq 0, -1, -2, \ldots$, and let the t_l be as in (5.9). Then the following are true:*

(i) $\check{\Gamma}_n^{(j)}(\delta) \to \infty$ *faster than n^σ for every $\sigma > 0$, i.e., Process II for $\varphi(t) = t^\delta$ is unstable.*

(ii) *Let $\varphi(t) = t^\delta H(t)$ with $H(t) \in C^\infty[0, t_j]$ and $H(t) \neq 0$ on $[0, t_j]$. Assume that*

$$|Y_n^{(j)}| \geq C_1 n^{\alpha_1} \quad \text{for all } n;\ C_1 > 0 \text{ and } \alpha_1 \text{ constants}. \tag{5.34}$$

Then $\Gamma_n^{(j)} \to \infty$ as $n \to \infty$ faster than n^σ for every $\sigma > 0$, i.e., Process II is unstable.

Proof. Substituting (5.33) in (5.30), we obtain

$$\check{\Gamma}_n^{(j)}(\delta) = \frac{n!}{|(\delta)_n|} \left(\frac{\tilde{t}_{jn}}{\check{t}_{jn}} \right)^{\delta-1} \check{\Gamma}_n^{(j)}(1). \tag{5.35}$$

Invoking the asymptotic equality

$$\frac{(a)_n}{(b)_n} = \frac{\Gamma(b)}{\Gamma(a)} \frac{\Gamma(n+a)}{\Gamma(n+b)} \sim \frac{\Gamma(b)}{\Gamma(a)} n^{a-b} \quad \text{as } n \to \infty,$$

in (5.35), we have for all large n

$$(5.36)\qquad \check{\Gamma}_n^{(j)}(\delta) \geq K(\delta) n^{1-\delta} \left(\frac{t_{j+n}}{t_j} \right)^{|\delta-1|} \check{\Gamma}_n^{(j)}(1) \quad \text{for some constant } K(\delta) > 0.$$

Now $t_{j+n}^{|\delta-1|} \sim c^{|\delta-1|} n^{-q|\delta-1|}$ as $n \to \infty$ and, by Theorem 5.4, $\check{\Gamma}_n^{(j)}(1) \to \infty$ as $n \to \infty$ faster than n^σ for every $\sigma > 0$. Consequently, $\check{\Gamma}_n^{(j)}(\delta) \to \infty$ as $n \to \infty$ faster than n^σ for every $\sigma > 0$ as well. The assertion about $\Gamma_n^{(j)}$ can now be proved by using this result in (5.31) along with (5.34) and the fact that $|H(\hat{t}_{jn})|^{-1} \geq \left(\max_{t \in [0,t_j]} |H(t)|\right)^{-1} > 0$ independently of n. □

The purpose of the next theorem is to give as good a bound as possible for $|A_n^{(j)} - A|$ in Process II. A convergence result can then be obtained by imposing suitable and liberal growth conditions on $\|B^{(n)}\|$ and $Y_n^{(j)}$ and recalling that $D_n^{(j)}\{t^{-1}\} = (-1)^n / \left(\prod_{i=0}^n t_{j+i}\right)$.

Theorem 5.9. *Assume that $B(t) \in C^\infty[0, t_j]$ and define $\|B^{(n)}\|$ as in (5.19). Let $\varphi(t)$ be as in Theorem 5.8. Then, for some constant $L > 0$,*

$$(5.37)\qquad |A_n^{(j)} - A| \leq L\, |Y_n^{(j)}| \left(\max_{t \in [t_{j+n}, t_j]} t^{\delta-1} \right) n^{1-\delta} \frac{\|B^{(n)}\|}{n!} \left(\prod_{i=0}^n t_{j+i} \right).$$

We note that it is quite reasonable to assume that $Y_n^{(j)}$ is bounded as $n \to \infty$. The "justification" for this assumption is that $Y_n^{(j)} = \Delta^{(n)}(t'_{jn}) / \psi^{(n)}(t''_{jn})$ for some t'_{jn} and $t''_{jn} \in (t_{j+n}, t_j)$, where $\Delta(t) = t^{-\delta}$ and $\psi(t) = 1/\varphi(t) = t^{-\delta}/H(t)$, and that $\Delta^{(n)}(t)/\psi^{(n)}(t) \sim H(0)$ as $t \to 0+$. Indeed, when $1/H(t)$ is a polynomial in t, we have precisely $D_n^{(j)}\{t^{-\delta}/H(t)\} \sim D_n^{(j)}\{t^{-\delta}\}/H(0)$ as $n \to \infty$, as can be shown with the help of the corollary to Lemma 5.5 from which $Y_n^{(j)} \sim H(0)$ as $n \to \infty$. See also Lemma 7.5. Next, with the t_l as in (5.9), we also have that $(\max_{t \in [t_{j+n}, t_j]} t^{\delta-1})$ grows at most like $n^{q|\delta-1|}$ as $n \to \infty$. Thus, the product $|Y_n^{(j)}|(\max_{t \in [t_{j+n}, t_j]} t^{\delta-1})$ in (5.37) grows at most like a power of n as $n \to \infty$, and, consequently the main behavior of $|A_n^{(j)} - A|$ as $n \to \infty$ is determined by $(\|B^{(n)}\|/n!) \left(\prod_{i=0}^n t_{j+i}\right)$. We also note that the strength of (5.37) is primarily due to the factor $\prod_{i=0}^n t_{j+i}$ that tends to zero as $n \to \infty$ essentially like $(n!)^{-q}$ when the t_l satisfy (5.9). We recall that what produces this important factor is Lemma 5.5.

6. Analysis with $\lim_{l\to\infty}(t_{l+1}/t_l) = \omega \in (0,1)$

As is clear from our results in Section 5, both Process I and Process II are unstable when $\varphi(t)$ is slowly changing and the t_l satisfy (5.9) or, at least in some cases, when the t_l satisfy even the weaker condition $\lim_{l\to\infty}(t_{l+1}/t_l) = 1$. These results also show that convergence will take place in Process II nevertheless under rather liberal growth conditions for $B^{(n)}(t)$. The implication of this is that a required level of accuracy in the numerically computed $A_n^{(j)}$ may be achieved by computing the $a(t_l)$ with sufficiently high accuracy. This strategy is quite practical and has been employed successfully in numerical calculation of multiple integrals.

In case the accuracy with which $a(t)$ is computed is fixed and the $A_n^{(j)}$ are required to have comparable numerical accuracy, we need to choose the t_l such that the $A_n^{(j)}$ can be computed stably. When $\varphi(t) = t^\delta H(t)$ with $H(0) \neq 0$ and

$H(t)$ continuous in a right neighborhood of $t = 0$, best results for $A_n^{(j)}$ and $\Gamma_n^{(j)}$ are obtained by choosing $\{t_l\}$ such that $t_l \to 0$ as $l \to \infty$ *exponentially* in l. There are a few ways of achieving this and each of them has been used successfully in various problems.

Our first results with such $\{t_l\}$ given in Theorem 6.1 below concern Process I and, like those of Theorems 5.1 and 5.3, they are best asymptotically. These results were given as Theorems 2.1 and 3.1 in [Si6], which in turn, are special cases of Theorems 2.2 and 2.4 in [Si5].

Theorem 6.1. *Let $\varphi(t) = t^\delta H(t)$ with δ in general complex and $\delta \neq 0, -1, -2, \dots$, and $H(t) \sim H(0) \neq 0$ as $t \to 0+$. Pick the t_l such that $\lim_{l\to\infty}(t_{l+1}/t_l) = \omega$ for some fixed $\omega \in (0,1)$. Define*

$$c_k = \omega^{\delta+k-1}, \quad k = 1, 2, \dots . \tag{6.1}$$

Then, for fixed n, (3.11) and (3.12) hold, and we also have

$$A_n^{(j)} - A \sim \left(\prod_{i=1}^{n} \frac{c_{n+\mu+1} - c_i}{1 - c_i}\right) \beta_{n+\mu}\varphi(t_j)t_j^{n+\mu} \quad as \quad j \to \infty, \tag{6.2}$$

where $\beta_{n+\mu}$ is the first nonzero β_i with $i \geq n$ in (1.2). This result is valid whether $n \geq -\Re\delta$ or not. Also

$$\lim_{j\to\infty} \sum_{i=0}^{n} \gamma_{ni}^{(j)} z^i = \prod_{i=1}^{n} \frac{z - c_i}{1 - c_i} \equiv \sum_{i=0}^{n} \rho_{ni} z^i, \tag{6.3}$$

so that $\lim_{j\to\infty} \Gamma_n^{(j)}$ exists and

$$\lim_{j\to\infty} \Gamma_n^{(j)} = \sum_{i=0}^{n} |\rho_{ni}| = \prod_{i=1}^{n} \frac{1 + |c_i|}{|1 - c_i|}, \tag{6.4}$$

hence Process I is stable.

We note that Theorem 6.1 is valid also when $\varphi(t)$ satisfies $\varphi(t) \sim h_0 t^\delta |\log t|^\gamma$ as $t \to 0+$ with arbitrary γ. Obviously, this is a weaker condition than the one imposed on $\varphi(t)$ in the theorem.

Upon comparing Theorem 6.1 with Theorem 5.1 we realize that the remarks that follow the proof of Theorem 5.1 and that concern the convergence of column sequences are valid without any changes also under the conditions of Theorem 6.1.

So far we do not have results on Process II with $\varphi(t)$ and $\{t_l\}$ as in Theorem 6.1. We are able to provide some analysis for the cases in which δ is real, however. This is the subject of the next section.

We are able to give very strong results on Process II for the case in which $\{t_l\}$ is a truly geometric sequence. The conditions we impose on $\varphi(t)$ in this case are extremely weak, in the sense that $\varphi(t) = t^\delta H(t)$ with δ complex in general and $H(t)$ not necessarily differentiable at $t = 0$.

Theorem 6.2. *Let $\varphi(t) = t^\delta H(t)$ with δ in general complex and $\delta \neq 0, -1, -2, \dots$, and $H(t) = H(0) + O(t^\theta)$ as $t \to 0+$, with $H(0) \neq 0$ and $\theta > 0$. Pick the t_l such that $t_l = t_0\omega^l$, $l = 0, 1, \dots$, for some $\omega \in (0,1)$. Define $c_k = \omega^{\delta+k-1}$, $k = 1, 2, \dots$. Then, for any fixed j, Process II is both stable and convergent whether $\lim_{t\to 0+} a(t)$ exists or not. In particular, we have $\lim_{n\to\infty} A_n^{(j)} = A$ with*

$$A_n^{(j)} - A = O(\omega^{\sigma n}) \quad as \quad n \to \infty, \quad for\ every \quad \sigma > 0, \tag{6.5}$$

and $\sup_n \Gamma_n^{(j)} < \infty$ *with*

$$\lim_{n\to\infty} \Gamma_n^{(j)} = \prod_{i=1}^{\infty} \frac{1+|c_i|}{|1-c_i|} < \infty. \tag{6.6}$$

The convergence result of (6.5) *can be refined as follows: With* $B(t) \in C[0,\hat{t}]$ *for some* $\hat{t} > 0$, *define*

$$\hat{\beta}_s = \max_{t\in[0,\hat{t}]} \left(|B(t) - \sum_{i=0}^{s-1} \beta_i t^i|/t^s \right), \ s = 0,1,\ldots, \tag{6.7}$$

or when $B(t) \in C^{\infty}[0,\hat{t}]$, *define*

$$\tilde{\beta}_s = \max_{t\in[0,\hat{t}]} (|B^{(s)}(t)|/s!), \ s = 0,1,\ldots . \tag{6.8}$$

If $\hat{\beta}_n$ *or* $\tilde{\beta}_n$ *is* $O(e^{\sigma n^{\tau}})$ *as* $n \to \infty$ *for some* $\sigma > 0$ *and* $\tau < 2$, *then, for any* $\epsilon > 0$ *such that* $\omega + \epsilon < 1$,

$$A_n^{(j)} - A = O\left((\omega+\epsilon)^{n^2/2} \right) \quad as \ \ n \to \infty. \tag{6.9}$$

We refer the reader to the paper Sidi [Si7] for the proofs of the results in (6.5), (6.6), and (6.9).

We would like to make the following observation about Theorem 6.2. We first note that all the results in this theorem are independent of θ, i.e., of the details of $\varphi(t) - H(0)t^{\delta}$ as $t \to 0+$. Next, (6.5) implies that all diagonal sequences $\{A_n^{(j)}\}_{n=0}^{\infty}$, $j = 0,1,\ldots$, converge, and the error $A_n^{(j)} - A$ tends to 0 as $n \to \infty$ faster than $e^{-\lambda n}$ for *every* $\lambda > 0$, i.e., the convergence is *superlinear*. Under the additional growth condition imposed on $\hat{\beta}_n$ or $\tilde{\beta}_n$ we have that $A_n^{(j)} - A$ tends to 0 as $n \to \infty$ at the rate of $e^{-\kappa n^2}$ for *some* $\kappa > 0$. Note that this condition is very liberal and is satisfied in most practical situations. It holds, for example, when $\hat{\beta}_n$ or $\tilde{\beta}_n$ are $O(\,(pn)!\,)$ as $n \to \infty$ for *some* $p > 0$. Also, it is quite interesting that $\lim_{n\to\infty} \Gamma_n^{(j)}$ is independent of j, as seen from (6.6).

Finally, we note that Theorem 6.1 pertaining to Process I holds under the conditions of Theorem 6.2 without any changes as $\lim_{l\to\infty}(t_{l+1}/t_l) = \omega$ is obviously satisfied since $t_{l+1}/t_l = \omega$ for all l.

7. Analysis with real δ and $t_{l+1}/t_l \leq \omega \in (0,1)$

In this section we would like to consider the convergence and stability properties of Process II when $\{t_l\}$ is not necessarily a geometric sequence as in Theorem 6.2 or $\lim_{l\to\infty}(t_{l+1}/t_l)$ does not necessarily exist as in Theorem 6.1. We are now concerned with the choice

$$t_{l+1}/t_l \leq \omega, \ l = 0,1,\ldots, \quad \text{for some fixed } \ \omega \in (0,1). \tag{7.1}$$

If $\lim_{l\to\infty}(t_{l+1}/t_l) = \lambda$ for some $\lambda \in (0,1)$, then given $\epsilon > 0$ such that $\omega = \lambda + \epsilon < 1$, there exists an integer $L > 0$ such that

$$\lambda - \epsilon < t_{l+1}/t_l < \lambda + \epsilon \ \text{ for all } \ l \geq L. \tag{7.2}$$

Thus, if $t_0, t_1, \ldots, t_{L-1}$ are chosen appropriately, the sequence $\{t_l\}$ automatically satisfies (7.1). Consequently, the results of this section apply also to the case in which $\lim_{l\to\infty}(t_{l+1}/t_l) = \lambda \in (0,1)$.

7.1. **The case $\varphi(t) = t$.** The case that has been studied most extensively under (7.1) is that of $\varphi(t) = t$, and we will treat this case first. The outcome of this treatment will prove to be very useful in the study of the general case.

We start with the stability problem. As was done in Lemma 5.7, we shall denote the $\Gamma_n^{(j)}$ corresponding to $\varphi(t) = t^\delta$ by $\check{\Gamma}_n^{(j)}(\delta)$, and its corresponding $\gamma_{ni}^{(j)}$ by $\check{\gamma}_{ni}^{(j)}(\delta)$.

Theorem 7.1. *With $\varphi(t) = t$ and the t_l as in (7.1), we have for all j and n*

$$\text{(7.3)}\qquad \check{\Gamma}_n^{(j)}(1) = \sum_{i=0}^{n} \left|\check{\gamma}_{ni}^{(j)}(1)\right| \leq \Lambda_n \equiv \prod_{i=1}^{n} \frac{1+\omega^i}{1-\omega^i} < \prod_{i=1}^{\infty} \frac{1+\omega^i}{1-\omega^i} < \infty.$$

Therefore, both Process I and Process II are stable. Furthermore, for each fixed i, we have $\lim_{n\to\infty} \check{\gamma}_{ni}^{(j)}(1) = 0$, with

$$\text{(7.4)}\qquad \check{\gamma}_{ni}^{(j)}(1) = O(\omega^{n^2/2+d_i n}) \quad \textit{as } n \to \infty, \ d_i \textit{ a constant.}$$

Proof. Let $\bar{t}_l = t_0\omega^l$, $l = 0, 1, \ldots$, and denote the $\gamma_{ni}^{(j)}$ and $\Gamma_n^{(j)}$ appropriate for $\varphi(t) = t$ and the $\bar{t}_l$, respectively, by $\bar{\gamma}_{ni}^{(j)}$ and $\bar{\Gamma}_n^{(j)}$. By (5.26), we have that

$$\text{(7.5)}\qquad \begin{aligned} |\check{\gamma}_{ni}^{(j)}(1)| &= \left(\prod_{k=0}^{i-1} \frac{1}{1 - t_{j+i}/t_{j+k}}\right)\left(\prod_{k=i+1}^{n} \frac{1}{t_{j+i}/t_{j+k} - 1}\right) \\ &\leq \left(\prod_{k=0}^{i-1} \frac{1}{1-\omega^{i-k}}\right)\left(\prod_{k=i+1}^{n} \frac{1}{\omega^{i-k}-1}\right) \\ &= \left(\prod_{k=0}^{i-1} \frac{1}{1 - \bar{t}_{j+i}/\bar{t}_{j+k}}\right)\left(\prod_{k=i+1}^{n} \frac{1}{\bar{t}_{j+i}/\bar{t}_{j+k} - 1}\right) = |\bar{\gamma}_{ni}^{(j)}|. \end{aligned}$$

Therefore, $\check{\Gamma}_n^{(j)}(1) \leq \bar{\Gamma}_n^{(j)}$. But $\bar{\Gamma}_n^{(j)} = \Lambda_n$ by Theorem 2.1 in [Si7]. The relation in (7.4) is a consequence of the fact that

$$\text{(7.6)}\qquad |\bar{\gamma}_{ni}^{(j)}| = \left(\prod_{k=1}^{n}(1-c_k)\right)^{-1} \sum_{1\leq k_1<\cdots<k_{n-i}\leq n} c_{k_1}\cdots c_{k_{n-i}},$$

with $c_k = \omega^k$, $k = 1, 2, \ldots$, that in turn follows from

$$\sum_{i=0}^{n} \bar{\gamma}_{ni}^{(j)} z^i = \prod_{k=1}^{n} (z - c_k)/(1 - c_k).$$

□

We mention here that the fact that $\check{\Gamma}_n^{(j)}(1)$ is bounded uniformly both in j and in n was originally proved by Laurent [Laure]. The refined bound in (7.3) was given without proof in [Si6].

Now that we have proved that Process I is stable, we can apply Theorem 3.5 and conclude that $\lim_{j\to\infty} A_n^{(j)} = A$ with

$$\text{(7.7)}\qquad A_n^{(j)} - A = O(t_j^{n+\mu+1}) \quad \text{as } j \to \infty,$$

without assuming that $B(t)$ is differentiable in a right neighborhood of $t = 0$.

Since Process II satisfies the conditions of the Silverman-Toeplitz theorem, see, e.g., Hardy [H] or Powell and Shah [PS], we also have $\lim_{n\to\infty} A_n^{(j)} = A$. We now turn to the convergence issue for Process II to provide realistic rates of convergence for it. We start with the following important lemma due to Bulirsch and Stoer [BS]. The proof of this lemma is very lengthy and difficult and we, therefore, refer the reader to the original paper.

Lemma 7.2. *With $\varphi(t) = t$ and the t_l as in (7.1), we have for each integer $s \in \{0, 1, \dots, n\}$*

$$\sum_{i=0}^{n} |\check{\gamma}_{ni}^{(j)}(1)|\, t_{j+i}^{s+1} \leq M \left(\prod_{k=n-s}^{n} t_{j+k} \right) \tag{7.8}$$

for some constant $M > 0$ independent of j, n, and s.

This lemma becomes very useful in the proof of the convergence of Process II.

Lemma 7.3. *Let us pick the t_l to satisfy (7.1). Then, with j fixed,*

$$\frac{|D_n^{(j)}\{B(t)\}|}{|D_n^{(j)}\{t^{-1}\}|} = O(\omega^{\sigma n}) \quad \text{as } n \to \infty, \quad \text{for every } \sigma > 0. \tag{7.9}$$

This result can be refined as follows: Define $\hat{\beta}_s$ exactly as in Theorem 6.2. If $\hat{\beta}_n = O(e^{\sigma n^\tau})$ as $n \to \infty$ for some $\sigma > 0$ and $\tau < 2$, then, for any $\epsilon > 0$ such that $\omega + \epsilon < 1$,

$$\frac{|D_n^{(j)}\{B(t)\}|}{|D_n^{(j)}\{t^{-1}\}|} = O(\, (\omega + \epsilon)^{n^2/2}) \quad \text{as } n \to \infty. \tag{7.10}$$

Proof. From (3.9) we have for each $s \leq n$

(7.11)

$$Q_n^{(j)} \equiv \frac{|D_n^{(j)}\{B(t)\}|}{|D_n^{(j)}\{t^{-1}\}|} \leq \sum_{i=0}^{n} |\check{\gamma}_{ni}^{(j)}(1)|\, E_s(t_{j+i})\, t_{j+i}\,; \quad E_s(t) \equiv |B(t) - \sum_{k=0}^{s-1} \beta_k t^k|.$$

By (1.2) there exist constants $\eta_s > 0$ such that $E_s(t) \leq \eta_s t^s$ when $t \in [0, \hat{t}]$ for some $\hat{t} > 0$ and also when $t = t_l > \hat{t}$ (note that there are at most finitely many $t_l > \hat{t}$). Therefore, (7.11) becomes

$$Q_n^{(j)} \leq \eta_s \sum_{i=0}^{n} |\check{\gamma}_{ni}^{(j)}(1)|\, t_{j+i}^{s+1} \leq M \eta_s \left(\prod_{k=n-s}^{n} t_{j+k} \right). \tag{7.12}$$

the last inequality being a consequence of Lemma 7.2. The result in (7.9) follows from (7.12) once we observe by (7.1) that $\prod_{k=n-s}^{n} t_{j+k} = O(\omega^{n(s+1)})$ as $n \to \infty$ with s fixed but arbitrary.

To prove the second part we use the definition of $\hat{\beta}_s$ to rewrite (7.11) (with $s = n$) in the form

$$Q_n^{(j)} \leq \sum_{t_{j+i} > \hat{t}} |\check{\gamma}_{ni}^{(j)}(1)|\, E_n(t_{j+i})\, t_{j+i} + \hat{\beta}_n \sum_{t_{j+i} \leq \hat{t}} |\check{\gamma}_{ni}^{(j)}(1)|\, t_{j+i}^{n+1}. \tag{7.13}$$

Since $E_n(t) \leq |B(t)| + \sum_{k=0}^{n-1} |\beta_k|\, t^k$, $|\beta_k| \leq \hat{\beta}_k$ for each k and $\hat{\beta}_n$ grows at most like $e^{\sigma n^\tau}$ for $\tau < 2$, $\check{\gamma}_{ni}^{(j)}(1) = O(\omega^{n^2/2 + d_i n})$ as $n \to \infty$ from (7.4), and there are at most finitely many $t_l > \hat{t}$, we have that the first summation on the right-hand side of (7.13) is $O(\,(\omega+\epsilon)^{n^2/2})$ as $n\to\infty$, for any $\epsilon>0$. Using Lemma 7.2, we obtain for the second summation

$$\hat{\beta}_n \sum_{t_{j+i}\leq \hat{t}} |\check{\gamma}_{ni}^{(j)}(1)|\, t_{j+i}^{n+1} \leq \hat{\beta}_n \sum_{i=0}^{n} |\check{\gamma}_{ni}^{(j)}(1)|\, t_{j+i}^{n+1} \leq M\hat{\beta}_n \left(\prod_{i=0}^{n} t_{j+i}\right),$$

which, by (7.1), is also $O(\,(\omega+\epsilon)^{n^2/2})$ as $n\to\infty$, for any $\epsilon>0$. Combining the above in (7.13), we obtain (7.10). □

The following theorem is a trivial rewording of Lemma 7.3.

Theorem 7.4. *Let $\varphi(t)=t$ and pick the t_l to satisfy* (7.1). *Then, with j fixed, $\lim_{n\to\infty} A_n^{(j)}=A$, and*

$$A_n^{(j)}-A=O(\omega^{\sigma n}) \quad \text{as } n\to\infty, \quad \text{for every } \sigma>0. \tag{7.14}$$

This result can be refined as follows: Define $\hat{\beta}_s$ exactly as in Theorem 6.2. If $\hat{\beta}_n=O(e^{\sigma n^\tau})$ as $n\to\infty$ for some $\sigma>0$ and $\tau<2$, then, for any $\epsilon>0$ such that $\omega+\epsilon<1$,

$$A_n^{(j)}-A=O(\,(\omega+\epsilon)^{n^2/2}) \quad \text{as } n\to\infty. \tag{7.15}$$

Theorem 7.4 first implies that all diagonal sequences $\{A_n^{(j)}\}_{n=0}^{\infty}$ converge to A and that $|A_n^{(j)}-A|\to 0$ as $n\to\infty$ faster than $e^{-\lambda n}$ for *every* $\lambda>0$. It next implies that with a suitable and liberal growth rate on the $\hat{\beta}_n$ it is possible to achieve that $|A_n^{(j)}-A|\to 0$ as $n\to\infty$ practically like $e^{-\kappa n^2}$ for *some* $\kappa>0$.

7.2. **The case $\varphi(t)=t^\delta H(t)$ with real δ and $t_{l+1}/t_l\leq\omega\in(0,1)$.** We now come back to the general case in which $\varphi(t)=t^\delta H(t)$ with δ real, $\delta\neq 0,-1,-2,\ldots$, and $H(t)\sim H(0)\neq 0$ as $t\to 0+$. We assume only that $H(t)\in C[0,\hat{t}]$ and $H(t)\neq 0$ when $t\in[0,\hat{t}]$ for some $\hat{t}>0$ and that $H(t)\sim\sum_{i=0}^{\infty} h_i t^i$ as $t\to 0+$, $h_0\neq 0$. Similarly, $B(t)\in C[0,\hat{t}]$ and $B(t)\sim\sum_{i=0}^{\infty}\beta_i t^i$ as $t\to 0+$, as before. We do not impose any differentiability conditions either on $B(t)$ or $H(t)$. Finally, unless stated otherwise, we require the t_l to satisfy

$$\nu\leq t_{l+1}/t_l\leq\omega,\ l=0,1,\ldots, \quad \text{for some fixed } \nu \text{ and } \omega,\ 0<\nu<\omega<1, \tag{7.16}$$

instead of (7.1) only. (We recall from the remark following the statement of Lemma 4.2 that the additional condition, $\nu\leq t_{l+1}/t_l$, is naturally satisfied, for example, when $\lim_{l\to\infty}(t_{l+1}/t_l)=\lambda\in(0,1)$, cf. also (7.2). It also enables us to overcome some problems in the proofs of our main results).

We start with the following lemma that is analogous to Lemma 4.2 and Lemma 5.2.

Lemma 7.5. *Let $g(t)\sim\sum_{i=0}^{\infty} g_i t^{\theta+i}$ as $t\to 0+$, where $g_0\neq 0$ and θ is real, such that $g(t)t^{-\theta}\in C[0,\hat{t}]$ for some $\hat{t}>0$, and pick the t_l to satisfy* (7.16). *Then the following are true:*

(i) *The nonzero members of $\{D_n^{(j)}\{t^{\theta+i}\}\}_{i=0}^{\infty}$ form an asymptotic sequence both as $j\to\infty$ and as $n\to\infty$.*

(ii) *$D_n^{(j)}\{g(t)\}$ has the bona fide asymptotic expansion*

$$D_n^{(j)}\{g(t)\}\sim{\sum_{i=0}^{\infty}}^{*} g_i D_n^{(j)}\{t^{\theta+i}\} \quad \text{as } j\to\infty, \tag{7.17}$$

where the star on the summation means that only those terms for which $D_n^{(j)}\{t^{\theta+i}\} \neq 0$, *i.e., for which* $\theta + i \neq 0, 1, \ldots, n-1$, *are taken into account.*

(iii) *When* $\theta \neq 0, 1, \ldots, n-1$, *we also have*

$$D_n^{(j)}\{g(t)\} \sim g_0 D_n^{(j)}\{t^\theta\} \quad \text{as } n \to \infty. \tag{7.18}$$

When $\theta < 0$, *the condition in* (7.1) *is sufficient for* (7.18) *to hold.*

Proof. Part (i) follows from the corollary to Lemma 5.5. For the proof of part (ii) we follow the steps of the proof of part (ii) of Lemma 4.2. For arbitrary m we have $g(t) = \sum_{i=0}^{m-1} g_i t^{\theta+i} + v_m(t) t^{\theta+m}$, where $|v_m(t)| \leq C_m$ for some constant $C_m > 0$, whenever $t \in [0, \hat{t}]$ and also $t = t_l > \hat{t}$ (recall that there are at most finitely many $t_l > \hat{t}$). Thus

$$D_n^{(j)}\{g(t)\} = \sum_{i=0}^{m-1} g_i D_n^{(j)}\{t^{\theta+i}\} + D_n^{(j)}\{v_m(t) t^{\theta+m}\}. \tag{7.19}$$

We therefore have to show that $D_n^{(j)}\{v_m(t)t^{\theta+m}\} = O(D_n^{(j)}\{t^{\theta+m}\})$ as $j \to \infty$ when $\theta + m \neq 0, 1, \ldots, n-1$.

By the fact that $\tilde{\gamma}_{ni}^{(j)}(1) = c_{ni}^{(j)} t_{j+i}^{-1} / D_n^{(j)}\{t^{-1}\}$, we have

$$\begin{aligned} |D_n^{(j)}\{v_m(t)t^{\theta+m}\}| &\leq \sum_{i=0}^{n} |c_{ni}^{(j)}|\, |v_m(t_{j+i})|\, t_{j+i}^{\theta+m} \\ &\leq C_m\, |D_n^{(j)}\{t^{-1}\}| \sum_{i=0}^{n} |\tilde{\gamma}_{ni}^{(j)}(1)|\, t_{j+i}^{\theta+m+1}. \end{aligned} \tag{7.20}$$

Now taking s to be any integer that satisfies $0 \leq s \leq \min\{\theta+m, n\}$, and applying Lemma 7.2, we obtain

$$\sum_{i=0}^{n} |\tilde{\gamma}_{ni}^{(j)}(1)|\, t_{j+i}^{\theta+m+1} \leq \left(\sum_{i=0}^{n} |\tilde{\gamma}_{ni}^{(j)}(1)|\, t_{j+i}^{s+1} \right) t_j^{\theta+m-s} \leq M \left(\prod_{k=n-s}^{n} t_{j+k} \right) t_j^{\theta+m-s}. \tag{7.21}$$

Consequently, under (7.1) only,

$$|D_n^{(j)}\{v_m(t)t^{\theta+m}\}| \leq M C_m |D_n^{(j)}\{t^{-1}\}| \left(\prod_{k=n-s}^{n} t_{j+k} \right) t_j^{\theta+m-s}. \tag{7.22}$$

Recalling that $|D_n^{(j)}\{t^{-1}\}| = (\prod_{k=0}^{n} t_{j+k})^{-1}$ and $t_{l+1} \geq \nu t_l$, and invoking (4.9) that is valid in the present case, we obtain from (7.22)

$$D_n^{(j)}\{v_m(t)t^{\theta+m}\} = O(t_j^{\theta+m-n}) = O(D_n^{(j)}\{t^{\theta+m}\}) \quad \text{as } j \to \infty. \tag{7.23}$$

This completes the proof of part (ii).

As for part (iii), we first note that, by the corollary to Lemma 5.5,

$$\lim_{n \to \infty} \sum_{i=0}^{m-1} g_i D_n^{(j)}\{t^{\theta+i}\} / D_n^{(j)}\{t^\theta\} = g_0.$$

Therefore, the proof will be complete if we show that

$$\lim_{n \to \infty} D_n^{(j)}\{v_m(t)t^{\theta+m}\} / D_n^{(j)}\{t^\theta\} = 0.$$

By (7.22) we have

$$(7.24)\qquad T_n^{(j)} \equiv \frac{|D_n^{(j)}\{v_m(t)t^{\theta+m}\}|}{|D_n^{(j)}\{t^\theta\}|} \le MC_m \frac{|D_n^{(j)}\{t^{-1}\}|}{|D_n^{(j)}\{t^\theta\}|}\left(\prod_{k=n-s}^{n} t_{j+k}\right) t_j^{\theta+m-s}.$$

By the corollary to Lemma 5.5 again,

$$(7.25)\qquad \frac{|D_n^{(j)}\{t^{-1}\}|}{|D_n^{(j)}\{t^\theta\}|} \le K_1 n^{1+\theta}\left(\max_{t\in[t_{j+n},t_j]} t^{-1-\theta}\right),$$

and by (7.1),

$$(7.26)\qquad \prod_{k=n-s}^{n} t_{j+k} \le K_2 t_j^s t_{j+n}\omega^{ns},$$

where K_1 and K_2 are some positive constants independent of n. Combining these in (7.24), we have

$$(7.27)\qquad T_n^{(j)} \le LV_n^{(j)} n^{1+\theta} t_{j+n}\omega^{ns};\quad V_n^{(j)} \equiv \max_{t\in[t_{j+n},t_j]} t^{-1-\theta},$$

for some constant $L>0$ independent of n. Now (a) for $\theta \le -1$, $V_n^{(j)} = t_j^{-1-\theta}$, (b) for $-1<\theta<0$, $V_n^{(j)} = t_{j+n}^{-1-\theta}$, while (c) for $\theta>0$, $V_n^{(j)} = t_{j+n}^{-1-\theta} \le t_j^{-1-\theta}\nu^{-n(1+\theta)}$ by (7.16). Thus, (a) if $\theta \le -1$, then $T_n^{(j)} = O(n^{1+\theta}t_{j+n}\omega^{ns}) = o(1)$ as $n\to\infty$, (b) if $-1<\theta<0$, then $T_n^{(j)} = O(n^{1+\theta}t_{j+n}^{|\theta|}\omega^{ns}) = o(1)$ as $n\to\infty$, and (c) if $\theta>0$, then $T_n^{(j)} = O(n^{1+\theta}\nu^{-n(1+\theta)}\omega^{ns}) = o(1)$ as $n\to\infty$, provided we take s sufficiently large in this case, which is possible since m is arbitrary and n tends to infinity. This completes the proof. □

Our first major result concerns Process I.

Theorem 7.6. *Let $B(t)$, $\varphi(t)$, and $\{t_l\}$ be as in the first paragraph of this subsection. Then $A_n^{(j)}-A$ satisfies (3.11) and (3.12), and hence $A_n^{(j)}-A = O(\varphi(t_j)t_j^{n+\mu})$ as $j\to\infty$. In addition, $\sup_j \Gamma_n^{(j)}<\infty$, i.e., Process I is stable.*

Proof. The assertions about $A_n^{(j)}-A$ follow by applying Lemma 7.5 to $B(t)$ and to $\psi(t)\equiv 1/\varphi(t)$. As for $\Gamma_n^{(j)}$, we proceed as follows. By (5.30) and (5.33) and (7.16), we first have that

$$(7.28)\qquad \check{\Gamma}_n^{(j)}(\delta) \le \frac{n!}{|(\delta)_n|}\left(\frac{t_j}{t_{j+n}}\right)^{|\delta-1|}\check{\Gamma}_n^{(j)}(1) \le \frac{n!}{|(\delta)_n|}\nu^{-n|\delta-1|}\check{\Gamma}_n^{(j)}(1).$$

By Theorem 7.1 it therefore follows that $\sup_j \check{\Gamma}_n^{(j)}(\delta)<\infty$. Next, by Lemma 7.5 again, we have that $Y_n^{(j)} \sim h_0$ as $j\to\infty$, and $|H(t)|^{-1}$ is bounded for all t close to 0. Combining these facts in (5.31), it follows that $\sup_j \Gamma_n^{(j)}<\infty$. □

As for Process II, we do not have a stability theorem for it under the conditions of Theorem 7.6. (The upper bound on $\check{\Gamma}_n^{(j)}(\delta)$ that is given in (7.28) tends to infinity as $n\to\infty$). We do, however, have a strong convergence theorem for Process II.

Theorem 7.7. *Let $B(t)$, $\varphi(t)$, and $\{t_l\}$ be as in the first paragraph of this subsection. Then, for any fixed j, Process II is convergent whether $\lim_{t\to 0+} a(t)$ exists or not. In particular, we have $\lim_{n\to\infty} A_n^{(j)} = A$ with*

$$A_n^{(j)} - A = O(\omega^{\sigma n}) \quad \text{as } n \to \infty, \quad \text{for every } \sigma > 0. \tag{7.29}$$

This result can be refined as follows: Define $\hat{\beta}_s$ exactly as in Theorem 6.2. If $\hat{\beta}_n = O(e^{\sigma n^\tau})$ as $n \to \infty$ for some $\sigma > 0$ and $\tau < 2$, then for any $\epsilon > 0$ such that $\omega + \epsilon < 1$

$$A_n^{(j)} - A = O(\,(\omega + \epsilon)^{n^2/2}) \quad \text{as } n \to \infty. \tag{7.30}$$

When $\delta > 0$, these results are valid under (7.1).

Proof. First, by (5.28) and part (iii) of Lemma 7.5, we have that $Y_n^{(j)} \sim H(0)$ as $n \to \infty$. The proof can now be completed by also invoking (5.33) and Lemma 7.3 in (5.32). We leave the details to the reader. □

8. Practical conclusions

We have analyzed the convergence and stability of $\text{GREP}^{(1)}$ for slowly varying functions $a(t)$ with different types of collocation sequences $\{t_l\}$. From the stability analyses we have given we can derive the following practical conclusions on how we can use $\text{GREP}^{(1)}$ in these cases.

In case we must stick with the choice of the t_l as in (5.9), and cannot allow a fast growth as in (7.1), we have seen that $\text{GREP}^{(1)}$ is not stable. In this case we can avoid the problem of loss of accuracy by doing all of our computations in high-precision floating-point arithmetic, if this is possible, as explained in Sidi [Si8]. It is also clear from Theorem 5.3 that $\Gamma_n^{(j)}$, in spite of being unbounded as $j \to \infty$, is small when $|\Im\delta|$ is large since it is proportional to $1/|(\delta)_n|$. In this case $|A_n^{(j)} - A|$ is small as well since it too is proportional to $1/|(\delta)_n|$ as $j \to \infty$. Thus, when $|\Im\delta|$ is large, we may be able to obtain high accuracy in the $A_n^{(j)}$ despite the fact that the extrapolation process is definitely not stable. In the case of the $d^{(1)}$-transformation, for example, we can ensure that the t_l satisfy (5.9) by choosing $t_l = 1/R_l$, $R_l = (l+1)$, $l = 0, 1, \ldots$.

When we are able to choose the t_l as in (7.1), we can attain high accuracy and good stability properties in $\text{GREP}^{(1)}$, as we have seen in Sections 6 and 7. (Numerical experience seems to suggest that more stability is achieved by decreasing ω in (7.1).) Now in most practical situations, as $t \to 0+$, either the computational effort spent in obtaining $a(t)$ increases drastically or the computation of $a(t)$ becomes prone to roundoff error. Therefore, we must make sure that the t_l do not decrease too quickly with l. This can be achieved by choosing ω not too small. For the $d^{(1)}$-transformation on infinite series this can be achieved by letting $t_l = 1/R_l$, where R_l are positive integers determined, for example, as in

$$R_0 = 1, \quad R_l = \lfloor \sigma R_{l-1} \rfloor + 1, \quad l = 1, 2, \ldots, \quad \sigma > 1 \ \text{some constant}. \tag{8.1}$$

(Note that when $\sigma = 1$, we will have $R_l = l + 1$ for all l, in which case the $d^{(1)}$-transformation reduces to the Levin u-transformation.) It is easy to see that, with $t_l = 1/R_l$ and $\omega = \sigma^{-1} < 1$, the t_l satisfy $\omega t_l/(1 + \omega t_l) \le t_{l+1} < \omega t_l$ for all l. We thus have a sequence of t_l's that satisfy $\nu t_l \le t_{l+1} \le \omega t_l$ with some $\nu < \omega$, and hence decrease like ω^l. In addition, they satisfy $\lim_{l\to\infty}(t_{l+1}/t_l) = \omega$. (Therefore, all the conclusions of Sections 6 and 7 are valid for the $d^{(1)}$-transformation with these R_l.) Note that since $a(t_l) = A_{R_l}$ for the $d^{(1)}$-transformation, it is essential that $1/t_l$ be integers that grow exponentially but in a mild fashion. As explained in [Si6], we can choose $\sigma \in [1.1, 1.5]$, for example. This provides us with a sequence of t_l's that decrease exponentially but at a reasonable rate. Note that the choice of the R_l we have described here was first suggested in Ford and Sidi [FS].

We would like to end this work by recalling the following facts concerning the acceleration of convergence of the logarithmic sequences discussed in Examples 2.2 and 2.3 by various well known methods. These sequences have formed a very active test ground for convergence acceleration methods. An extensive numerical study that included many known methods was carried out by Smith and Ford in [SF1] and [SF2]. These authors conclude in [SF1] that, as far as logarithmic sequences are concerned, the Levin u-transformation is the across-the-board winner, followed by the θ-algorithm of Brezinski [B]. Both transformations are prone to roundoff error propagation and hence are unstable. In other words, in finite-precision arithmetic, their accuracies increase up to a certain point only. Following that, they decrease and are destroyed completely. Another method whose performance is comparable to those of the u-transformation and the θ-algorithm is the ρ-algorithm of Wynn [W], but it works only when $a_n \sim \sum_{i=0}^{\infty} \nu_i n^{-i-2}$ as $n \to \infty$, i.e., when $\gamma = -2, -3, \dots$, in Example 2.2. When γ is not an integer in Example 2.2, it can be shown rigorously that no convergence acceleration is achieved by the ρ-algorithm. That is, the ρ-algorithm has a very limited scope. We recall that the ρ-algorithm is derived from the continued fraction of Thiele [T] that is an elegant implementation of interpolation by rational functions. On the other hand, as we mentioned in the previous paragraph, the $d^{(1)}$-transformation with R_l as in (8.1) is stable and produces more accuracy without deterioration when applied to logarithmic sequences. This is the conclusion reached in the comparative numerical study of Van Tuyl [V] as well. See also the numerical results given in Example 5.1 and Table 5.1.2 of Sidi [Si6], where the $d^{(1)}$-transformation is applied to the infinite series $\sum_{k=1}^{\infty} k^{0.1-10\mathrm{i}}$ with R_l as in (8.1) and $\sigma = 1.2$. (This series diverges and its antilimit is $\zeta(-0.1+10\mathrm{i})$, where $\zeta(z)$ is the Riemann zeta function.) In quadruple-precision arithmetic (approximately 35 decimal digits) this strategy achieves an accuracy of 29 significant figures, while the u-transformation achieves an accuracy of 22 significant figures. This suggests that, with such R_l, the $d^{(1)}$-transformation is a most effective convergence acceleration method for logarithmic sequences in that it produces the highest accuracy in finite-precision arithmetic.

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